Payout policy and bubbles

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Abstract

Bubbles are believed to arise only under special circumstances, and hence to be fragile. Kocherlakota (2008) showed, however, that with endogenous solvency constraints deriving from limited commitment, as in Alvarez and Jermann (2000), and with complete markets, bubbles are a robust feature. In fact, an arbitrary bubble component can be injected in equilibrium asset prices, without affecting the consumption choices of the agents and the pricing kernel. We extend the scope of Kocherlakota's (2008) results to economies with incomplete markets, and show that any non-negative process that does not alter the set of pricing kernels can be injected in an equilibrium. Moreover, if the assets represent shares of firms that are allowed to trade, bubbles are a natural consequence of firms believing that they can manipulate their share price through an active payout policy. Active trading by firms can lead to equilibria equivalent, from the point of view of consumption allocations, to an equilibrium where firms' trading is disallowed, but where share prices have a bubble component. The total value of the firm can be preserved or not, hence the Modigliani-Miller theorem can fail.

1 Introduction

A bubble is defined as the price of an asset in excess of the discounted present value of its dividends. While developing (collapsing) price bubbles are a favorite explanation for stock market run-ups (crashes), their existence in standard stochastic dynamic general equilibrium models is possible only under special conditions. Santos and Woodford (1997) showed that bubbles can be ruled out on assets in positive supply, when the present value of aggregate consumption is finite.[1] This is always the case if, for example, there is at least an asset that grows at a (long-run) rate greater or equal to the growth rate of aggregate consumption. The outline of their argument is that optimizing agents do not allow their financial wealth to exceed the present value of their future consumption. Thus the aggregate financial wealth becomes arbitrarily small in present value terms, which is incompatible with the existence of a bubble on an asset in positive supply.

In a representative agent economy, Montrucchio and Privileggi (2001) corroborate the results of Santos and Woodford (1997) and show that under mild assumptions on agent's preferences, bubbles

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[1] They prove that there exists a discount factor (pricing kernel) compatible with the absence of arbitrage opportunities such that the fundamental value of the asset computed under this discount factor equals its price. Moreover, if the agents are sufficiently impatient, in the sense that they are always willing to trade a fixed fraction of all future consumption in exchange for the current aggregate endowment, then the price of an asset in positive supply is always equal to its fundamental value, irrespective of the choice of a discount factor compatible with the absence of arbitrage.

cannot exist. The absence of bubbles follows even without assuming the existence of a sufficiently productive asset.

The apparent fragility of bubbles was recently turned on its head by Kocherlakota (2008). His insight was that arbitrary bubbles can be injected in asset prices, while leaving agents' consumption unchanged, as long as the solvency constraints of the agents are allowed to be adjusted upwards by their initial endowment of the assets multiplied with the bubble term. The modified solvency constraints bind in exactly the same dates and states. The introduction of a bubble gives consumers a windfall proportional to their initial holding of the asset, which can be sterilized, leaving their budgets unaffected, by an appropriate tightening of the solvency constraints. He refers to this result as "the bubble equivalence theorem". Assuming that agents have the option to default on debt and receive an outside option, Alvarez and Jermann (2000) build a theory of endogenous constraints by arguing that the markets select the largest credit limits that prevent default.[2] It turns out that the modified solvency constraints resulting from the injection of a bubble are the endogenous solvency constraints allowing for maximal credit expansion and preventing default. Hence bubbles are a robust and intrinsic feature of economies where solvency constraints arise endogenously from enforcement limitations. In fact, endogenous solvency constraints à la Alvarez and Jermann (2000) are determined only up to a bubble. This was shown by Hellwig and Lorenzoni (2009) for the case when the default punishment is the interdiction to borrow, and extended by Bejan and Bidian (2010) to the general case where upon default agents receive an arbitrary exogenous continuation utility.

A major limitation of Kocherlakota's (2008) results is the assumption that that agents can trade in a full set of state-contingent claims to consumption next period, in addition to the existing finite number of infinitely lived securities. Hence one might infer that bubble injections are associated to knife-edge situations, and they might not apply even to economies with dynamically complete markets (rather than Arrow-Debreu complete).

We prove that the bubble equivalence theorem holds even when markets are incomplete, making Kocherlakota's (2008) results robust. Incomplete markets models with limited enforcement warrant study since they can capture better the limited extent of risk-sharing and the positive relation between credit limits and income in the data (Ábrahám and Cárceles-Poveda 2010). We show that any positive process that does not distort the set of pricing kernels can be injected in the asset prices as a bubble. Gain processes associated to a large class of trading strategies satisfy this condition. We also allow for more general punishments after default than in Kocherlakota (2008). In particular, we cover the case where upon default the agents are forbidden to carry debt (Bulow and Rogoff 1989, Hellwig and Lorenzoni 2009). For this outside option, the agents' punishment continuation utilities after default depend on asset prices, since lending is still allowed, and could be affected by a bubble injection. It turns out, however, that this is not the case.

Next we allow firms to trade and extend the results of DeMarzo (1988) to infinite horizon economies. Thus we show that consumers can undo the effects of firms' trading by adjusting their portfolio holdings and keeping their consumption unaltered. However, with an infinite horizon, bubbles are a natural consequence of firms believing that they can manipulate their share price by changing their payout policy. While DeMarzo (1988) ignored the limited liability of shareholders, we give conditions under which firms' trading results in positive dividends and prices. Active trading by firms can lead to equilibria equivalent from the point of view of consumption allocations to an

[2] Due to limited enforceability of contracts, agents can default on their debt at any date and state and leave the economy, receiving a fixed continuation utility that can be date and state dependent. The solvency constraints are set in each period to the maximum level so that repayment is always individually rational given future debt limits.

equilibrium where firms' trading is disallowed, but where share prices have a bubble component even though no bubble existed under no-trading. We construct such equilibria with trading where the total value of the firms is the same as under no-trading, hence the Modigliani-Miller theorem holds. However there are equilibria with trading in which the total value of the firms can exceed the their total value under no-trading by a bubble, in which case the Modigliani-Miller theorem fails.

Section 2 presents the model and extends the results of Kocherlakota's (2008) to incomplete markets. Section 3 allows firms to trade and show how this can lead to bubbles. The appendixes contain technical results. Appendix A gives necessary and sufficient conditions on a process, which if added to asset prices, will not distort the pricing kernels (and the one-period asset spans). Appendix B shows that gain processes associated to a large class of strategies satisfy those conditions. Appendix C investigates the effect of a bubble injection on the volume of trade. Finally, Appendix D discusses the existence of equilibria with endogenous solvency constraints à la Alvarez and Jermann (2000).

2 Bubble injections

We consider a stochastic, discrete-time, infinite horizon economy. The time periods are indexed by the set $\mathbb{N} := \{0, 1, \ldots\}$. The uncertainty is described by a probability space $(\Omega, \mathcal{F}, P)$ and by the filtration $(\mathcal{F}_t)_{t=0}^{\infty}$, which is an increasing sequence of σ-algebras on the set of states of the world Ω, generating $\mathcal{F}$, that is such that $\mathcal{F} = \sigma(\cup_t \mathcal{F}_t)$. We interpret $\mathcal{F}_t$ as the information available at period t. We assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and that $\mathcal{F}_t$ is a finite σ-algebra, for all t. For $\omega \in \Omega$ and $t \in \mathbb{N}$, we let $\mathcal{F}_t(\omega) := \cap\{A \in \mathcal{F}_t \mid \omega \in A\}$.[3]

A sequence $x = (x_t)_{t\in\mathbb{N}}$ of random variables ($\mathcal{F}$-measurable real-valued functions) is a *stochastic process adapted to* $(\mathcal{F}_t)_{t=0}^{\infty}$ ("process" henceforth) if for each $t \in \mathbb{N}$, x_t is $\mathcal{F}_t$-measurable. We let X be the set of all stochastic processes, and denote by X_+ (respectively X_{++}) the processes $x \in X$ such that $x_t \geq 0$ P-almost surely (respectively $x_t > 0$ P-almost surely) for all $t \in \mathbb{N}$. Throughout the paper, all statements, equalities, and inequalities involving random variables are assumed to hold only "P-almost surely", and we will omit adding this qualifier. When $K, L \in \mathbb{N}\backslash\{0\}$, let $X^{K\times L}$ be the set of vector (or matrix) processes $(x^{ij})_{1\leq i\leq K, 1\leq j\leq L}$ with $x^{ij} \in X$. For $x \in X^{K\times L}$, we write $x \geq 0$ (respectively $x > 0, x = 0$) if for all $1 \leq i \leq K, 1 \leq j \leq L$ and $t \in \mathbb{N}$, $x_t^{ij} \geq 0$ (respectively $x_t^{ij} > 0, x_t^{ij} = 0$). Similarly $x \gneqq 0$ means that $x \geq 0$ but $x \neq 0$. The set of non-negative processes $x \in X^{K\times L}$ (that is, such that $x \geq 0$) is denoted by $X_+^{K\times L}$.

There is a single consumption good and a finite number, I, of consumers. An agent $i \in \{1, 2, \ldots, I\}$ has endowments $e^i \in X_+$, and his preferences are represented by a utility $U : X_+ \to \mathbb{R}$ represented by $U^i(c) = E\sum_{t=0}^{\infty} u_t^i(c_t)$, where $u_t^i : \mathbb{R}_+ \to \mathbb{R}$ is continuous, increasing and concave and $E(\cdot)$ is the expectation operator with respect to probability P. The conditional expectation given the information available at t, $\mathcal{F}_t$, is denoted by $E_t(\cdot)$. Given the absence of information at period 0, $E_0(\cdot) = E(\cdot)$. We denote by $U_t^i(c) := E_t \sum_{s\geq t} u_s^i(c_s)$ the continuation utility of agent i after t provided by a consumption stream $c \in X_+$.

There is a finite number J of infinitely lived, disposable securities, traded at every date. The securities are shares of firms, having singleton production sets.[4] The earnings of firm $j \in \{1, 2, \ldots, J\}$

[3]Using the usual "event tree" terminology, $\mathcal{F}_t(\omega)$ is the date t node containing state ("leaf") ω (for the parallel between the stochastic processes vs. event tree language, see Leroy and Werner 2001, chapter 21).

[4]Thus we study the effect of firms' trading conditional on their choice of a production plan.

are described by the process $d^j \in X_+$. In this section, firms are regarded as passive agents that pay out their earnings exclusively in the form of dividends, assumption which will be relaxed in the next section. The ex-dividend price per share of firm j is a process $p^j \in X_+$. We denote by $d = (d^1, \ldots, d^J) \in X_+^{1\times J}$ the dividend vector process, and by $p = (p^1, \ldots, p^J) \in X_{++}^{1\times J}$ the price vector process.

Consumer i has an initial endowment $\theta_{-1}^i \in \mathbb{R}^J$ of securities and his trading strategy is represented by a process $\theta^i \in X^{J\times 1}$. We assume, without loss of generality, that firms are in unit supply, thus $\sum_{i=1}^I \theta_{-1}^i = \mathbf{1} := (1, \ldots, 1)' \in \mathbb{R}^J$. Fix some wealth bounds $w^i \in X$ for agent i and define the budget constraint and indirect utility of an agent i from period $s \geq 0$ onward, when faced with prices $p \in X_+^{1\times J}$, solvency bounds $w^i \in X$ and having an initial wealth $\nu_s : \Omega \to \mathbb{R}$ which is $\mathcal{F}_s$-measurable, as:[5]

$$B_s^i(\nu_s, w^i, p, d) = \{(c^i, \theta^i) \in X_+ \times X^{J\times 1} \mid c_s^i + p_s\theta_s^i \leq e_s^i + \nu_s, \tag{2.1}$$

$$c_t^i + p_t\theta_t^i \leq e_t^i + (p_t + d_t)\theta_{t-1}^i, (p_t + d_t)\theta_{t-1}^i \geq w_t^i, \forall t > s\}$$

$$V_s^i(\nu_s, w, p, d) = \max_{(c^i,\theta^i)\in B_s^i(\nu_s,w^i,p,d)} U_t^i(c^i). \tag{2.2}$$

Definition 1. *A vector $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I\right)$ consisting of[6] a security price process $\bar{p} \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, solvency constraints $\bar{w}^i \in X$, a consumption process $\bar{c}^i \in X_+$ and a trading strategy $\bar{\theta}^i \in X^{J\times 1}$ is an equilibrium with exogenous solvency constraints if the following conditions are met:*

i. The consumption and trading strategies of each agent i are feasible, that is $(\bar{c}^i, \bar{\theta}^i) \in B_0((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d)$, and optimal,

$$U^i(\bar{c}^i) = V_0^i\left((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d\right). \tag{2.3}$$

ii. Markets for goods and firms' shares clear,

$$\sum_{i=1}^I \bar{c}_t^i = \sum_{i=1}^I e_t^i + d_t \cdot \mathbf{1}, \forall t \in \mathbb{N}, \tag{2.4}$$

$$\sum_{i=1}^I \theta_t^i = \mathbf{1}, \forall t \in \mathbb{N}. \tag{2.5}$$

Consider an equilibrium $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ with exogenous solvency constraints. The absence of arbitrage opportunities implies the existence of $a \in X_{++}$ such that (see, for example, Santos and Woodford 1997)

$$a_t p_t = E_t\left[a_{t+1}(p_{t+1} + d_{t+1})\right], \forall t \geq 0. \tag{2.6}$$

We denote by $A(p, d)$ the set of all processes $a \in X$ satisfying equation (2.6), and we call them *deflators*. Strictly positive deflators belonging to $A_{++}(p, d) := A(p, d) \cap X_{++}$ will be called *state*

[5] $B_s^i(\nu_s, w^i, p, d)$ depends solely on $(w_t^i)_{t>s}, (p_t)_{t\geq s}$, rather than on the full processes w, p. We prefer this simplified notation to, for example, $B_s^i(\nu_s, (w_t^i)_{t>s}, (p_t)_{t\geq s}, (d_t)_{t\geq s})$, for reasons of simplicity.

[6] Although solvency constraints are exogenous here, we include them in the equilibrium outcome for ease of the exposition.

price densities, or (interchangeably) *pricing kernels*. Equation (2.6) implies that

$$p_t = \frac{1}{a_t} E_t \sum_{s>t} a_s d_s + \lim_{T\to\infty} \frac{1}{a_t} E_t a_T p_T.$$

Thus

$$b_t(a,p) := \frac{1}{a_t} \lim_{T\to\infty} E_t a_T p_T \tag{2.7}$$

is well defined and non-negative, and for all $t \in \mathbb{N}$, $a_t b_t(a,p) = E_t a_{t+1} b_{t+1}(a,p)$. Thus $a \cdot b(a,p)$ is a non-negative martingale, and $b(a,p) = 0$ if and only if $b_0(a,p) = \frac{1}{a_0} \lim_{t\to\infty} E a_t p_t = 0$. We interpret the discounted present value of dividends d under under the state price density a, that is $f_t(a,d) := \frac{1}{a_t} E_t \sum_{s>t} a_s d_s$, as the *fundamental value* of d. Thus $b(a,p)$ represents the part of asset prices in excess of fundamental value. Following Santos and Woodford (1997), we say that the equilibrium price process p *ambiguously involves a bubble* if $b_0(a,p) > 0$ for some $a \in A_{++}(p,d)$, while $b_0(a',p) = 0$ for some other $a' \in A_{++}(p,d)$. If $b_0(a,p) > 0$ for all $a \in A_{++}(p,d)$, the equilibrium price process *unambiguously involves a bubble* component.[7]

Kocherlakota (2008) assumed that in addition to trading in firms' shares, agents can also trade in each period a full set of state-contingent claims to consumption next period. Given an equilibrium without bubbles in which the asset prices are p and the state price density[8] is a, and given an arbitrary process $\varepsilon \in X_+^{1\times J}$ such that $a\cdot\varepsilon$ is a martingale, he showed that an "equivalent" equilibrium with asset prices $p + \varepsilon$, pricing kernel a and identical consumption paths for the agents can be constructed. Moreover, in the new equilibrium, the solvency constraints bind in exactly the same dates and states as in the original equilibrium. He dubbed this result the "bubble equivalence theorem", since the process ε "injected" in the asset prices is the bubble component for the price process $p + \varepsilon$, that is $\varepsilon = b(a, p + \varepsilon)$.

We show that Kocherlakota's (2008) bubble equivalence theorem holds in our incomplete markets framework, if the class of candidate processes to be injected in asset prices is restricted to non-negative elements of the set

$$\begin{aligned} M^J(p,d) = \big\{\varepsilon \in X^{1\times J} \mid \exists \Lambda \in X^{J\times J}, a \in A_{++}(p,d) \text{ s.t. } \forall t > 0, \varepsilon_t = (p_t + d_t)\Lambda_{t-1}, \\ \mathbf{I} + \Lambda_{t-1} \text{ is non-singular and } a_{t-1}\varepsilon_{t-1} = E_{t-1} a_t \varepsilon_t\big\}, \end{aligned} \tag{2.8}$$

where $\mathbf{I}$ denotes the J-dimensional identity matrix. Let also $M_+^J(p,d) := M^J(p,d) \cap X_+^{1\times J}$. As shown in the Appendix A (Lemmas 2 and 3, and Proposition 4), a process ε belongs to $M^J(p,d)$ if and only if the set of deflators associated to prices p and $p+\varepsilon$ coincide, that is $A(p,d) = A(p+\varepsilon,d)$, or equivalently, if and only if p and $p + \varepsilon$ generate the same one-period asset spans and $a \cdot \varepsilon$ is a martingale for any deflator $a \in A(p,d)$. It is well known that the gain process of any security or trading strategy becomes a martingale when deflated by any $a \in A(p,d)$ (see, for example, Leroy and Werner 2001, p.261). Lemma 4 in the Appendix A shows that the gain process associated with the shares of the J firms belongs to $M_+^J(p,d)$, which is thus a non-empty, and potentially very large set. For any $\varepsilon \in M^J(p,d)$, we let

$$\Lambda(\varepsilon,p,d) := \big\{\Lambda \in X^{J\times J} \mid \forall t \geq 1, \varepsilon_t = (p_t + d_t)\Lambda_{t-1} \text{ and } \mathbf{I} + \Lambda_{t-1} \text{ is non-singular}\big\}. \tag{2.9}$$

[7] In Section 3, we apply the notation $b(a',p')$ and $f(a',d')$ to arbitrary processes $a' \in X_{++}$, and $p',d' \in X_+^{1,k}$ for some $k \in \mathbb{N} \setminus \{0\}$.

[8] The pricing kernel is unique when markets are complete .

We prove first that agents' feasible consumption paths remain unchanged when prices are inflated by a bubble in $M_+^J(p,d)$, if the solvency constraints are adjusted upwards by the bubble multiplied by initial wealth.

Proposition 1. *Consider an agent i, having beginning-of-period-t wealth equal to ν_t ($\mathcal{F}_t$-measurable). Then for any $\bar{\theta}_{-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_t$-measurable and any $\varepsilon \in M_+^J(p,d)$,*

$$(c^i, \theta^i) \in B_t^i\left(\nu_t, w^i, p, d\right) \Longleftrightarrow (c^i, \hat{\theta}^i) \in B_t^i\left(\nu_t + \varepsilon_t \bar{\theta}_{-1}, w^i + \varepsilon \bar{\theta}_{-1}, p + \varepsilon, d\right),$$

where $\hat{\theta}_s^i = (\mathbf{I} + \Lambda_s)^{-1}\left(\theta_s^i + \Lambda_s \bar{\theta}_{-1}\right)$ for every $s \geq t$, and $\Lambda \in \Lambda(\varepsilon, p, d)$.

Proof. Notice that for all $s \geq t$,

$$\begin{aligned}\hat{\theta}_s^i = (\mathbf{I}+\Lambda_s)^{-1}\theta_s^i + (\mathbf{I}+\Lambda_s)^{-1}\Lambda_s\bar{\theta}_{-1} &= \left(I - (\mathbf{I}+\Lambda_s)^{-1}\Lambda_s\right)\theta_s^i + (\mathbf{I}+\Lambda_s)^{-1}\Lambda_s\bar{\theta}_{-1} \\ &= \theta_s^i + (\mathbf{I}+\Lambda_s)^{-1}\Lambda_s(\bar{\theta}_{-1} - \theta_s^i).\end{aligned}$$

Let $\lambda_s^i := (\mathbf{I}+\Lambda_s)^{-1}\Lambda_s(\bar{\theta}_{-1} - \theta_s^i)$. Therefore, for any $s \geq t+1$,

$$(p_s + d_s + \varepsilon_s)\lambda_{s-1}^i = (p_s + d_s)(\mathbf{I} + \Lambda_{s-1})\lambda_{s-1}^i = (p_s + d_s)\Lambda_{s-1}(\bar{\theta}_{-1} - \theta_s^i) = \varepsilon_s(\bar{\theta}_{-1} - \theta_{s-1}^i),$$

and thus

$$(p_{s-1} + \varepsilon_{s-1})\lambda_{s-1}^i = E_s \frac{a_s}{a_{s-1}}(p_s + d_s + \varepsilon_s)\lambda_{s-1}^i = \varepsilon_{s-1}(\bar{\theta}_{-1} - \theta_{s-1}^i).$$

It follows that

$$\nu_t + \varepsilon_t\bar{\theta}_{-1} - (p_t + \varepsilon_t)\hat{\theta}_t^i = \nu_t - p_t\theta_t^i + \varepsilon_t(\bar{\theta}_{-1} - \theta_t^i) - (p_t + \varepsilon_t)\lambda_t = \nu_t - p_t\theta_t^i$$

and (for $s \geq t+1$)

$$\begin{aligned}(p_s + d_s + \varepsilon_s)\hat{\theta}_{s-1}^i - (p_s + \varepsilon_s)\hat{\theta}_s^i &= \\ &= [(p_s + d_s + \varepsilon_s)\theta_{s-1}^i - (p_s + \varepsilon_s)\theta_s^i] + [(p_s + d_s + \varepsilon_s)\lambda_{s-1}^i - (p_s + \varepsilon_s)\lambda_s^i] \\ &= [(p_s + d_s)\theta_{s-1}^i + \varepsilon_s\theta_{s-1}^i - p_s\theta_s^i - \varepsilon_s\theta_s^i] + [\varepsilon_s(\bar{\theta}_{-1} - \theta_{s-1}^i) - \varepsilon_s(\bar{\theta}_{-1} - \theta_s^i)] \\ &= (p_s + d_s)\theta_{s-1}^i - p_s\theta_s^i.\end{aligned}$$

Moreover, for $s \geq t+1$,

$$\begin{aligned}(p_s + d_s + \varepsilon_s)\hat{\theta}_{s-1}^i &= (p_s + d_s + \varepsilon_s)\theta_{s-1}^i + (p_s + d_s + \varepsilon_s)\lambda_{s-1}^i \\ &= (p_s + d_s)\theta_{s-1}^i + \varepsilon_s\theta_{s-1}^i + \varepsilon_s(\bar{\theta}_{-1} - \theta_{s-1}^i) \\ &= (p_s + d_s)\theta_{s-1}^i + \varepsilon_s\bar{\theta}_{-1}.\end{aligned}$$

Thus $(p_s + d_s + \varepsilon_s)\hat{\theta}_{s-1}^i \geq w_s^i + \varepsilon_s\theta_{-1}^i$ if and only if $(p_s + d_s)\theta_{s-1}^i \geq w_s^i$, and the conclusion follows. □

The intuition for the proposition is as follows. With bubble-inflated prices, the initial owners of the asset receive a windfall in the form of higher initial wealth. Tightening their future solvency constraints by the bubble weighted by initial asset holdings will force them to save the initial windfall in order to meet the more stringent borrowing requirements, leading thus to equivalent budget constraints.

Given an equilibrium without bubbles with asset prices p, for any process $\varepsilon \in M_+^J(p,d)$, we show that there is an equivalent equilibrium with prices $p+\varepsilon$, identical consumption and state price densities, and in which the solvency constraints bind in exactly the same date and states (even though they differ). Moreover ε is the bubble component in the prices $p+\varepsilon$ for any state price density $a \in A(p+\varepsilon,d)(=A(p,d))$, that is $\varepsilon = b(a, p+\varepsilon)$, hence the new equilibrium unambiguously involves a bubble.

Theorem 2. *Let $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ be an equilibrium with exogenous solvency constraints. Choose $\varepsilon \in M_+^J(p,d)$ and $\Lambda \in \Lambda(\varepsilon,p,d)$. Then $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i,\hat{\theta}^i)_{i=1}^I\right)$ is an equilibrium with exogenous solvency constraints, where*

$$\hat{p} = p+\varepsilon, \quad \hat{\theta}^i = (\mathbf{I}+\Lambda)^{-1}\left(\theta^i + \Lambda\theta^i_{-1}\right), \quad \hat{w}^i = w^i + \varepsilon\theta^i_{-1}. \tag{2.10}$$

Proof. Optimality of $(c^i,\hat{\theta}^i)$ in the set $B_0^i((\hat{p}_0+d_0)\theta^i_{-1}, \hat{w}, \hat{p}, d)$ follows from the optimality of (c^i,θ^i) in $B_0^i((p_0+d_0)\theta^i_{-1}, w, p, d)$, and the equality of these two budgets (Proposition 1). Notice that $\sum_i \hat{\theta}^i_t = (\mathbf{I}+\Lambda)^{-1}(\mathbf{1}+\Lambda\mathbf{1}) = \mathbf{1}$, since $\sum_i \theta^i_t = \sum_i \theta^i_{-1} = \mathbf{1}$. Thus the market clearing conditions are satisfied. □

While the injection of the bubble leaves agents' consumption unchanged, it can affect asset price returns. Therefore it can potentially reconcile the high volatility of returns to the relative stable consumption data (the "equity premium puzzle). The injection of a bubble may affect also the volume of trade. This is discussed in Appendix C. The "bubble equivalence" theorem above compares equilibria with different solvency constraints. This seems artificial, if the borrowing constraints are viewed as exogenously given. We allow for the *endogenous* determination of borrowing constraints driven by limited commitment/imperfect enforcement as in Alvarez and Jermann (2000), and show that the bubble inflated constraints in the equivalent equilibrium are also compatible with the endogenous mechanism determining the allowed borrowing limits.

Assume that at any period t, when facing prices p (and dividends d), consumer i can choose to default on his beginning of period debt[9] and leave the economy, receiving a continuation utility after default $\widetilde{V}_t^i(p,d)$ (assumed $\mathcal{F}_t$-measurable). Thus the "punishment" continuation utility for each agent i is described by a mapping $\widetilde{V}^i : X_+^{1\times J} \times X_+^{1\times J} \to X$. We allow this continuation utility mapping to depend on prices and dividends explicitly, since in the next section dividends will be endogenous and will depend on the trading strategies of the firm. Moreover the mapping $\widetilde{V}^i$ can depend on agents' endowments and other exogenous primitives of the economy. Alvarez and Jermann (2000), following Kehoe and Levine (1993), worked under the assumption that agents are banned from trading following default, hence for each agent i,

$$\widetilde{V}_t^i(p,d) := U_t^i(e^i). \tag{2.11}$$

Alternatively, Hellwig and Lorenzoni (2009), building on the work of Bulow and Rogoff (1989), assume that agents can continue to lend but not to borrow following default, thus in their case

$$\widetilde{V}_t^i(p,d) := V_t^i(0,0,p,d), \tag{2.12}$$

where the second argument in $V_t^i(0,0,p,d)$ is the process in X equal to zero at any date and state.

Some of the results that follows rely on continuation utilities after default being of the form (2.11) or (2.12), in which case we will impose the following:

[9] This is equal to $(p_t+d_t)\theta^i_{t-1}$ if his trading strategy is $\theta^i \in X^{J\times 1}$.

Assumption 1. *One of the following holds:*

(i) For each $i \in \{1, \ldots, I\}$, there exists $\widetilde{U}^i \in X$ such that $\widetilde{V}^i(p,d) = \widetilde{U}^i$ for all p, d.

(ii) For each $i \in \{1, \ldots, I\}$, there exists $\widetilde{w}^i \in X$ such that $\widetilde{V}_t^i(p,d) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p, d)$ for all p, d.

As in Alvarez and Jermann (2000), the option to default endogenizes the solvency constraints in each period to the maximum level so that repayment is always individually rational given future debt limits. We present the definition of solvency constraints that are *not too tight*.

Definition 3. *The solvency constraints w^i faced by the agent i are not too tight (NTT) given prices p, dividends d and the punishment continuation utility described by $\widetilde{V}^i$ if and only if*

$$V_t^i(w_t^i, w^i, p, d) = \widetilde{V}_t^i(p,d), \forall t.$$

The definition captures the idea that the bounds w^i have to be "tight enough" to prevent default ($V_t^i(w_t^i, w^i, p, d) \geq \widetilde{V}_t^i(p,d)$), but they should allow for maximum credit expansion (thus one should not have $V_t^i(w_t^i, w^i, p, d) > \widetilde{V}_t^i(p,d)$ on a positive probability set). One can envision the NTT solvency constraints as being imposed to agents, who cannot trade directly with each other, by competitive financial intermediaries. The intermediaries set constraints such that default is prevented, but credit is not restricted unnecessarily, since competing intermediaries could relax them and increase their profits (see Ábrahám and Cárceles-Poveda (2010) for such a model in an economy with production).

We extend our definition of equilibrium to allow for the endogenous determination of solvency constraints, in the presence of an outside option to default. An *Alvarez-Jermann equilibrium* (*AJ-equilibrium*, for short) with no trade by firms is a vector $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ consisting of a security price process $\bar{p} \in X_+^{1\times J}$, and for each agent $i \in \{1, \ldots, I\}$, solvency constraints $\bar{w}^i \in X$, a consumption process $\bar{c}^i \in X_+$, a trading strategy $\bar{\theta}^i \in X^{J\times 1}$ and a mapping $\widetilde{V}^i$ from prices and dividends into continuation utilities after default such that $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I\right)$ is an equilibrium, and w^i are not too tight given penalties $\widetilde{V}^i(p,d)$ for default. The existence of Alvarez-Jermann equilibria is discussed in Appendix D.

We show next that bubble injections as in Theorem 2 preserve the NTT condition on solvency constraints, under a mild assumption on the form of penalties for default.

Theorem 4. *Let $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ be an AJ-equilibrium. Choose $\varepsilon \in M_+^J(p,d)$ and $\Lambda \in \Lambda(\varepsilon, p, d)$. If $\widetilde{V}^i(p+\varepsilon, d) = \widetilde{V}^i(p,d)$ for all agents $i \in \{1, \ldots, I\}$, then $\left(\hat{p}, (\hat{w}^i)_{i=1}^I, (c^i, \hat{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ is an AJ-equilibrium, where $\hat{p}, \hat{\theta}^i, \hat{w}^i$ are given by* (2.10).

Proof. We need to prove that $\hat{w}^i = w^i + \varepsilon\theta_{-1}^i$ are not too tight for prices $\hat{p}$. This is a consequence of Proposition 1, since

$$\widetilde{V}^i(\hat{p}, d) = \widetilde{V}^i(p,d) = V_t^i(w_t^i, w^i, p, d) = V_t^i(w_t^i + \varepsilon_t\theta_{-1}^i, w^i + \varepsilon\theta_{-1}^i, p+\varepsilon, d) = V_t^i(\hat{w}_t^i, \hat{w}^i, \hat{p}, d).$$

□

The condition $\widetilde{V}^i(p+\varepsilon, d) = \widetilde{V}^i(p,d)$ in the Theorem is trivially satisfied for default punishments satisfying Assumption 1, Part (i), since in this case $\widetilde{V}^i$ does not depend on prices and dividends. It holds also for default punishments satisfying Assumption 1, Part (ii), since, by Proposition 1, with $\nu_t := \widetilde{w}_t^i$, $\theta_{-1} := 0 \in \mathbb{R}^J$ and $w^i := \widetilde{w}^i$, $V_t^i(\widetilde{w}_t^i + \varepsilon_t \cdot 0, \widetilde{w}^i, p+\varepsilon, d) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p, d)$, and therefore

$$\widetilde{V}_t^i(p+\varepsilon, d) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p+\varepsilon, d) = V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, p, d) = \widetilde{V}_t^i(p, d). \tag{2.13}$$

3 Bubble injections with non-trivial payout policies

In this section, we consider the case in which firms can allocate their earnings $d = (d^1, \ldots, d^j) \in X^{1\times J}$ among dividend payouts, share repurchases and portfolio investments in the shares of the other firms. Firms' trading affect simultaneously their dividends and their security prices, since the dividends depend on the prices of the securities, which in turn depend on the dividends. The trading strategy of firm j is described by a vector $\gamma^j \in X^{J\times 1}$, representing holdings of shares of the J firms. Thus for $k \in \{1, \ldots, J\}$ and $t \in \mathbb{N}$, $\gamma_t^{k,j}$ represents firm j's holdings of shares of firm k at period t. We let $\gamma = (\gamma^1, \ldots, \gamma^J) \in X^{J\times J}$ and assume throughout that $\gamma_{-1} := 0 \in \mathbb{R}^{J\times J}$. We impose several conditions on trading strategies γ, so they lead to well-defined dividends and security prices. At a minimum, the trading strategies have to result in prices and dividends that satisfy the budget constraints of the firms, and respect portfolio constraints designed to prevent firms from running Ponzi schemes that would enable them to offer arbitrarily large dividend streams. We assume that firms are subject to no-debt requirements. Define Γ to be the set of strategies γ such that

(i) There exist positive prices $p \in X_+^{1\times J}$ and dividends $\delta \in X_+^{1\times J}$ such that firms' budget constraints and No-Ponzi game conditions are satisfied, that is for all $j \in \{1, \ldots, J\}$,

$$\delta_t^j = d_t^j + (p_t + \delta_t)\gamma_{t-1}^j - p_t\gamma_t^j, \forall t \geq 0, \tag{3.1}$$

$$(p_t + \delta_t)\gamma_{t-1}^j \geq 0, \forall t \geq 1. \tag{3.2}$$

(ii) γ is *regular*, that is $\mathbf{I} - \gamma_t$ is nonsingular, for all $t \geq 0$.[10]

A regular trading strategy γ produces unambiguous dividends for a given p, as seen from (3.1). Denote the set of prices and dividends compatible with firm j trading according to $\gamma^j \in X^{J\times 1}$ by

$$PD^j(\gamma^j) := \left\{(p, \delta) \in X_+^{1\times J} \times X_+^{1\times J} \mid (3.1) - (3.2) \text{ hold}\right\}, \tag{3.3}$$

and for $\gamma = (\gamma^1, \ldots, \gamma^J) \in X^{J\times J}$, let $PD(\gamma) = \cap_{j=1}^J PD^j(\gamma^j)$.

We analyze first equilibria induced by a fixed financial policy of the firms. An *Alvarez-Jermann equilibrium corresponding to a fixed payout policy* γ (henceforth an $AJ(\gamma)$-equilibrium) is a vector $\left(\bar{p}(\gamma), \bar{d}(\gamma), (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ consisting of security prices and dividend processes $\bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$, and for all $i \in \{1, \ldots, I\}$, solvency constraints $\bar{w}^i \in X$, consumption processes $\bar{c}^i \in X_+^J$, trading strategies $\bar{\theta}^i \in X^{J\times 1}$, and default continuation utilities $\widetilde{V}^i$ such that

(i) For each agent i, $(\bar{c}^i, \bar{\theta}^i)$ is optimal in $B_0^i((\bar{p}_0(\gamma) + \bar{d}_0(\gamma))\theta_{-1}^i, \bar{w}^i, \bar{p}(\gamma), \bar{d}(\gamma))$.

(ii) Constraints $\bar{w}^i$ are not too tight given penalties $\widetilde{V}^i(\bar{p}(\gamma), \bar{d}(\gamma))$ for default,

[10] Such strategies are called *proper* by DeMarzo (1988) or *regular* by Duffie (1988, p.122).

(iii) Prices and dividends are compatible with γ, that is $(\bar{p}(\gamma), \bar{d}(\gamma)) \in PD(\gamma)$.

(iv) Markets clear, i.e.

$$\sum_{i=1}^{I} \bar{c}_t^i = \sum_{i=1}^{I} e_t^i + \sum_{j=1}^{J} d_t^j, \ \forall t \in \mathbb{N}, \tag{3.4}$$

$$\sum_{i=1}^{I} \bar{\theta}_t^i + \sum_{j=1}^{J} \gamma_t^j = \mathbf{1}, \ \forall t \in \mathbb{N}. \tag{3.5}$$

Notice that for each $\gamma \in \Gamma$ and $(p, \delta) \in PD(\gamma)$, by equation (3.1), for any $a \in A(p, \delta)$,

$$E_t a_{t+1}(p_{t+1} + \delta_{t+1}) = E_t a_{t+1} \left(\delta_{t+1} + (p_{t+1} + \delta_{t+1})\gamma_t + p_{t+1}(\mathbf{I} - \gamma_{t+1})\right),$$

and therefore

$$a_t p_t(\mathbf{I} - \gamma_t) = E_t a_{t+1} \left(p_{t+1}(\mathbf{I} - \gamma_{t+1}) + \delta_{t+1}\right), \quad \forall t \geq 0, \forall a \in A(p, \delta). \tag{3.6}$$

This suggest the next result, in which we show, in the spirit of Modigliani-Miller theorem, that any financial trading by the firms can be "undone" by consumers.

Theorem 5. *Let $\gamma \in \Gamma$ and $\bar{p}, \bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$ such that $(\bar{p}_t(\gamma) + \bar{d}_t(\gamma))\gamma_{t-1} \geq 0$ for all $t > 0$ and*

$$\bar{p}_t(\gamma) = \bar{p}_t(\mathbf{I} - \gamma_t)^{-1}, \forall t \geq 0, \tag{3.7}$$

$$\bar{p}_t(\gamma) + \bar{d}_t(\gamma) = (\bar{p}_t + d_t)(\mathbf{I} - \gamma_{t-1})^{-1}, \forall t \geq 0. \tag{3.8}$$

Consider some $(\widetilde{V}^i)_{i=1}^I$ satisfying Assumption 1. Then $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ is an AJ-equilibrium (with no trade by firms) if and only if $\left(\bar{p}(\gamma), \bar{d}(\gamma), (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i(\gamma))_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ is an $AJ(\gamma)$-equilibrium, where

$$\bar{\theta}_t^i(\gamma) := (\mathbf{I} - \gamma_t)\bar{\theta}_t^i, \forall t \geq 0, \forall i = 1, \ldots, I. \tag{3.9}$$

Moreover, $A(\bar{p}(\gamma), \bar{d}(\gamma)) = A(\bar{p}, d)$.

Proof. It is immediate to check that $(\bar{p}(\gamma), \bar{d}(\gamma)) \in PD(\gamma)$ and that $(\bar{c}^i, \bar{\theta}^i) \in B_0((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d)$ if and only if $(\bar{c}^i, \bar{\theta}^i(\gamma)) \in B_0((\bar{p}_0(\gamma) + \bar{d}_0(\gamma))\theta_{-1}^i, \bar{w}^i, \bar{p}(\gamma), \bar{d}(\gamma))$. Moreover, since $(\tilde{c}^i, \tilde{\theta}^i) \in B_0((\bar{p}_0(\gamma) + \bar{d}_0(\gamma))\theta_{-1}^i, \bar{w}^i, \bar{p}(\gamma), \bar{d}(\gamma))$ if and only if $(\tilde{c}^i, ((\mathbf{I} - \gamma_t)^{-1}\tilde{\theta}_t^i)_t) \in B_0((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d)$, it follows that $(\bar{c}^i, \bar{\theta}^i)$ is optimal in $B_0((\bar{p}_0 + d_0)\theta_{-1}^i, \bar{w}^i, \bar{p}, d)$ if and only if $(\bar{c}^i, \bar{\theta}^i(\gamma))$ is optimal in $B_0((\bar{p}_0(\gamma) + \bar{d}_0(\gamma))\theta_{-1}^i, \bar{w}^i, \bar{p}(\gamma), \bar{d}(\gamma))$.

Clearly, $\left((\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I\right)$ is feasible when firms do not trade if and only if $\left((\bar{c}^i)_{i=1}^I, (\bar{\theta}^i(\gamma))_{i=1}^I\right)$ is feasible when firms adopt the trading strategy γ. Moreover, $V_t^i(\widetilde{w}_t^i, \widetilde{w}^i, \bar{p}, d) = \widetilde{V}_t^i(\widetilde{w}_t^i, \widetilde{w}^i, \bar{p}(\gamma), \bar{d}(\gamma))$ and thus, $\widetilde{V}^i(\bar{p}, d) = \widetilde{V}^i(\bar{p}(\gamma), \bar{d}(\gamma))$ if $\widetilde{V}^i$ is as in Assumption 1, (ii). The same equality holds trivially if $\widetilde{V}^i$ satisfies Assumption 1, (i). Hence, solvency constraints are not too tight at $(\bar{p}, d)$ if and only if they are not too tight at $(\bar{p}(\gamma), \bar{d}(\gamma))$.

To prove the equality of price deflators it is enough to observe that $\bar{p}(\gamma) + \bar{d}(\gamma) = (\bar{p} + d)(\mathbf{I} - \gamma_{t-1})^{-1}$. Thus $a \in A(\bar{p}, d)$ is equivalent to

$$a_t \bar{p}_t(\gamma) = a_t \bar{p}_t(\mathbf{I} - \gamma_{t-1})^{-1} = E_t \left(a_{t+1}(\bar{p}_{t+1} + d_{t+1})(\mathbf{I} - \gamma_t)^{-1}\right) = E_t \left(a_{t+1}(\bar{p}_{t+1}(\gamma) + \bar{d}_{t+1})\right),$$

and thus $a \in A(\bar{p}(\gamma), \bar{d}(\gamma))$. □

In the equilibrium with trading constructed above, agents have identical consumption allocations and *total holdings of shares* as in the equilibrium with no trading by firms.[11] The literature on the firm's objective under incomplete markets suggests that the discount process used by firms to value future cash flows should be some weighted average of the subjective discount rates (intertemporal marginal rates of substitution) of its shareholders (for a comprehensive discussion, see DeMarzo 1988). Thus if different shareholders control the firm in the equivalent equilibrium with non-trivial financial policy for the firms, the assumption that the discount process is unchanged might not be valid. The above argument shows that this concern is not justified, since it turns out that the "real" shareholder composition of each firm is unchanged.

Given a $\bar{p} \in X_+^{1\times J}$, Theorem 5 raises the question whether there exists a non-trivial $\gamma \in \Gamma$ (that is, $\gamma \neq 0 \in X^{J\times J}$) and $\bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$ such that $(\bar{p}_t(\gamma) + \bar{d}_t(\gamma))\gamma_{t-1} \geq 0$ for all $t > 0$ and (3.7)-(3.8) hold. We show that this is indeed the case, by identifying non-trivial strategies γ with the desired properties. Moreover we show that there exist trading strategies γ that lead to asset price bubbles in the $AJ(\gamma)$-equilibrium of Theorem 5, even though the equivalent no-trade equilibrium does not have a bubble.

Proposition 2. *Let* $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *be an AJ-equilibrium. Assume that there exist a pricing kernel* $a \in A_{++}(\bar{p}, d)$ *under which* $\bar{p}$ *has no bubbles, that is* $b_0(a,\bar{p}) = 0$.[12] *There exists* $\gamma \in X^{J\times J}\setminus\{\mathbf{0}\}$ *such that* $\bar{p}(\gamma), \bar{d}(\gamma)$ *defined by* (3.7)-(3.8) *satisfy* $(\bar{p}(\gamma), \bar{d}(\gamma)) \in PD(\gamma)$ *(hence are non-negative) and such that* $\bar{p}(\gamma)$ *has bubbles under* a*, that is* $b_0(a, \bar{p}_t(\gamma)) \neq 0$.

Proof. We construct a γ with the desired properties in which only one firm trades (firm 1), thus $\gamma = (\gamma^1, 0, \ldots, 0) \in X^{J\times J}$. The non-negativity of prices and dividends associated to a general γ in which all firms trade is considerably harder to investigate. Notice that

$$(\mathbf{I} - \gamma_t)^{-1} = \mathbf{I} + \frac{1}{1-\gamma_t^{1,1}}\gamma_t. \qquad (3.10)$$

Therefore if $\gamma^1 \in X_+^{J\times 1}$ and $\gamma_t^{1,1} < 1$ for all t, then $\bar{p}(\gamma)$ given by (3.7) is nonnegative, and (3.2) holds. The non-negativity of $d_t(\gamma)$ given by (3.8) follows by continuity, if γ_t is sufficiently close to γ_{t-1} (if $d_t \gneqq 0$ then γ_t can be chosen different than γ_{t-1} and still guarantee that $d_t(\gamma) \geq 0$). To construct bubble-inducing trading strategies in the simplest way, we assume further that firm 1 trades only in its own shares, that is $\gamma^{j,k} = 0$ for all $(j,k) \neq (1,1)$. Thus there is a bubble in the price $\bar{p}^1(\gamma)$ if and only if

$$0 < \lim_{t\to\infty} Ea_t\bar{p}_t^1(\gamma) = \lim_{t\to\infty} Ea_t\bar{p}_t^1(1-\gamma_t^{1,1})^{-1}.$$

Since $d \geq 0$, it follows that $\bar{d}(\gamma) \geq 0$ if and only if $\bar{d}^1(\gamma) \geq 0$, which holds if and only if

$$(1-\gamma_t^{1,1})^{-1} \leq (1-\gamma_{t-1}^{1,1})^{-1}\left(1 + \frac{d_t^1}{\bar{p}_t^1}\right), \forall t \geq 0.$$

[11] Intuitively,

$$\bar{\theta}^i = (I-\gamma)^{-1}\bar{\theta}^i(\gamma) = \bar{\theta}^i(\gamma) + \gamma\bar{\theta}^i(\gamma) + \gamma^2\bar{\theta}^i(\gamma) + \ldots.$$

The right hand side of the equation above represents the total ownership of shares of an agent owning $\theta^i(\gamma)$ shares in firms with a trading portfolio γ, since it takes into account *indirect holdings* of securities through the portfolio γ owned by the firms.

[12] $b_0(a,\bar{p}) = \lim_{t\to\infty} \frac{1}{a_0} Ea_t\bar{p}_t$ (see section 2).

Choose a sequence of real numbers $(\beta_t)_{t=0}^{\infty}$ such that $\beta_t \geq 0$ and $\sum_{t=0}^{\infty} \beta_t < \infty$. Let $\alpha_t := (1+\beta_t)^{-1}$. It follows that $\alpha := \prod_{t=0}^{\infty} \alpha_t \in (0, 1]$, since

$$\exp\left(\sum_{i=0}^{t} \beta_i\right) \geq \prod_{i=0}^{t}(1+\beta_i) \geq 1 + \sum_{i=0}^{t} \beta_i, \forall t \geq 0.$$

Construct the strategy $\gamma^{1,1}$ recursively, satisfying $\gamma^{1,1}_{-1} = 0$ and

$$(1-\gamma^{1,1}_t)^{-1} = \alpha_t(1-\gamma^{1,1}_{t-1})^{-1}\left(1+\frac{d^1_t}{\bar{p}^1_t}\right), \forall t \geq 0.$$

By construction, $\bar{p}^1(\gamma), \bar{d}^1(\gamma) \geq 0$, and $(1-\gamma^{1,1}_t)^{-1} = \prod_{i=0}^{t} \alpha_i \left(1+\frac{d^1_i}{\bar{p}^1_i}\right)$. Since $a \in A_{++}(\bar{p}, d)$, it follows that the process $\left(a_t\bar{p}^1_t \prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right)\right)_{t=0}^{\infty}$ is a martingale. There is a bubble in the price $\bar{p}^1(\gamma)$ if and only if $0 < \lim_{t\to\infty} Ea_t\bar{p}^1_t(\gamma)$. Notice that

$$\lim_{t\to\infty} Ea_t\bar{p}^1_t(\gamma) = \lim_{t\to\infty} Ea_t\bar{p}^1_t(1-\gamma^{1,1}_t)^{-1} = \lim_{t\to\infty} \prod_{i=0}^{t} \alpha_i \cdot Ea_t\bar{p}^1_t \prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right) = \alpha(p^1_0 + d^1_0).$$

Thus firm 1 can introduce bubbles by trading in its shares according to $\gamma^{1,1}$. The largest such bubble is when $\alpha = 1$, which happens when firms uses its earnings entirely for share buy-back and pays no dividends. Notice also that

$$\lim_{t\to\infty}(1-\gamma^{1,1}_t)^{-1} = \alpha \cdot \lim_{t\to\infty} \prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right).$$

Therefore if $\prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right) \to \infty$ then $\gamma^{1,1}_t \to 0$ and thus firm is buying back asymptotically its outstanding shares entirely. If there is only one firm in the economy and the prices $\bar{p}$ exclude bubbles under any pricing kernel in $A_{++}(\bar{p}, d)$, then $\prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right) \to \infty$. Indeed, $\bar{a} \in X_{++}$ given by $\bar{a}_0 = 1$ and $\bar{a}_t^{-1} := \frac{p_t}{p_0}\prod_{i=1}^{t}\left(1+\frac{d^1_i}{p^1_i}\right)$ belongs to $A_{++}(\bar{p}^1, d^1)$ and has the property that $\lim_{t\to\infty} E\bar{a}_t p^1_t > 0$ if $\prod_{i=0}^{t}\left(1+\frac{d^1_i}{\bar{p}^1_i}\right) < \infty$ on a set of positive probability. □

The proof shows that there is a large class of trading strategies γ such that for any $\bar{p} \in X^{1\times J}_{+}$, the prices and dividends $\bar{p}(\gamma), \bar{d}(\gamma)$ given by (3.7)-(3.8) are non-negative. The bubbles arising from trading, as in Proposition 2, do not affect the total market value of the firms. Indeed, $(\bar{p}_t(\gamma) + \bar{d}_t(\gamma))(\mathbf{I} - \gamma_{t-1})$ is the vector of firms' date-t total market value at the $AJ(\gamma)$-equilibrium $\left(\bar{p}(\gamma), \bar{d}(\gamma),\ (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i(\gamma))_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$, since it represents the value of firms' shares minus the value of their financial holdings.[13] By (3.8), this is equal to $\bar{p}_t + d_t$ which represents firms' date-t values at the no-trade equilibrium $\left(\bar{p}, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$. Hence Theorem 5 implies that for every equilibrium with no trading by firms there exists an equilibrium with trading in which firms' values remain unchanged. This does *not* imply that firms' cannot affect their total market values by changing their payout policies. Indeed, combining Theorems 4 and 5, we get the following:

[13] In the familiar case when firms' trading is restricted to debt instruments, then the total merket value becomes equity plus debt.

Corollary 6. *Let* $\left(p, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *be an AJ-equilibrium and* $\varepsilon \in M_+^J(\bar{p}, d)$. *Let* $\bar{p} := p + \varepsilon$ *and let* $\gamma \in \Gamma$ *and* $\bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$ *such that* $\bar{p}_t(\gamma) + \bar{d}_t(\gamma))\gamma_{t-1} \geq 0$ *for all* $t > 0$ *and (3.7)-(3.8) hold. Then* $\left(\bar{p}(\gamma), \bar{d}(\gamma), (\bar{w}^i(\gamma))_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i(\gamma))_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is an AJ(γ)-equilibrium, where for all* $i \in \{1, \ldots, I\}$,

$$\bar{w}^i(\gamma) := \bar{w}^i + \varepsilon\theta_{-1}^i, \quad \bar{\theta}^i(\gamma) := (\mathbf{I} - \gamma)(\mathbf{I} + \Lambda)^{-1}\bar{\theta}^i, \tag{3.11}$$

with $\Lambda \in \Lambda(\varepsilon, p, d)$. *Moreover,* $A(\bar{p}(\gamma), \bar{d}(\gamma)) = A(\bar{p}, d) = A(p, d)$.

The validity of Modigliani-Miller theorem, which states that the total value of the firm is not affected by trading, can only be discussed in a fully specified model in which firms choose a payout policy optimally. The theorem may fail, for example, if a no-bubble equilibrium is selected in the no-trade case and an equilibrium in which firms' values are bubble-inflated is selected after a particular trading strategy γ. The preferred trading strategy of each firm depends on its objective and its beliefs (conjectures) about the value of a dividend stream. Following DeMarzo (1988), we assume that each firm j maximizes its initial value and has a *market valuation functional* $\Pi^j : X \to X$, assigning a value at period t equal to $\Pi_t^j(\delta)$ to a dividend stream $\delta \in X$. Firms are forward looking, in the sense that for any $t \in \mathbb{N}$, $\omega \in \Omega$ and $\delta, \delta' \in X$ such that $\delta_s = \delta'_s$ on $\mathcal{F}_t(\omega)$ for all $s > t$, $\Pi_t^j(\delta) = \Pi_t^j(\delta')$. We assume that firms associate positive value to positive cashflows, that is $\Pi^j(X_+) \subset X_+$. Additionally, we require the market valuation functionals of each firm to be consistent with the absence of arbitrage. In a finite horizon setting, DeMarzo (1988) used the same assumption on firms' beliefs to show that the Modigliani-Miller theorem holds, even when firms trade and markets are incomplete.

Definition 7. *A market valuation functional* $\Pi : X \to X$ *is arbitrage-free if for any* $k \in \mathbb{N}$ *and* $\Delta = (\delta^1, \ldots, \delta^k) \in X^{1\times k}$, *there is no* $\theta \in X^{k\times 1}$ *such that*[14]

$$(\Pi_t(\Delta) + \Delta_t)\theta_{t-1} \geq 0 \text{ for all } t \geq 1, \tag{3.12}$$

and

$$(\Pi_t(\Delta) + \Delta_t)\theta_{t-1} - \Pi_t(\Delta)\theta_t \geq 0, \forall t \in \mathbb{N}, \text{ with at least one } \gneq . \tag{3.13}$$

Lemma 1. *The market valuation functional* $\Pi : X \to X$ *is arbitrage-free if and only if there exists a unique* $a \in X_{++}$ *such that*

$$\Pi_t(\delta) = E_t \frac{a_{t+1}}{a_t} \left(\Pi_{t+1}(\delta) + \delta_{t+1}\right), \forall t \geq 0, \forall \delta \in X. \tag{3.14}$$

Proof. If $\Pi : X \to X$ is arbitrage-free, then in particular, for any $\Delta = (\delta^1, \ldots, \delta^k) \in X^{1\times k}$, there is no $\theta \in X^{k\times 1}$ such that $\theta_s = 0$ for all $s \neq t$ and (3.12) and (3.13) hold (in other words there are no one-period arbitrages at t). This is equivalent with the existence of an $\mathcal{F}_{t+1}$-measurable random variable $q_{t+1} > 0$ such that (3.14) holds, with $\frac{a_{t+1}}{a_t}$ replaced by q_{t+1} (Santos and Woodford 1997). Moreover q_{t+1} is unique (since k can be chosen arbitrarily large, Δ_{t+1} is arbitrary, and $\Pi_{t+1}(\Delta)$ depends only on Δ_s with $s > t + 1$). Repeating this reasoning at each $t \geq 0$, we can construct the desired process a by letting $a_0 := 1$ and $a_t := q_1 \cdot \ldots q_t$ for $t \geq 1$.

Conversely, assume that there exists $a \in X_{++}$ such that (3.14) holds. Suppose, by contradiction, that there exists $\Delta \in X^{1\times k}$ and $\theta \in X^{k\times 1}$ such that (3.12) and (3.13) hold. Multiplying (3.13)

[14] We assume that $\theta_{-1} := 0 \in \mathbb{R}^k$ and $\Pi_t(\Delta) := (\Pi_t(\delta^1), \ldots, \Pi_t(\delta^k))$.

written for t by a_t and taking expectations, then summing the resulting expressions for all $t \leq T$, we get $-Ea_T\Pi_T(\Delta)\theta_T \geq 0$ for all $T \in \mathbb{N}$, with at least one $\gneqq$. However, multiplying the No-Ponzi condition (3.12) written at t by a_t and taking expectations (for all t), we obtain $Ea_T\Pi_T(\Delta)\theta_T \geq 0$ for all T, which is a contradiction. □

The proof makes clear that Lemma 1 holds true for any no-Ponzi game conditions on firms strategies that satisfy $\limsup_{t\to\infty} Ea_t\Pi_t(\Delta)\theta_t \geq 0$ for all θ, Δ and $a \in X_{++}$ compatible with the absence of one-period arbitrages, rather than the specific form imposed in (3.12). Lemma 1 implies that every market valuation functional can be written as (see footnote 7)

$$\Pi_t(\delta) = \frac{1}{a_t} E_t \sum_{s>t} a_s \delta_s + b_t(a, \Pi(\delta)). \tag{3.15}$$

We will make explicit the deflator a associated to a market valuation functional Π by using the notation (Π, a) when referring to Π.

We define next the notion of a strategic equilibrium in which firms choose their financial strategy optimally.

Definition 8. *A vector* $\left(\bar{\gamma}, (\Pi^j, a^j)_{j=1}^J, \bar{p}(\bar{\gamma}), \bar{d}(\bar{\gamma}), (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is an AJ-equilibrium with trade by firms if the following conditions hold:*

(i) $\left(\bar{p}(\bar{\gamma}), \bar{d}(\bar{\gamma}), (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is an* $AJ(\bar{\gamma})$*-equilibrium.*

(ii) For all j, (Π^j, a^j) *is consistent with equilibrium prices, that is* $\Pi^j(\bar{d}^j(\bar{\gamma})) = \bar{p}^j(\bar{\gamma})$.

(iii) For all j, *let* $\gamma^j \in X^{J\times 1}$ *and* $\delta^j \in X^{1\times J}$ *arbitrary. Let* $p^j(\gamma) := \Pi^j(\delta^j)$, $d^j(\gamma) := \delta^j$, *and for* $k \neq j$, *let* $p^k(\gamma) := \bar{p}^k(\bar{\gamma}))$ *and* $d^k(\gamma) := \bar{d}^k(\bar{\gamma})$. *If* $(p(\gamma), d(\gamma)) \in PD^j(\gamma^j)$, *then*

$$\bar{p}_0^j(\bar{\gamma}) + \bar{d}_0^j(\bar{\gamma}) \geq p_0^j(\gamma) + d_0^j(\gamma). \tag{3.16}$$

Notice that part (iii) of the definition requires each firm to be "unsophisticated" and believe that its trading strategy cannot affect the prices and dividends of the other firms, even though their portfolios contain its shares.[15]

For any strategy γ^j that firm j contemplates choosing such that there exists $(p, \delta) \in PD^j(\gamma^j)$ with $p^j = \Pi^j(\delta^j)$ (that is γ^j is compatible with firm j's budget and solvency constraints and its market valuation functional), equation (3.1) implies that

$$E \sum_{s\geq 0} a_s^j \delta_s^j = E \sum_{s\geq 0} a_s^j d_s^j - \lim_{T\to\infty} Ea_T^j p_T \gamma_T^j. \tag{3.17}$$

By the no-Ponzi condition (3.2), $\lim_{T\to\infty} Ea_T^j p_T \gamma_T^j \geq 0$ and thus $\delta_0^j + f_0(a, \delta^j) \leq d_0^j + f_0(a, d^j)$. Therefore firm j cannot increase its cum-dividend fundamental value by trading. On the other

[15] Alternatively, we could have required each firm j to understand that its choice of a different strategy affects the prices and dividends of all the firms, and restricts the firm j to strategies with non-empty sets of prices and dividends compatible with them. In this case we would need to replace (iii) by

(iii') For all j, $\bar{\gamma}^j$ is *optimal*, that is for every $\gamma^j \in X^{J\times 1}$ such that $\gamma := (\gamma^j, \bar{\gamma}^{-j}) \in \Gamma$ and for every $(p(\gamma), d(\gamma)) \in PD(\gamma)$ with the property that $p(\gamma) = \Pi^j(d(\gamma))$, equation (3.16) holds.

All results hold under this alternative notion of strategic equilibrium with trade.

hand, firm j's conjectured initial total market value is

$$\delta_0^j + \Pi_0^j(\delta) = E\sum_{s\geq 0} a_s^j d_s^j + b_0(a^j, \Pi^j(\delta^j)) - \lim_{T\to\infty} E a_T^j p_T \gamma_T^j. \tag{3.18}$$

Hence firms have a strict incentive to trade only if they believe they can induce a bubble in their share price, which is high enough to compensate for the present value of its asymptotic financial wealth and for the bubble that might exist in earnings. If firms' market valuation functionals exclude bubbles, then not trading is a (weakly) optimal strategy for each firm.

We construct next a (strategic) AJ-equilibrium with bubbles, in which trading is strictly preferred to no-trading. Let $\left(p, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ be an AJ-equilibrium without bubbles. Let $a \in A_{++}(p,d)$, $\varepsilon \in M_+^J(p,d)$, $\bar{p} := p + \varepsilon$, $\gamma \in \Gamma$ and $\bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$ such that (3.2) and (3.7)-(3.8) hold. For each firm j, we construct a market valuation functional (Π^j, a) which attaches the value $\bar{p}_t^j(\gamma)$ to cashflows that coincide with $\bar{d}^j(\gamma)$ after t, and the fundamental value under a to cashflows that do not ultimately coincide with $\bar{d}^j(\gamma)$. Concretely, fix some $\delta \in X_+$. For every $\omega \in \Omega$, define the stopping time

$$\tau(\omega) := \inf\{t \mid \delta_s = \bar{d}_s^j(\gamma) \text{ on } \mathcal{F}_t(\omega), \forall s > t\}.$$

Fix some $\omega \in \Omega$ and $t \in \mathbb{N}$. If $t \geq \tau(\omega)$ then define $\Pi_t^j(\delta) := \bar{p}_t^j(\gamma)$ on $\mathcal{F}_t(\omega)$. If $t < \tau(\omega) < +\infty$ then define $\Pi_t^j(\delta) := \frac{1}{a_t} E_t[a_{\tau(\omega)} \Pi_{\tau(\omega)}^j(\delta)]$ on $\mathcal{F}_t(\omega)$. If $\tau(\omega) = +\infty$ then $\Pi_t^j(\delta) := \frac{1}{a_t} E_t \sum_{s>t} a_s \delta_s$. We have the following:

Proposition 3. *Let* $\left(p, (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *be an AJ-equilibrium and* $\varepsilon \in M_+^J(\bar{p}, d)$. *Let* $\bar{p} := p + \varepsilon$ *and let* $\gamma \in \Gamma$ *and* $\bar{p}(\gamma), \bar{d}(\gamma) \in X_+^{1\times J}$ *such that* $\bar{p}_t(\gamma) + \bar{d}_t(\gamma))\gamma_{t-1} \geq 0$ *for all* $t > 0$ *and* (3.7)-(3.8) *hold. For each firm* j, *let* Π^j *be defined as above. If* $\lim_{T\to\infty} E a_T \bar{p}_T(\gamma)\gamma_T = 0$, *then* $\left(\gamma, (\Pi^j, a)_{j=1}^J, \bar{p}(\gamma), \bar{d}(\gamma), (\bar{w}^i)_{i=1}^I, (\bar{c}^i)_{i=1}^I, (\bar{\theta}^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ *is a (strategic) AJ-equilibrium with trade by firms, where for all* $i \in \{1, \ldots, I\}$,

$$\bar{w}^i(\gamma) := \bar{w}^i + \varepsilon\theta_{-1}^i, \quad \bar{\theta}^i(\gamma) := (\mathbf{I} - \gamma)(\mathbf{I} + \Lambda)^{-1}\bar{\theta}^i, \tag{3.19}$$

with $\Lambda \in \Lambda(\varepsilon, p, d)$.

Proof. By Corollary 6, it is enough to check condition (iii) in Definition 8. By construction, $\Pi^j(\bar{d}^j(\gamma) = \bar{p}^j(\gamma)$ and firm j's initial total market value at γ is $d_0^j + f_0(a, d^j) + \varepsilon_0^j$. The market valuation functional Π^j also satisfies

$$\Pi_t^j(\delta) \leq \frac{1}{a_t} E_t \sum_{s>t} a_s \delta_s + b_t(a, \Pi^j(\bar{d}^j(\gamma))), \forall \delta \in X_+, \forall t \geq 0, \tag{3.20}$$

and the inequality is strict on a set of positive probability if $(\delta_{t+1}, \delta_{t+2}, \ldots) \neq (\bar{d}_{t+1}^j(\gamma), \bar{d}_{t+2}^j(\gamma), \ldots)$.

Firm j's value at an alternative trading strategy $\hat{\gamma}^j$ and prices and dividends $(p(\hat{\gamma}), d(\hat{\gamma})) \in PD^j(\hat{\gamma}^j)$ (with $\hat{\gamma} := (\hat{\gamma}^j, \gamma^{-j})$) satisfies

$$d_0^j(\hat{\gamma}) + \Pi_0^j(d^j(\hat{\gamma})) < f_0(d^j(\hat{\gamma})) + b_0(a, \Pi^j(\bar{d}(\gamma))) \leq d_0^j + f_0(a, d^j) + \varepsilon_0^j, \tag{3.21}$$

where the first inequality follows from (3.20) and the last equality from (3.18). This completes the proof.

□

4 Conclusion

We showed that bubbles can be injected in economies with incomplete markets, extending thus the results of Kocherlakota (2008), which were limited to environments with complete sets of period-ahead state-contingent securities. Any non-negative process that does not change the set of pricing kernels can be made into a bubble in an equivalent equilibrium with identical consumption for the agents, but tighter solvency constraints. Moreover, if the solvency constraints are endogenized as in Alvarez and Jermann (2000) to reflect some inherent participation constraints the agents are subject to, then the modified solvency constraints in an equilibrium with bubble injected prices still arise endogenously from the existence of participation constraints.

Moreover, the bubble injection results can be extended to situations where firms are allowed to trade. In this setup, bubbles can be interpreted as arising from firms' self-fulfilling conviction that they can influence their share prices by manipulating their payout policy adequately. Some equilibria with trading preserve the total value of the firms while others do not, and hence violate the Modigliani-Miller theorem.

A Characterization of the set $M^J(p,d)$

Lemma 2. *Let $p, d \in X_+^{1\times J}$ such that $A_{++}(p,d) \neq \emptyset$ and $\varepsilon \in X^{1\times J}$. The following are equivalent:*

(i) There exists $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t+d_t)\Lambda_{t-1}$ for all $t \geq 1$ and there exists $a \in A_{++}(p,d)$ such that $a \cdot \varepsilon$ is a martingale.

(ii) There exists $\Lambda \in X^{J\times J}$ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ for all $t \geq 1$ and $\varepsilon_t = p_t\Lambda_t$, for all $t \geq 0$.

(iii) $A(p,d) \subset A(p+\varepsilon, d)$

(iv) For each $a \in A(p,d)$, $a \cdot \varepsilon$ is a martingale.

Proof. (i) $\Rightarrow$ (ii) The conclusion is immediate, since for all $t \geq 0$,

$$\varepsilon_t = E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})\Lambda_t = p_t\Lambda_t.$$

(ii) $\Rightarrow$ (iii) Let $a \in A(p,d)$. The conclusion follows, since

$$E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) = E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1})(\mathbf{I} + \Lambda_t) = p_t(\mathbf{I} + \Lambda_t) = p_t + \varepsilon_t.$$

(iii) $\Rightarrow$ (iv) Let $a \in A(p,d)$. Thus $a \in A(p+\varepsilon, d)$. It follows that for all $t \geq 0$,

$$p_t + E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1} \overset{a\in A(p,d)}{=} E_t \frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1}) \overset{a\in A(p+\varepsilon,d)}{=} p_t + \varepsilon_t,$$

and thus $\varepsilon_t = E_t \frac{a_{t+1}}{a_t}\varepsilon_{t+1}$.

(iv) $\Rightarrow$ (i) Assume that $m \in X$ is such that $a \cdot m$ is a martingale, for any $a \in A(p,d)$. Pick an arbitrary date t event $\mathcal{F}_t(\omega)$ with $\omega \in \Omega$, with $\mathcal{F}_t(\omega)$ representing the cell of the partition $\mathcal{F}_t$

containing ω) (in other words pick a date t node in the uncertainty tree). Assume that there $\mathcal{F}_{t+1}$ has S subsets of $\mathcal{F}_t(\omega)$ (i.e. there are S branches leaving the fixed node). Then the returns r_{t+1} conditional on the event $\mathcal{F}_t(\omega)$ can be viewed as an $S \times J$ matrix R. Similarly m_{t+1}/m_t conditional on $\mathcal{F}_t(\omega)$ is represented by a vector $M \in \mathbb{R}^S$. If $\mu \in \mathbb{R}^S$ is interpreted as conditional state price process a_{t+1}/a_t times conditional probabilities, it follows that for any $\mu \in \mathbb{R}^S$ such that $\mathbf{1}' = \mu' R$, it must be the case that $1 = \mu' M$. We can transform this statement into an incompatibility of a system of equations statement, in order to be able to use alternative theorems. Specifically, the following system is incompatible

$$\begin{cases} \mu'(R^1 - M) < 0 \\ \mu'(R^1 - R^j) = 0, \quad j \in \{2, \dots, J\}. \end{cases}$$

By Motzkin's alternative theorem (Motzkin 1951), there exist $\alpha_2, \dots, \alpha_J \in \mathbb{R}$ such that $R^1 - M = \sum_{j=2}^{J} \alpha_j (R^1 - R^J)$. Therefore M can be written as a linear combination of the columns of R, and thus there exists $\lambda \in X^{J\times 1}$ such that $m_t = (p_t + d_t)\lambda_{t-1}$, for all $t \geq 1$.

Each component ε^j of $\varepsilon = (\varepsilon^1, \dots, \varepsilon^J)$ is a martingale when deflated by any $a \in A(p, d)$. As proven above, for each j there exists $\lambda^j \in X^{J\times 1}$ such that $\varepsilon_t^j = (p_t + d_t)\lambda_{t-1}^j$ for all $t \geq 1$. The conclusion follows by setting $\Lambda = (\lambda^1, \dots, \lambda^J)$. □

For each $t \geq 1$, let $\mathcal{S}_t(p, d)$ be the set of attainable payoffs at t given the price and dividend processes $p, d \in X_+^{1\times J}$:

$$\mathcal{S}_t(p, d) := \{(p_t + d_t)\lambda \mid \lambda : \Omega \to \mathbb{R}^J \text{ and } \lambda \text{ is } \mathcal{F}_{t-1} - \text{measurable}\}. \tag{A.1}$$

We will refer to $\mathcal{S}_t(p, d)$ as the period t *asset span.* We say that *there are no redundant securities* at $t - 1$, given prices p, if there is no $\lambda : \Omega \to \mathbb{R}^J$ such that λ is $\mathcal{F}_{t-1}$–measurable, $\lambda \neq 0$ and $(p_t + d_t)\lambda = 0$.

Lemma 3. *Let $p, d, \varepsilon \in X_+^{1\times J}$. Fix a $t \geq 1$. Then the following hold:*

(i) $\mathcal{S}_t(p + \varepsilon, d) \subset \mathcal{S}_t(p, d)$ if and only if there exists $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$.

(ii) If there exists $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ and $\mathbf{I} + \Delta_{t-1}$ is non-singular, then $\mathcal{S}_t(p + \varepsilon, d) = \mathcal{S}_t(p, d)$.

(iii) If there exists t such that $\mathcal{S}_t(p + \varepsilon, d) = \mathcal{S}_t(p, d)$ and there are no redundant securities at $t - 1$, then there exists $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$ and $\mathbf{I} + \Delta_{t-1}$ is non-singular.

Proof. (i) Assume $\mathcal{S}_t(p + \varepsilon, d) \subset \mathcal{S}_t(p, d)$. Therefore for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, there exists $\lambda'_{t-1} : \Omega \to \mathbb{R}^J$, $\mathcal{F}_{t-1}$-measurable, such that $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)\lambda'_{t-1}$. It follows that $\varepsilon_t \lambda_{t-1} = (p_t + d_t)(\lambda'_{t-1} - \lambda_{t-1})$, and since λ_{t-1} was arbitrary, we conclude that each of the J components of ε_t belongs to $\mathcal{S}_t(p, d)$ and therefore $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$, for some $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable. Conversely, for any $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ which is $\mathcal{F}_{t-1}$-measurable, $(p_t + d_t + \varepsilon_t)\lambda_{t-1} = (p_t + d_t)(\mathbf{I} + \Delta_{t-1}\lambda_{t-1} \in \mathcal{S}_t(p, d)$.

(ii) Given the previous part, it is enough to prove $\mathcal{S}_t(p, d) \subset \mathcal{S}_t(p + \varepsilon, d)$. Let $\lambda_{t-1} : \Omega \to \mathbb{R}^J$ be $\mathcal{F}_{t-1}$-measurable. The conclusion follows from

$$(p_t + d_t)\lambda_{t-1} = (p_t + d_t + \varepsilon_t)(\mathbf{I} + \Lambda_{t-1})^{-1}\lambda_{t-1} \in \mathcal{S}_t(p + \varepsilon, d).$$

(iii) Since $\mathcal{S}_t(p+\varepsilon,d) \subset \mathcal{S}_t(p,d)$, by Part (i), there exists $\Lambda_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$. Using $\mathcal{S}_t(p,d) = \mathcal{S}_t(p+\varepsilon+(-\varepsilon),d) \subset \mathcal{S}_t(p+\varepsilon,d)$, together with Part (i), there exists $\Gamma_{t-1} : \Omega \to \mathbb{R}^{J\times J}$ which is $\mathcal{F}_{t-1}$-measurable and such that $\varepsilon_t = (p_t + d_t + \varepsilon_t)\Gamma_{t-1}$. Therefore

$$\varepsilon_t(\mathbf{I} - \Gamma_{t-1}) = (p_t + d_t)\Gamma_{t-1} = (p_t + d_t)\left(\Delta_{t-1}(\mathbf{I} - \Gamma_{t-1}) - \Gamma_{t-1}\right) = 0.$$

Since there are no redundant securities, we conclude that $\Delta_{t-1}(\mathbf{I} - \Gamma_{t-1}) - \Gamma_{t-1} = 0$, which is equivalent to $(\mathbf{I} + \Delta_{t-1})(\mathbf{I} - \Gamma_{t-1}) = \mathbf{I}$, hence $\mathbf{I} + \Delta_{t-1}$ is indeed non-singular. □

Proposition 4. *Assume that there are no redundant securities at any period t. The following are equivalent:*

(i) $\varepsilon \in M^J(p,d)$.

(ii) $A(p,d) = A(p+\varepsilon,d)$.

(iii) $\mathcal{S}_t(p,d) = \mathcal{S}_t(p+\varepsilon,d)$ *for all natural t and $a \cdot \varepsilon$ is a martingale, for some $a \in A(p,d)$.*

Proof. (i) ⇒ (ii) Notice that Lemma 2 implies that $A(p,d) \subset A(p+\varepsilon,d)$. Choose $a \in A(p+\varepsilon,d)$. Then for any $t \geq 0$,

$$\begin{aligned}E_t\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1}) = E_t\frac{a_{t+1}}{a_t}(p_{t+1} + d_{t+1} + \varepsilon_{t+1})(\mathbf{I} + \Lambda_t)^{-1} = (p_t + \varepsilon_t)(\mathbf{I} + \Lambda_t)^{-1}\\ = (p_t + p_t\Lambda_t)(\mathbf{I} + \Lambda_t)^{-1} = p_t(\mathbf{I} + \Lambda_t)(\mathbf{I} + \Lambda_t)^{-1} = p_t.\end{aligned}$$

Thus $a \in A(p,d)$ and the conclusion follows.

(ii) ⇒ (iii) $A(p,d) \subset A(p+\varepsilon,d)$ implies the existence of Δ such that $\varepsilon_t = (p_t + d_t)\Lambda_{t-1}$. We just need to show that $\mathbf{I} + \Delta_{t-1}$ is non-singular. Since $A(p+\varepsilon,d) \subset A(p+\varepsilon+(-\varepsilon),d)$, there exists Γ such that $\varepsilon_t = (p_t + d_t)\Gamma_{t-1}$, and it follows that $(\mathbf{I} + \Delta_{t-1})(\mathbf{I} - \Gamma_{t-1}) = \mathbf{I}$, hence $\mathbf{I} + \Delta_{t-1}$ is indeed non-singular.

(iii) ⇒ (i) Follows from Lemma 3, part (iii).

□

B Gain processes belong to $M_+^J(p,d)$

We give conditions under which the *gain process* associated to a J-dimensional vector of trading strategies belongs to $M^J(p,d)$. The return from t to $t+1$ on a trading strategy θ is defined as

$$r_{t+1} := \frac{(p_{t+1} + d_{t+1})\theta_t}{p_t\theta_t}.$$

Fix a trading strategy $\bar{\theta} \in X_+^{J\times 1}$ having a positive return $\bar{r}$.[16] Define the *discount factor* process ρ as

$$\rho_t = \prod_{s=1}^{t} \bar{r}_s^{-1}, \forall t > 0.$$

[16]If dividends and prices are positive, any $\bar{\theta} \in X_+^{J\times 1}$ generates a positive return.

The *gain process* $g(\theta) \in X$ associated to a trading strategy $\theta \in X^{1\times J}$ is defined as (Leroy and Werner 2001, p. 259)

$$g_t(\theta) := p_t\theta_t + \rho_t^{-1}\sum_{s=1}^{t}\rho_s\left((p_s+d_s)\theta_{s-1} - p_s\theta_s\right), \forall t.$$

Thus $g_t(\theta)$ represents the gain realized by the trading strategy θ from date 0 to date t measured in units of date t consumption; it is computed as the sum of payoffs of the strategy θ up to date t reinvested in each period at the rate of return generated by $\bar{\theta}$. Given a J-dimensional vector of trading strategies $\Theta = (\theta^1, \ldots, \theta^J) \in X^{J\times J}$, we let $g(\Theta) := (g(\theta^1), \ldots, g(\theta^J)) \in X^{1\times J}$ be the gain process associated to Θ. It follows that $g(\Theta)$ has the same expression as that for $g(\theta)$ given above, if θ is replaced by Θ. The gain process g^j associated to security j is defined as the gain of a buy-and-hold portfolio consisting of a unit of security j. Thus

$$g_t^j = p_t^j + \rho_t^{-1}\sum_{s=1}^{t}\rho_s d_s^j, \text{ and } g_t = p_t + \rho_t^{-1}\sum_{s=1}^{t}\rho_s d_s.$$

We let $g := (g^1, \ldots, g^J)$. Notice that $g = g(\mathbf{I})$. For any trading strategy $\Theta \in X^{J\times J}$, it is immediate to check that $g(\Theta)$ is a martingale when deflated by an $a \in A(p, d)$ (for the case $\Theta = I$ and a discount factor generated by risk free returns, see Theorem 26.4.1, Leroy and Werner 2001). Moreover,

$$\begin{aligned} g_t(\Theta) &= (p_t+d_t)\Theta_{t-1} + \bar{r}_t\rho_{t-1}^{-1}\sum_{s=1}^{t-1}\rho_s\left((p_s+d_s)\Theta_{s-1} - p_s\Theta_s\right) \\ &= (p_t+d_t)\left(\Theta_{t-1} + \bar{\theta}_{t-1}(p_{t-1}\bar{\theta}_{t-1})^{-1}\rho_{t-1}^{-1}\sum_{s=1}^{t-1}\rho_s\left((p_s+d_s)\Theta_{s-1} - p_s\Theta_s\right)\right). \end{aligned}$$

Define $\lambda_{t-1}(\Theta) \in X^{J\times 1}$ by

$$\lambda_{t-1}(\Theta) := \left((p_{t-1}\bar{\theta}_{t-1})^{-1}\rho_{t-1}^{-1}\sum_{s=1}^{t-1}\rho_s\left((p_s+d_s)\Theta_{s-1} - p_s\Theta_s\right)\right)'.$$

It is immediate to check that

$$\left(\mathbf{I} + \Theta_{t-1} + \bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)\right)^{-1} = (\mathbf{I} + \Theta_{t-1})^{-1} - \frac{(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}}{1 + \lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1}},$$

whenever $\mathbf{I} + \Theta_{t-1}$ is non-singular and $1 + \lambda'_{t-1}(\Theta)(\mathbf{I}+\Theta_{t-1})^{-1}\bar{\theta}_{t-1} \neq 0$. This suggests that $g(\Theta) \in M^J(p, d)$ for a large set of Θ's. In particular, for $\Theta = I$,[17]

$$\left(\mathbf{I} + \Theta_{t-1} + \bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)\right)^{-1} = \frac{1}{2}\mathbf{I} + \frac{1}{4}\frac{\bar{\theta}_{t-1}\lambda'_{t-1}(I)}{1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2}.$$

[17] In the case $\Theta = I$, the determinant of the matrix $\mathbf{I} + \Theta_{t-1} + \bar{\theta}_{t-1}\lambda'_{t-1}(\Theta)$ is easy to compute, and equals $2^J(1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2)$. Thus the matrix is non-singular if and only if $1+\lambda'_{t-1}(I)\bar{\theta}_{t-1}/2 \neq 0$, and we arrived to the same conclusion, but without deriving the explicit expression for the inverse.

A sufficient condition (but not necessary) for $1 + \lambda'_{t-1}(I)\bar{\theta}_{t-1}/2 \neq 0$, taking into account the fact that $\lambda'_{t-1}(I) = (p_{t-1}\bar{\theta}_{t-1})^{-1}\rho_{t-1}^{-1}\sum_{s=1}^{t-1}\rho_s d_s \geq 0$ is for $\bar{\theta} \in X_+^{J\times 1}$, since in that case $\lambda'_{t-1}(I)\bar{\theta}_{t-1} \geq 0$.

Thus we gave conditions on a vector of trading strategies $\Theta \in X^{J\times J}$ such that the associated gain process $g(\Theta)$ belongs to $M^J(p,d)$. In particular we obtained the following:

Lemma 4. *The security gain process g belongs to $M_+^J(p,d)$.*

C Volume of trading effects of bubble injections

Bubble episodes (that is periods of asset prices too high to be justified by fundamentals) are typically associated with large increases in both share trading volume and dollar trading volume for those assets (Cochrane 2002). We investigate the effect the injection of a bubble has on the volume of trade. We compare the *volume of trade* for the two "equivalent" equilibria of Theorem 2, the bubble-free equilibrium $\left(p,(w^i)_{i=1}^I,(c^i)_{i=1}^I,(\theta^i)_{i=1}^I\right)$, and the bubbly equilibrium $\left(\hat{p},(\hat{w}^i)_{i=1}^I,(c^i,\hat{\theta}^i)_{i=1}^I\right)$. In the first equilibrium, the number of shares of each asset bought, respectively sold, by agent i at t is $(\theta_t^i-\theta_{t-1}^i)^+$, respectively $(\theta_t^i-\theta_{t-1}^i)^-$ (the positive part and the negative part of the change in portfolio are applied component-wise). Notice that the total number of shares of each asset bought and sold at t are equal, since

$$\sum_i(\theta_t^i-\theta_{t-1}^i)^+ = \sum_i(\theta_t^i-\theta_{t-1}^i)^- = \frac{1}{2}\sum_i|\theta_t^i-\theta_{t-1}^i|.$$

Thus we can measure the *share volume of trade* (total number of shares traded) at t as $\frac{1}{2}\mathbf{1}'\sum_i|\theta_t^i-\theta_{t-1}^i|$, and the *dollar volume of trade* by $\frac{1}{2}p\sum_i|\theta_t^i-\theta_{t-1}^i|$. We compare the total volumes of trade at period 0 (when the bubble is injected), since it leads to comparatively simpler expressions. The initial dollar volume of trade, in the bubble free equilibrium, is $\frac{1}{2}p_0\sum_i|\theta_0^i-\theta_{-1}^i|$. In the bubbly equilibrium,

$$\begin{aligned}\frac{1}{2}\hat{p}_0\sum_i|\hat{\theta}_0^i-\theta_{-1}^i| &= \frac{1}{2}p_0(\mathbf{I}+\Lambda_0)\sum_i\left|(\mathbf{I}+\Lambda_0)^{-1}(\theta_0^i+\Lambda_0\theta_{-1}^i)-\theta_{-1}^i\right| \\ &= \frac{1}{2}p_0(\mathbf{I}+\Lambda_0)\sum_i\left|(\mathbf{I}+\Lambda_0)^{-1}(\theta_0^i-\theta_{-1}^i)\right|.\end{aligned}$$

Of course, the share volume of trade in the two equilibria are $\frac{1}{2}\mathbf{1}'\sum_i|\theta_0^i-\theta_{-1}^i|$, respectively $\frac{1}{2}\mathbf{1}'\sum_i\left|(\mathbf{I}+\Lambda_0)^{-1}(\theta_0^i-\theta_{-1}^i)\right|$. Therefore the introduction of the bubble affects in general the volume of trade. At this level of generality, it is difficult to ascertain the direction of change of the volume of trade. However, when there exists a single security ($J=1$), the *dollar* volume of trade is unaffected by the presence of the bubble, in the period when the bubble is injected.

D Existence of Alvarez-Jermann equilibria

Existence of an equilibrium $\left(p,(w^i)_{i=1}^I,(c^i)_{i=1}^I,(\theta^i)_{i=1}^I\right)$, as defined by (2.3)-(2.5) in section 2, is a delicate problem, due to the presence of incomplete markets, real (long-lived) securities and infinite horizon. The dependence of the rank of the matrix of returns (at each date and state) on asset prices can create discontinuities in demand and lead to existence failures (for a two period environment where an equilibrium does not exist, due to the "drop in rank" problem, see Hart 1975). Hernandez and Santos (1996) prove that an equilibrium exists in our environment for a dense subset of endowment and dividend processes, if agents are "sufficiently impatient" (see footnote 1), they have a non-negative initial holding of securities and if the solvency constraints are chosen equal to the maximum amount that an agent can borrow, if he must hold non-negative wealth after some finite date, that is[18]

$$w_t^i = -\inf_{a\in A_{++}(p,d)} E_t \sum_{s\geq t} \frac{a_s}{a_t} e_s^i, \forall i, t. \tag{D.1}$$

Let $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ be an equilibrium with solvency constraints as above. If $V_t^i(w_t^i, w^i, p, d)$ is well defined,[19] then $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ with $\widetilde{V}_t^i := V_t^i(w_t^i, w^i, p, d)$ is trivially an Alvarez-Jermann equilibrium. Similarly, $\left(p, (\bar{w}^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ with $\bar{w}_t^i := (p_t + d_t)\theta_{t-1}^i$ and $\widetilde{V}_t^i := U_t^i(c^i)$ is an Alvarez-Jermann equilibrium. The exogenous penalties for default in both these two Alvarez-Jermann equilibria satisfy the condition in Theorem 4 (are unaffected by the injection of a bubble in asset prices).

The existence of Alvarez-Jermann equilibria in which the penalty of default is predetermined, as in (2.11) for example (where agents are banned from trading upon default), is a harder problem. With complete markets, Kehoe and Levine (1993) showed that Arrow-Debreu equilibria where agents' consumption sets are restricted to consumption paths with continuation utilities higher than autarky exist. Such a Kehoe-Levine equilibrium can be implemented as an Alvarez-Jermann equilibrium (Alvarez and Jermann 2000). With incomplete markets, the most we can hope is to prove the existence of an equilibrium having the property that agents' continuation utilities are higher than in autarky, and then show that the solvency constraints associated to this equilibrium can be modified so that they become not too tight. Of course the existence step is problematic, as we pointed out above. We know (generic) existence for equilibria subject to solvency constraints of type (D.1). Assume that $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I\right)$ is such an equilibrium, and that $U_t^i(c^i) \geq U_t^i(e^i)$ for all i, t. We show that there exist solvency constraints $(\bar{w}^i)$ such that $\left(p, (w^i)_{i=1}^I, (c^i)_{i=1}^I, (\theta^i)_{i=1}^I, (\widetilde{V}^i)_{i=1}^I\right)$ with $\widetilde{V}_t^i$ given by (2.11) is an Alvarez-Jerman equilibrium. We construct $\bar{w}^i$ as the limit of an increasing sequence of solvency constraints $(w^{i,(k)})_{k\in\mathbb{N}}$. Let $w^{i,(0)} := w^i \leq 0$. There exists $w^{i,(1)} \in X$ such that for all t, $V_t^i(w_t^{i,(1)}, w^{i,(0)}, p, d) = U_t^i(e^i)$, due to the concavity (and hence continuity) of V_t^i in the first argument. Since $U_t^i(e^i) \leq U_t^i(c^i) = V_t^i((p_t + d_t)\theta_{t-1}^i, w^{i,(0)}, p, d)$, it follows that $(p_t + d_t)\theta_{t-1}^i \geq w_t^{i,(1)}$, for all t. We can also show that $\phi^{i,(1)} \leq 0$. This follows from

$$V_t^i(w_t^{i,(1)}, w^{i,(0)}, p, d) = U_t(e^i) \leq V_t^i(0, w^{i,(0)}, p, d), \tag{D.2}$$

[18] Hernandez and Santos (1996) actually work with *borrowing limits* that limit end of period wealth, that is an agent's i trading strategy θ^i must satisfy

$$p_t\theta_t^i \geq -\inf_{a\in A_{++}(p,d)} E_t \sum_{s\geq t+1} \frac{a_s}{a_t} e_s^i,$$

when faced with prices p. Florenzano and Gourdel (1993) show that agents subject to these borrowing limits have identical budget constraints to the situation where they are subjected to *debt constraints* $(p_t + d_t)\theta_{t-1}^i \geq w_t^i$ that limit the beginning of period wealth, with w^i given by (D.1). Therefore all the results of Hernandez and Santos (1996) apply to the corresponding environment with debt constraints (D.1).

[19] This is always the case if the markets are dynamically complete and agents' utility of the zero consumption path is finite.

since starting with wealth zero at t and facing negative solvency constraints always makes autarky (i.e. e^i) feasible. Moreover, $w^{i,(1)} \geq w^{i,(0)}$, since $V^i_t(w^{i,(1)}_t, w^{i,(0)}, p, d) \geq V^i_t(w^{i,(0)}_t, w^{i,(0)}, p, d)$. Construct next the bounds $w^{i,(2)} \in X$ such that $V^i_t(w^{i,(2)}_t, w^{i,(1)}, p, d) = U^i_t(e^i)$. From

$$U^i_t(e^i) = V^i_t(w^{i,(1)}_t, w^{i,(0)}, p, d) \geq V^i_t(w^{i,(1)}_t, w^{i,(1)}, p, d)$$

we infer $w^{i,(2)} \geq w^{i,(1)}$. Since $B^i_t((p_t + d_t)\theta^i_{t-1}, w^{i,(1)}, p, d) \subset B^i_t((p_t + d_t)\theta^i_{t-1}, w^{i,(0)}, p, d)$, it follows that $(p_t + d_t)\theta^i_{t-1} \geq w^{i,(2)}_t$. Equation (D.2) (with $w^{i,(0)}$ replaced by $w^{i,(1)}$ and $w^{i,(1)}$ by $w^{i,(2)}$) implies $w^{i,(2)} \leq 0$. Repeating the construction, we obtain the monotone increasing bounds $w^{i,(k)}_t$, dominated from above by $\min\{(p_t + d_t)\theta^i_{t-1}, 0\}$, hence converging to some $\bar{w}^i_t$, and such that

$$V^i_t(w^{i,(k+1)}_t, w^{i,(k)}, p, d) = U^i_t(e^i).$$

Continuity in the product topology of V^i_t in the first and second arguments (following from concavity) implies that by letting $k \to \infty$, $V^i_t(\bar{w}^i_t, \bar{w}^i, p, d) = U^i_t(e^i)$, which means that $\bar{w}^i$ are not too tight.

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