Reduced equivalent form of a financial structure

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Abstract

Under a fairly general regularity assumption on wealth transfers, we show that every financial structure is *equivalent* in terms of *consumption equilibria* to its *reduced* form. Building upon the equilibrium existence result for reduced financial economies (Aouani and Cornet, 2009), existence of financial equilibrium is shown under standard assumptions on the consumption side and under the aforementioned regularity assumption on the financial side.

Keywords: Restricted participation, financial exchange economy, reduced financial structure, equivalent financial structure, arbitrage-free prices, consumption equilibrium.

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1. Introduction

Restricted participation to financial markets refers to the fact that agents face exogenous constraints on their portfolio holdings. These constraints are usually described by a collection of subsets (one for each agent) of the space of all possible portfolios. Each subset represents the set of portfolios that the corresponding agent is allowed to hold. The economic relevance and interest in considering restricted participation is known since a long time and dates back to the seminal paper of Radner (1972) where agents face short sales constraints. It is also well known that the presence of such portfolio constraints is a natural cause of market incompleteness – even if there exist enough assets to hedge all risks – and allows to capture a wide range of imperfections in the financial markets, such as collateral requirements, margin requirements, "combo" sales, short selling constraints, and more generally institutional constraints. We refer the reader to Elsinger and Summer (2001) who provide an extensive discussion of these institutional constraints and how to model them in a general financial framework.

The equilibrium existence problem in this context has been enjoying a growing interest since the first work by Siconolfi (1989), and Cass (1984, 2006). Linear equality constraints are considered by Balasko et al. (1990) with nominal assets, and by Polemarchakis and Siconolfi (1997) with real assets, whereas Aouani and Cornet (2009) study linear equality and *inequality* constraints with either nominal or numéraire assets. More recently, the "general" case of portfolio sets which are closed and convex subsets of the space of all possible portfolios, as in Siconolfi (1989), is considered by Angeloni and Cornet (2006) and Aouani and Cornet (2009) for real assets, and by Martins-da-Rocha and Triki (2005), Hahn and Won (2007), and Cornet and Gopalan (forthcoming) when assets are nominal.

A key step in the proof of existence of a financial equilibrium with nominal, numéraire or real assets is to show that equilibrium portfolios can be, a priori, chosen in a bounded set. This is an easy task if there are no redundant assets, or equivalently if the return matrix has full column rank. With unrestricted participation, there is no loss of generality in making this "Full Rank Assumption". Indeed, one can easily show that a situation in which redundant assets are available can be converted into one in which they are not by removing the redundant assets (which practically amounts to deleting redundant columns from the return matrix). Matters are much more complicated when agents' participation to financial markets is constrained. In fact, as emphasized by Balasko et al. (1990), one significant source of restricted participation is financial intermediation which typically involves redundancy. So there are no a priori grounds for the standard Full Rank Assumption in the presence of restricted participation. This fact constitutes a major obstacle to the application of fixed point theorems which are usually used to show existence of equilibrium.

With portfolio sets defined by linear equality constraints, Balasko et al. (1990) develop a procedure to overcome this obstacle. They show how to transform agents' financial opportunities to obtain a financial economy in which each agent's portfolio choice set is a subspace having the same dimension as the wealth space it generates; a non-redundancy-type condition used by Siconolfi (1989) to show existence of equilibrium. Moreover, every equilibrium in the transformed economy leads to an equilibrium in the original one (of course, the transformation would be of no avail if the latter result did not hold). It is not possible to follow this procedure when portfolio sets are not linear. However, in Aouani and Cornet (2009) we are able to extend the above analysis to polyhedral portfolio sets i.e. sets that are defined by linear equality and inequality constraints. By appropriately modifying agents' portfolio sets, we obtain a new − say *reduced* − financial structure satisfying a non-redundancy-type condition weaker than the one in Siconolfi (1989), keeping the correspondence between the equilibria. Furthermore, we show existence of equilibrium for reduced financial economies.

The main purpose of this paper is to go beyond polyhedral portfolio sets. Although existence of equilibrium was the driving force of this work, the proof of existence of equilibria will be merely an immediate consequence of the conjunction of our main result concerning existence of reduced equivalent financial structures and the equilibrium existence result for reduced financial economies in Aouani and Cornet (2009). This approach allows generalizing all previous existence results for nominal or numéraire assets, and represents the first contribution of this paper to the literature. The key contribution of this paper is to provide a novel approach to the problem posed by dealing with redundant assets. Since simply removing redundant assets in the presence of portfolio restrictions would considerably change the nature of the market by altering wealth transfer sets, we propose instead, to remove some of the redundant portfolios. In Aouani and Cornet (2008), these portfolios are labeled as *useless* and a justification for this term is provided. More precisely, we show that under a very mild assumption – **Closedness** in the text – every financial structure is *equivalent*, in terms of financial possibilities and in terms of consumption equilibria, to another structure in which there are no useless portfolios (its reduced form). It is the purpose of future research to investigate the risk-sharing role of those redundant portfolios that cannot be dispensed with, without losing either of the properties *equivalent* or *reduced*. One can think of those portfolios as being *useful*.

The paper is organized as follows. In Section 2, we describe the financial exchange economy, introduce an equivalence relation on the set of financial structures, define the reduced form of a financial structure, state our results, and provide a proof for the equilibrium existence theorem (Theorem 3). Section 3 is devoted to the proof of our main result (Theorem 1) as a consequence of a sharper result (Theorem 4). Section 2.6 provides sufficient conditions for the **Closedness** Assumption to hold and an example showing that **Closedness** is strictly weaker than the sufficient conditions presented in Propositions 2 and 3. Section 4 is devoted to the proof of Theorem 2. The Appendix (Section 6) gathers the proofs of all the propositions stated in the text.

2. The two-period model and the main result

2.1. The stochastic financial exchange economy

[2] We consider the basic stochastic model with two dates: $t = 0$ (today) and $t = 1$ (tomorrow). At the second date, there is a nonempty finite set $\mathcal{S} := \{1, \ldots, S\}$ of states of nature, one of which

[2]We shall use hereafter the following notations. If $\mathcal{I}$ is a finite set, whose number of elements is I, the space $\mathbb{R}^I$ (identified to the space $\mathbb{R}^{\mathcal{I}}$ of functions $x : \mathcal{I} \to \mathbb{R}$ whenever necessary) is endowed with the scalar product $x \cdot y := \sum_{i=1}^{I} x_i y_i$, and we denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm, $B_I(x, r) := \left\{y \in \mathbb{R}^I : \|y - x\| \leq r\right\}$, the closed ball centered at $x \in \mathbb{R}^I$ of radius $r > 0$. For $x = (x_i)$ and $y = (y_i)$ in $\mathbb{R}^I$, the notation $x \geq y$ (resp. $x > y$, $x \gg y$) means that, for every i, $x_i \geq y_i$ (resp. $x \geq y$ and $x \neq y$, resp. $x_i > y_i$) and we let $\mathbb{R}^I_+ = \left\{x \in \mathbb{R}^I : x \geq 0\right\}$, $\mathbb{R}^I_{++} = \left\{x \in \mathbb{R}^I : x \gg 0\right\}$. Let $X \subset \mathbb{R}^I$, the span of X is the linear subspace of $\mathbb{R}^I$, denoted $< X >$, which is the set of all the K-linear combinations $\sum_{k=1}^{K} \alpha_k x_k$ of vectors $x_k \in X$ for every integer K, and we denote by int X, cl X, respectively, the interior and the closure of X. Consider a $I \times J$-matrix A with I rows and J columns, with entries $A_i^j (i \in I, j \in J)$, we denote by A_i the i-th row of A (hence a row vector, i.e., a $(1 \times J)$-matrix, often identified to a vector in $\mathbb{R}^J$ when there is no risk of confusion) and A^j denotes the j-th column of A (hence a column vector, i.e., a $I \times 1$-matrix, which

prevails at time $t = 1$ and is only known at time $t = 1$. For convenience, $s = 0$ denotes the state of the world (known with certainty) at period 0 and we let $\bar{\mathcal{S}} = \{0\} \cup \mathcal{S} = \{0, 1, \ldots, S\}$. At each state, today and tomorrow, there is a spot market for a positive number ℓ of divisible physical goods and we assume that the goods are perishable, i.e., each good does not last more than one period. In this model a commodity is a couple (h, s), specifying the physical good $h = 1, \ldots, \ell$ and the state $0, 1, \ldots, S$ at which it is available. Thus the commodity space is $\mathbb{R}^L$, where $L = \ell(1 + S)$. An element x (resp. p) in $\mathbb{R}^L$ is called a consumption (resp. a price) and we will use the notation $x = (x(s))_{s \in \bar{\mathcal{S}}} \in \mathbb{R}^L$, where $x(s) = (x_1(s), \ldots, x_\ell(s)) \in \mathbb{R}^\ell$, denotes the spot consumption at node $s \in \bar{\mathcal{S}}$.

In the exchange economy, there is nonempty finite set $\mathcal{I} := \{1, \ldots, I\}$ of consumers. Each consumer $i \in \mathcal{I}$ is endowed with a consumption set $X_i \subset \mathbb{R}^L$, a preference correspondence P_i, from $\prod_{k \in \mathcal{I}} X_k$ to X_i, and an endowment vector $e_i \in \mathbb{R}^L$. The set X_i is the set of her possible consumptions, and for $x \in \prod_{i \in \mathcal{I}} X_i$, $P_i(x)$ is the set of consumption plans in X_i which are strictly preferred to x_i by consumer i, given the consumption plans $(x_{i'})_{i' \neq i}$ of the other agents. Finally $e_i = (e_i(s))_{\bar{\mathcal{S}}}$ lists the state endowment $e_i(s)$ across states, with $e_i(0)$ being known with certainty and $e_i(s)$ $(s \neq 0)$ being available only if state s prevails at $t = 1$. The exchange economy can thus be summarized by $\mathcal{E} = (X_i, P_i, e_i)_{i \in \mathcal{I}}$.

Agents may operate financial transfers across states in $\bar{\mathcal{S}}$ (i.e. across the two periods and across the states of the second period) by exchanging a finite number of assets $j \in \mathcal{J} := \{1, \cdots, J\}$, which define the financial structure of the model.[3] The assets are traded at the first period $(t = 0)$ and yield payoffs at the second period $(t = 1)$, contingent on the realization of the state of nature $s \in \mathcal{S}$. The payoff of asset $j \in \mathcal{J}$, when state $s \in \mathcal{S}$ is realized, is $V_s^j(p)$ (for a given commodity price $p \in \mathbb{R}^L$). So, the payoff of asset j across tomorrow states is described by the mapping $p \mapsto V^j(p) := (V_s^j(p))_{s \in S} \in \mathbb{R}^S$. The financial structure is described by the payoff matrix mapping

may similarly be identified to a vector in $\mathbb{R}^I$). If there is no risk of confusion, we will use the same notation for the $I \times J$-matrix A and the associated linear mapping $A : \mathbb{R}^J \to \mathbb{R}^I$. We shall denote by $\ker A := \left\{x \in \mathbb{R}^J : Ax = 0\right\}$ the kernel of A, by $\operatorname{Im} A := \left\{Ax : x \in \mathbb{R}^J\right\}$ the image of A, and by rank A the rank of the matrix A, that is, the dimension of $\operatorname{Im} A$. We also denote $\ker A$ by $\{A = 0\}$ and we let $\{A \geq 0\} := \left\{x \in \mathbb{R}^J : Ax \geq 0\right\}$. The transpose matrix of A, denoted by A^T, is the $J \times I$-matrix whose rows are the columns of A, or equivalently, is the unique linear mapping $A^T : \mathbb{R}^I \to \mathbb{R}^J$, satisfying $Ax \cdot y = x \cdot A^T y$ for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^I$.

[3] The case of no financial assets – i.e., $\mathcal{J}$ is empty – is called pure spot markets.

$V : p \mapsto V(p)$, where $V(p)$ is the $S \times J$-matrix, whose columns are the payoffs $V^j(p)$ $(j = 1, \ldots, J)$ of the J assets. A portfolio $z = (z_j) \in \mathbb{R}^J$ specifies quantities $|z_j|$ $(j \in \mathcal{J})$ of each asset j, with the convention that the asset j is bought if $z_j > 0$ and sold if $z_j < 0$. Thus $V(p)z$ is its random financial return across states at time $t = 1$, and $V_s(p) \cdot z$ is its return if state s prevails.

We assume that each agent i is restricted to choose her portfolio within a portfolio set $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are (institutionally) admissible for agent i. This general framework allows us to address, for example, the following important situations:

(i) $Z_i = \mathbb{R}^J$ (unconstrained portfolios),

(ii) $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales),

(iii) $Z_i = B_J(0, 1)$ (bounded portfolio sets),

(iv) Z_i is a vector space (linear equality constraints),

(v) Z_i is polyhedral and contains 0 (linear equality and inequality portfolio constraints).

Note that the polyhedral case covers the cases (i)-(iv) (with an appropriate choice of the norm in (iii)). Throughout the paper we make the following assumption which covers all the above cases:

F1. For every $i \in \mathcal{I}$, Z_i is closed convex and contains 0, and $V : \mathbb{R}^L \to \mathbb{R}^{S\times J}$ is continuous.

We summarize by $\mathcal{F} = \left(V, (Z_i)_{i\in\mathcal{I}}\right)$ the financial characteristics, referred to as the financial structure. We will say that the financial structure $\mathcal{F}$ is *standard* if it satisfies Assumption **F1**.

2.2. Financial equilibria and no-arbitrage

Given commodity and asset prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, the budget set of consumer i is[4]

$$\begin{aligned} B^i_{\mathcal{F}}(p,q) &= \left\{ (x_i, z_i) \in X_i \times Z_i : \begin{array}{l} p(0)\cdot x_i(0) + q\cdot z_i \le p(0)\cdot e_i(0) \\ p(s)\cdot x_i(s) \le p(s)\cdot e_i(s) + V_s(p)\cdot z_i, \quad \forall s \in \mathcal{S} \end{array} \right\} \\ &= \{(x_i, z_i) \in X_i \times Z_i : \quad p \,\square\, (x_i - e_i) \le W(p,q) z_i\}, \end{aligned}$$

where $W(p, q)$ denotes the total payoff matrix, that is, the $(1 + S) \times J$-matrix $\begin{pmatrix} -q \\ V(p) \end{pmatrix}$.

We now introduce the standard equilibrium notion in this model.

[4]For every $p = (p(s))$, $x = (x(s))$ in $\mathbb{R}^L$, we denote by $p \,\square\, x$ the vector $\left(p(s)\cdot x(s)\right)_{s\in\bar{\mathcal{S}}}$.

Definition 1. *An equilibrium of the economy* $(\mathcal{E}, \mathcal{F})$ *is a list* $(\bar{p}, \bar{q}, \bar{x}, \bar{z}) \in \mathbb{R}^L \times \mathbb{R}^J \times (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I$ *such that*

(i) for every i, $(\bar{x}_i, \bar{z}_i)$ *maximizes the preference* P_i *under the budget constraint, that is*

$$(\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \text{ and } B^i_{\mathcal{F}}(\bar{p}, \bar{q}) \cap \Big(P_i(\bar{x}) \times Z_i\Big) = \emptyset,$$

(ii) [Market Clearing] $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$ *and* $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

A consumption equilibrium in the financial exchange economy $(\mathcal{E}, \mathcal{F})$ *is a list* $(\bar{p}, \bar{x}) \in \mathbb{R}^L \times (\mathbb{R}^L)^I$ *such that there exist* $(\bar{q}, \bar{z}) \in \mathbb{R}^J \times (\mathbb{R}^J)^I$ *with* $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ *is an equilibrium in* $(\mathcal{E}, \mathcal{F})$.

We make the following standard assumptions **C1-C6** on the consumption side. We denote by $\mathcal{A}(\mathcal{E})$ the set of attainable allocations of the economy, that is, $\mathcal{A}(\mathcal{E}) = \{(x_i)_{i \in} \in \prod_{i \in \mathcal{I}} X_i : \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i\}$.

Consumption Assumption C *For every* $i \in \mathcal{I}$ *and for every* $x = (x_i)_{i \in \mathcal{I}} \in \prod_i X_i$

C1 Consumption Sets: X_i *is a closed, convex, bounded below subset of* $\mathbb{R}^L$*;*

C2 Continuity: *The correspondence* P_i*, from* $\prod_{k \in \mathcal{I}} X_k$ *to* X_i*, is lower semicontinuous* [5] *with open values in* X_i *(for the relative topology of* X_i*);*

C3 Convexity: $P_i(x)$ *is convex;*

C4 Irreflexivity: $x_i \notin P_i(x)$*;*

C5 Local Non-Satiation LNS:

(a) $\forall x \in \mathcal{A}(\mathcal{E})$, $\forall s \in S$, $\exists\, x_i'(s) \in \mathbb{R}^\ell$, $(x_i'(s), x_i(-s)) \in P_i(x)$,[6]

(b) $\forall y_i \in P_i(x)$, $(x_i, y_i] \subset P_i(x)$*;*

C6 Consumption Survival CS: $e_i \in \operatorname{int} X_i$.

We note that these assumptions are standard in a model with nonordered preferences; the assumptions on P_i are satisfied in particular when agents' preferences are represented by utility functions that are continuous, strongly monotonic, and quasi-concave. An exchange economy

[5] Let Φ be a correspondence from X to Y, that is, Φ is a mapping from X to 2^Y. Then Φ is said to be lower semicontinuous (l.s.c.) at $x_o \in X$, if for every open set $V \subset Y$ such that $\Phi(x_o) \cap V \neq \emptyset$, there exists an open neighborhood U of x_o in X such that $\Phi(x) \cap V \neq \emptyset$ for all every $x \in U$. The correspondence Φ is said to be l.s.c. if it is l.s.c.at every point of X. Finally, we denote by $G(\Phi) := \{(x, y) \in X \times Y : y \in \Phi(x)\}$ the graph of Φ.

[6] Given $x_i \in X_i$ and $s \in \bar{S}$, we denote $x_i(-s) := (x_i(s'))_{s' \neq s}$.

$\mathcal{E}$ satisfying Assumption **C** will be called *standard*. We now recall that equilibrium asset prices preclude arbitrage opportunities under the above Non-Satiation Assumption. We denote by $\boldsymbol{A}Z$ the asymptotic cone [7] of a nonempty set $Z \subset \mathbb{R}^J$.

Proposition 1. *Assume* **LNS** *and the convexity of the portfolio sets Z_i ($i \in \mathcal{I}$). If $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then*

(i) $(\bar{q}, \bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$ in the sense that for every $i \in \mathcal{I}$, there is no $z_i \in Z_i$ such that $W(\bar{p}, \bar{q})z_i > W(\bar{p}, \bar{q})\bar{z}_i$.

(ii) $\bar{q}$ is arbitrage-free at $\bar{p}$, in the sense that

$$W(\bar{p}, \bar{q})(\bigcup_i \boldsymbol{A}Z_i) \cap \mathbb{R}^{\bar{S}}_+ = \{0\}\,.$$

We denote by $Q_{\mathcal{F}}(p)$ the set of arbitrage-free asset prices at $p \in \mathbb{R}^L$.

2.3. Equivalent and reduced financial structures

We introduce an equivalence relation on the set of all financial structures. We will say that two financial structures are equivalent if they are indistinguishable in terms of consumption equilibria. The intuition behind this definition is the following. Financial structures allow agents to transfer wealth across states of nature and thereby give them the possibility to enlarge their budget set. Hence if, regardless of the standard exchange economy $\mathcal{E}$, equilibrium consumption allocations and equilibrium commodity price vectors are the same when agents carry out their financial activities through two different structures, then we say that these two financial structures are equivalent.

Definition 2. *Consider two financial structures $\mathcal{F} = (V, (Z_i)_i)$ and $\mathcal{F}' = (V', (Z'_i)_i)$. We say that $\mathcal{F} \sim \mathcal{F}'$ (read $\mathcal{F}$ is equivalent to $\mathcal{F}'$) if for every standard exchange economy $\mathcal{E}$, the financial exchange economies $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}, \mathcal{F}')$ have the same consumption equilibria.*

Definition 3. *Let $\mathcal{F} = (V, (Z_i)_i)$ be a standard financial structure. The financial structure $\mathcal{F}$ is said to be reduced if one of the following condition is satisfied.*

For every $p \in \mathbb{R}^L$, $\mathcal{L}_{\mathcal{F}}(p) := \boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} Z_i \cap \{V(p) \geq 0\}\Big) \cap -\boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} Z_i \cap \{V(p) \geq 0\}\Big) = \{0\}$.

[7] The asymptotic cone of a nonempty subset Z of $\mathbb{R}^J$ is the set $\boldsymbol{A}Z := \{\lim_n \lambda^n z^n : (\lambda^n)_n \downarrow 0 \text{ and } z^n \in Z \text{ for all } n\}$. As a consequence from the definition, one has $\boldsymbol{A}(\mathrm{cl}Z) = \boldsymbol{A}Z$ and we refer to Debreu (1959) for a general reference. When Z is additionally assumed to be convex, then $\boldsymbol{A}Z = 0^+(\mathrm{cl}Z)$, where $0^+(C) := \left\{\zeta \in \mathbb{R}^J : \zeta + C \subset C\right\}$ is the recession cone of the convex set $C \subset \mathbb{R}^J$ (see Rockafellar (1997)). When Z is convex, the inclusion $0^+(Z) \subset \boldsymbol{A}Z$ holds but may be strict when Z is not closed.

The equivalence between the above conditions is established in Aouani and Cornet (2008).

2.4. The main result

Before stating our main results we introduce our central assumption. It is worthwhile to note that this assumption is new to the literature on existence of equilibrium.

Assumption F2: (a) Uniformity Assumption: The set $\mathcal{A}_{\mathcal{F}}(p) := A\Big(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq 0\})\Big)$ does not depend on p (hence is simply denoted $\mathcal{A}_{\mathcal{F}}$ hereafter);

(b) Closedness Assumption: Fro all $p \in \mathbb{R}^L$, the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed, where

$$\mathcal{G}_{\mathcal{F}}(p) := \left\{(V(p)z_1, \cdots, V(p)z_I, \sum_{i\in\mathcal{I}} z_i) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : (z_i)_{i\in\mathcal{I}} \in \prod_i Z_i\right\}.$$

It is important to notice that every reduced financial structure satisfies Closedness. Sufficient conditions for the Closedness Assumption to hold are provided in Section 2.6. Given the financial structure $\mathcal{F} = \Big(V, (Z_i)_{i\in\mathcal{I}}\Big)$, we denote $Z_{\mathcal{F}} :=< \sum_{i\in\mathcal{I}} Z_i >$ the linear space spanned by $\sum_{i\in\mathcal{I}} Z_i$, that is the space where financial activity takes place. We can now state the first result of this paper.

Theorem 1. *Let $\mathcal{F} = \Big(V, (Z_i)_i\Big)$ be a standard financial structure satisfying the Uniformity and Closedness assumptions. Then there exists a standard and reduced financial structure $\mathcal{F}'$ satisfying Uniformity, such that, for every standard exchange economy $\mathcal{E}$, every consumption equilibrium of $(\mathcal{E}, \mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E}, \mathcal{F})$. Moreover we can choose $\mathcal{F}'$ so that*

P *(i)* $\forall p \in \mathbb{R}^L, \mathrm{cl}Q_{\mathcal{F}'}(p) \cap Z_{\mathcal{F}'} \subset \mathrm{cl}Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}}$, *and*

P *(ii)* $\forall p \in \mathbb{R}^L, \forall q \in \mathrm{cl}Q_{\mathcal{F}'}(p) \cap Z_{\mathcal{F}'}, \forall i \in \mathcal{I}, \forall z_i \in Z_i, \exists\, z_i' \in Z_i', q \cdot z_i = q \cdot z_i'$.

The proof of Theorem 1 is given in Section 3. We end this section by a converse to Theorem 1.

F0. For all $p \in \mathbb{R}^L$, and for all $i \in \mathcal{I}$, there exists $\zeta_{i,p} \in AZ_i$ such that $V(p)\zeta_{i,p} \gg 0$.

Theorem 2. *Let $\mathcal{F} = \Big(V, (Z_i)_i\Big)$ be a standard financial structure satisfying Assumption* **F0**, *and the Uniformity and Closedness assumptions. Then there exists a reduced financial structure $\mathcal{F}'$ such that $\mathcal{F}$ and $\mathcal{F}'$ are equivalent.*

The proof of Theorem 2 is given in Section 4.

2.5. Existence of equilibria in the nominal and numéraire case

If the financial structure $\mathcal{F}$ is *nominal*, the matrix $V(p)$ of financial payoffs does not depend on the commodities price vector p and is denoted R.

A *numéraire asset* is defined as follows. Let us choose a commodity bundle $\nu \in \mathbb{R}^\ell$, a typical example being $\nu = (0, \ldots, 0, 1)$, when the ℓ-th good is chosen as numéraire. A numéraire asset j is a real asset which delivers the commodity bundle $A^j_s = R^j_s \nu \in \mathbb{R}^\ell$ at state s of date $t = 1$ if this state s prevails. Thus the payoff at state s is $(V_\nu)^j_s(p) = (p(s) \cdot \nu)R^j_s$ for the commodity price $p = (p(s)) \in \mathbb{R}^L$. For a numéraire financial structure, i.e., all the assets are numéraire assets (for the same commodity bundle ν), we denote R the $S \times J$-matrix with entries R^j_s and, for $p \in \mathbb{R}^L$, we denote $V_\nu(p)$ the associated $S \times J$-payoff-matrix, which has for entries $(V_\nu)^j_s(p) = (p(s) \cdot \nu)R^j_s$, i.e.

$$V_\nu(p) = \begin{pmatrix} p(1)\cdot\nu & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p(S)\cdot\nu \end{pmatrix} R.$$

In the *nominal* case, the set $Q(p)$ of arbitrage-free prices, that is, the set of asset prices q satisfying

$$\begin{pmatrix} -q \\ R \end{pmatrix} \Big(\bigcup_i AZ_i\Big) \cap \mathbb{R}^{\bar{S}}_+ = \{0\} \tag{2.1}$$

does not depend on the price p, hence is simply denoted Q_R. In the *numéraire* case, under the Desirability Assumption (made in **FN(a)**) below, if $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium, then $\bar{p}(s) \cdot \nu > 0$ for all $s \in \mathcal{S}$ (see the proof of Lemma 2.1 in Aouani and Cornet (2009)) and we notice that, if $\bar{p}(s) \cdot \nu > 0$ for all $s \in \mathcal{S}$, then $Q(\bar{p}) = Q_R$ as defined above by (2.1). Thus, every equilibrium asset price $\bar{q}$ belongs to Q_R (by Proposition 1) in the nominal case and in the numéraire case.

To state our second result, we need the following general assumptions on the financial side. We refer to Aouani and Cornet (2009) for a thorough discussion of these assumptions.

FN Financial Assumption in the Nominal-Numéraire case:

(a) *The financial structure $\mathcal{F}$ is either* **(i)** nominal*, i.e., $V(p) = R$ is independent of p,* **or (ii)** numéraire*, i.e., $V(p) = V_\nu(p)$, for every agent i, the correspondence P_i has an open graph and the commodity bundle $\nu \in \mathbb{R}^\ell$ is* desirable *at every state $s \in \mathcal{S}$, i.e., for all $x \in \mathcal{A}(\mathcal{E})$, for all $t > 0$, $(x_i(s) + t\nu, x_i(-s)) \in P_i(x)$;*

(b) *The financial structure $\mathcal{F}$ is standard and satisfies the Closedness Assumption. That is, for all $i \in \mathcal{I}$, Z_i is a closed convex set, $0 \in Z_i$, $V : \mathbb{R}^L \to \mathbb{R}^{S\times J}$ is continuous, and for all $p \in \mathbb{R}^L$ the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed;*

(c) Financial Survival[8] $\forall i \in \mathcal{I}$, $\forall p \in \mathbb{R}^L$, $p(0) = 0$, $\forall q \in \mathrm{cl}Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}}$, $q \neq 0$, $\exists\, \zeta_i \in Z_i$, $q \cdot \zeta_i < 0$.

Theorem 3. *Let the economy $(\mathcal{E}, \mathcal{F})$ satisfy assumptions* **C** *and* **FN**, *then it admits an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ such that $\|\bar{p}(s)\| = 1$ for $s \in \mathcal{S}$.*

Remark 1. Using the financial structure $\mathcal{F}_\pi$ given by Theorem 4 below one can easily show that equilibrium commodity and asset prices $\bar{p}$ and $\bar{q}$ satisfy the same conclusion as in the existence result of Aouani and Cornet (2009), that is, $\|\bar{p}(0)\| + \|\bar{q}\| = 1$ and $\|\bar{p}(s)\| = 1$ for $s \in \mathcal{S}$.

The proof of Theorem 3 is given in Section 2.7. We can now state some consequences to Theorem 3. The following Corollary 2 and 3 allow to extend to the case of consumers with nonordered preferences the existence results of Cass (1984), Duffie (1987), and Werner (1985) in the nominal case and Geanakoplos and Polemarchakis (1986) in the numéraire case.

Corollary 1. *The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumptions* **C**, **FN (a)** *and* **(b)** *if*

- $0 \in \mathrm{int}Z_i$ *for all i.*

Corollary 2. *The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumption* **C** *if*

- *$\mathcal{F}$ consists of nominal assets;*
- *$Z_i = \mathbb{R}^J$ for all i.*

Corollary 3. *The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumption* **C** *if*

- *$\mathcal{F}$ consists of numéraire assets and satisfies* **FN(a)(ii)**;
- *$Z_i = \mathbb{R}^J$ for all i.*

Corollary 4. *The economy $(\mathcal{E}, \mathcal{F})$ admits an equilibrium under Assumptions* **C**, **FN (a)** *and* **(c)** *if*

- *For all i, Z_i is a polyhedral set.*[9]

[8] This assumption can be weakened to the following: $\forall i \in \mathcal{I}$, $\forall q \in \mathrm{cl}Q_R \cap Z_{\mathcal{F}}$, $q \neq 0$, $\exists \zeta_i \in Z_i$, $q \cdot \zeta_i < 0$.

[9] We say that $Z \subset \mathbb{R}^J$ is a convex polyhedral set if it can be defined by finitely many linear inequalities, i.e., $Z := \left\{z \in \mathbb{R}^J : Bz \geq b\right\}$ for some $K \times J$-matrix B and some $b \in \mathbb{R}^K$.

2.6. Examples of restrictions satisfying the Closedness Assumption

As shown by the following Propositions 2 and 3, the Closedness Assumption holds true in many situations. Indeed, Closedness is fulfilled when restrictions on portfolio choices are given by a finite number of linear inequalities, that is, when all portfolio sets are finite intersections of half spaces. In particular, Closedness is fulfilled when portfolio sets are linear subspaces, when portfolio sets are unconstrained, or when there is an exogenous bound on portfolio short sales. Furthermore, the Closedness Assumption holds true under the no mutually compatible potential arbitrage condition (Page, 1987) that is when the family $\{\boldsymbol{A}Z_i \cap \ker V, i \in \mathcal{I}\}$ is positively semi-independent[10], in particular Closedness holds true under Siconolfi (1989)'s assumption ($\boldsymbol{A}Z_i \cap \ker V = \{0\}$ for all $i \in \mathcal{I}$), when portfolio sets are bounded, or when there are no redundant assets i.e. $\operatorname{rank} V = J$.

Proposition 2. *The Closedness Assumption holds true under anyone of the following conditions.*

(a) For all $i \in \mathcal{I}$, $Z_i = \mathbb{R}^J$ (unconstrained portfolios).

(b) For all $i \in \mathcal{I}$, Z_i is a linear subspace.

(c) For all $i \in \mathcal{I}$, $Z_i = \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales).

(d) For all $i \in \mathcal{I}$, Z_i is polyhedral.

(e) For all $i \in \mathcal{I}$, $Z_i = B_J(0,1)$ (bounded portfolio sets).

(f) For all $i \in \mathcal{I}$, $Z_i = K_i + P_i$ where K_i is nonempty compact and convex, and P_i is polyhedral.

The proof of Proposition 2 is given in Section 6.2.

Proposition 3. *The Closedness Assumption holds true under each of the following conditions.*

(g) There are no redundant assets i.e. $\forall p \in \mathbb{R}^L$, $\operatorname{rank}(V) = J$, or equivalently, $\ker V(p) = \{0\}$.

(h) For all $p \in \mathbb{R}^L$ and for all $i \in \mathcal{I}$, $\boldsymbol{A}Z_i \cap \ker V(p) = \{0\}$.

(i1) For all $p \in \mathbb{R}^L$, $\boldsymbol{A}\left(\sum_{i\in\mathcal{I}} Z_i \cap \{V(p) \geq 0\}\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathcal{I}} Z_i \cap \{V(p) \geq 0\}\right) = \{0\}$.

(i2) For all $p \in \mathbb{R}^L$, $\boldsymbol{A}\left(\sum_{i\in\mathcal{I}} Z_i \cap \ker V(p)\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathcal{I}} Z_i \cap \ker V(p)\right) = \{0\}$.

(i3) For all $p \in \mathbb{R}^L$, $\left(\sum_{i\in\mathcal{I}} \boldsymbol{A}Z_i \cap \{V(p) \geq 0\}\right) \cap -\left(\sum_{i\in\mathcal{I}} \boldsymbol{A}Z_i \cap \{V(p) \geq 0\}\right) = \{0\}$.

[10] A finite collection $\{C_i, i \in \mathcal{I}\}$ of nonempty convex cones in $\mathbb{R}^n$ is positively semi-independent if $c_i \in C_i$, for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} c_i = 0$, imply that for all $i \in \mathcal{I}$, $c_i = 0$.

(i4) For all $p \in \mathbb{R}^L$, $\left(\sum_{i\in\mathcal{I}} \boldsymbol{A}Z_i \cap \ker V(p)\right) \cap -\left(\sum_{i\in\mathcal{I}} \boldsymbol{A}Z_i \cap \ker V(p)\right) = \{0\}$.

(j1) For all $p \in \mathbb{R}^L$, the family $\{\boldsymbol{A}Z_i \cap \{V(p) \geq 0\} : i \in \mathcal{I}\}$ is positively semi-independent.

(j2) For all $p \in \mathbb{R}^L$, the family $\{\boldsymbol{A}Z_i \cap \ker V(p) : i \in \mathcal{I}\}$ is positively semi-independent.

(k1) For all $p \in \mathbb{R}^L$, the family $\{\boldsymbol{A}Z_i \cap \{V(p) \geq 0\}, i \in \mathcal{I}\}$ is weakly positively semi-independent.[11]

(k2) For all $p \in \mathbb{R}^L$, the family $\{\boldsymbol{A}Z_i \cap \ker V(p), i \in \mathcal{I}\}$ is weakly positively semi-independent.

The proof of Proposition 3 is given in Section 6.3. In Section 2.6.1, we provide an example of a financial structure satisfying The Closedness Assumption without satisfying any of the above conditions of Proposition 2 and Proposition 3.

2.6.1. An example

We now provide an example of a financial structure satisfying Assumption **F2** without satisfying any of the conditions in Proposition 2 and Proposition 3. Consider two agents, two states of nature at the second period, and three assets i.e. $I = S = 2$, and $J = 3$. Let the return matrix be given by

$$V = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Let agents' portfolio sets be given by (note that Z_1 is not polyhedral)

$$\begin{aligned} Z_1 &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3^2 \leq (z_1 + 1)z_2\right\}, \\ Z_2 &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \leq 0, z_3 = 0\right\}. \end{aligned}$$

Then the collection $\{\boldsymbol{A}Z_i \cap \{V \geq 0\} : i \in 1, 2\}$ is not **WPSI** and $\mathcal{F} = \left(V, (Z_i)_{i=1,2}\right)$ satisfies the Closedness Assumption. Details are provided in Section 6.4

2.6.2. Equivalent formulations of the Closedness Assumption

Let $\mathcal{F} = (V, (Z_i)_i)$ be a financial structure and, for $p \in \mathbb{R}^L$, denote

$$\mathcal{G}'_{\mathcal{F}}(p) := \left\{\left((v_i)_i, \sum_{i\in\mathcal{I}} z_i\right) \in (\mathbb{R}^S)^I \times \mathbb{R}^J : \forall i \in \mathcal{I}, z_i \in Z_i, V(p)z_i \geq v_i\right\}.$$

[11] A finite collection $\{C_i, i \in \mathcal{I}\}$ of nonempty convex cones of $\mathbb{R}^n$ is weakly positively semi-independent if $c_i \in C_i$ for all $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}} c_i = 0$ imply that for all $i \in \mathcal{I}$, $c_i \in C_i \cap -C_i$.

Proposition 4. *The set $\mathcal{G}_{\mathcal{F}}(p)$ is closed if and only if the set $\mathcal{G}'_{\mathcal{F}}(p)$ is closed.*

The proof of Proposition 4 is given in Section 6.5. Consider a financial structure $\mathcal{F} = (V, (Z_i)_i)$. Given $p \in \mathbb{R}^L$, let Φ_p and Ψ_p be the correspondences from $\mathbb{R}^{SI}$ into $\mathbb{R}^J$ defined by

$$\begin{aligned} \Phi_p(v_1, \cdots, v_I) &:= \sum_{i \in \mathcal{I}} \Big(Z_i \cap \{V(p) = v_i\} \Big), \\ \Psi_p(v_1, \cdots, v_I) &:= \sum_{i \in \mathcal{I}} \Big(Z_i \cap \{V(p) \geq v_i\} \Big). \end{aligned}$$

Then the graph of the correspondence Φ_p is precisely the set $\mathcal{G}_{\mathcal{F}}(p)$ i.e. $G(\Phi_p) = \mathcal{G}_{\mathcal{F}}(p)$. Similarly, one has $G(\Psi_p) = \mathcal{G}'_{\mathcal{F}}(p)$. Hence, the Closedness Assumption is equivalent to Φ_p and Ψ_p having closed graphs for all $p \in \mathbb{R}^L$.

2.7. Proof of the equilibrium existence result (Theorem 3)

2.7.1. The nominal case

This section considers the case of a financial structure $\mathcal{F} = (V, (Z_i)_i)$ with nominal assets satisfying Assumption **FN**, and in fact the more general case of a standard financial structure satisfying the Uniformity, Closedness and Financial Survival assumptions. This more general framework is needed in Section 2.7.2 to treat the case of numéraire assets.

By Theorem 1, there exists a standard and reduced financial structure $\mathcal{F}'$ satisfying Uniformity and property **P**, such that every consumption equilibrium of $(\mathcal{E}, \mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E}, \mathcal{F})$. Claim 2.1 below, shows that $\mathcal{F}'$ satisfies the conditions of Theorem 2 in Aouani and Cornet (2009). This allows us to apply the latter existence result to the financial exchange economy $(\mathcal{E}, \mathcal{F}')$ and to conclude to the existence of an equilibrium in $(\mathcal{E}, \mathcal{F}')$. Then $(\mathcal{E}, \mathcal{F})$ has an equilibrium since every consumption equilibrium of $(\mathcal{E}, \mathcal{F}')$ is a consumption equilibrium of $(\mathcal{E}, \mathcal{F})$.

Claim 2.1. *If $\mathcal{F} = (V, (Z_i)_i)$ is standard and satisfies Uniformity, Closedness and Financial Survival, then the financial structure $\mathcal{F}'$ provided by Theorem 1 is standard and satisfies: Financial Survival, the set $\boldsymbol{A}_{\mathcal{F}'}(p)$ does not depend on p, and $\boldsymbol{A}_{\mathcal{F}'} \cap -\boldsymbol{A}_{\mathcal{F}'} = \{0\}$.*

Proof. Standard: The financial structure $\mathcal{F}'$ is clearly standard (as a result from Theorem 1).

Financial Survival: Let $q \in \mathrm{cl}\, Q_{\mathcal{F}'}(p) \cap Z_{\mathcal{F}'} \setminus \{0\}$. Then, by Theorem 1 (more precisely, by property **P**(i)), $q \in \mathrm{cl}\, Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}} \setminus \{0\}$ and by Financial Survival in $\mathcal{F}$, for every $i \in \mathcal{I}$, there exists

$z_i \in Z_i$ such that $q \cdot z_i < 0$. Hence, again by Theorem 1 (more precisely, by property **P**(ii)), for each $i \in I$, there exists $z_i' \in Z_i'$ such that $q \cdot z_i' = q \cdot z_i < 0$.

For every $p \in \mathbb{R}^L$, $\boldsymbol{A}_{\mathcal{F}'}(p) \cap -\boldsymbol{A}_{\mathcal{F}'}(p) = \{0\}$: From Theorem 1, we have $\mathcal{A}_{\mathcal{F}'}(p) \cap -\mathcal{A}_{\mathcal{F}'}(p) = \{0\}$. Recalling that the latter condition is equivalent to $\boldsymbol{A}_{\mathcal{F}'}(p) \cap -\boldsymbol{A}_{\mathcal{F}'}(p) = \{0\}$ (see Aouani and Cornet (2008)), we get the desired result.

$\boldsymbol{A}_{\mathcal{F}'}(p)$ does not depend on p: From the previous step $\boldsymbol{A}_{\mathcal{F}'}(p) \cap -\boldsymbol{A}_{\mathcal{F}'}(p) = \{0\}$, thus the cones $\boldsymbol{A}Z_i' \cap \{V(p) \geq 0\}$ are weakly positively semi-independent. Hence $\boldsymbol{A}_{\mathcal{F}'}(p) = \mathcal{A}_{\mathcal{F}'}(p)$ (see Theorem 9.1 page 73 in Rockafellar (1997)). Recalling that, by Theorem 1, $\mathcal{F}'$ satisfies Uniformity, we conclude that $\boldsymbol{A}_{\mathcal{F}'}(p)$ does not depend on p. ∎

2.7.2. *The numéraire case*

Consider a financial economy with numéraire assets $(\mathcal{E}, \mathcal{F})$ satisfying Assumption **FN** (Part (ii) with numéraire assets). The proof of Theorem 3 consists in applying the result of the previous section to a modified financial economy $(\mathcal{E}, \mathcal{F}^\varepsilon)$ (for $\varepsilon > 0$ small enough), chosen so that (i) the financial structure $\mathcal{F}^\varepsilon$ is standard and satisfies Uniformity, Closedness and Financial Survival, and (ii) the equilibria of $(\mathcal{E}, \mathcal{F}^\varepsilon)$ are also equilibria of the original financial economy $(\mathcal{E}, \mathcal{F})$.

Step 0. We define the modified financial structure $\mathcal{F}^\varepsilon = (V^\varepsilon, (Z_i)_i)$ for $\varepsilon > 0$, by taking the same portfolio sets Z_i as for $\mathcal{F}$ and defining the modified payoff matrix V^ε, by

$$V^\varepsilon(p) = \begin{pmatrix} \max\{\varepsilon, p(1) \cdot v\} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \max\{\varepsilon, p(S) \cdot v\} \end{pmatrix} R.$$

The financial structure $\mathcal{F}^\varepsilon$ is standard and satisfies Uniformity, Closedness and Financial Survival whenever the financial structure $\mathcal{F}$ satisfies Assumption **FN**. Indeed, $\{V^\varepsilon(p) \geq 0\} = \{R \geq 0\}$ for every $p \in \mathbb{R}^L$, hence $\mathcal{F}^\varepsilon$ satisfies Uniformity. Closedness is obviously satisfied, and Financial Survival in $\mathcal{F}_\pi$ is a consequence of Financial Survival in $\mathcal{F}$ and the fact that $Q_{\mathcal{F}^\varepsilon}(p) = Q_R$ for every p. The relationship between the equilibria of the economies $(\mathcal{E}, \mathcal{F}^\varepsilon)$ and $(\mathcal{E}, \mathcal{F})$ is then given by the following lemma, the proof of which can be found in Aouani and Cornet (2009).

Lemma 2.1. *For $\varepsilon > 0$ small enough, every equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ of $(\mathcal{E}, \mathcal{F}^{\varepsilon})$ such that $\|\bar{p}(s)\| = 1$ for $s \in \mathcal{S}$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$.*

3. Proof of Theorem 1

3.1. A sharper result

Let $\mathcal{F} = (V, (Z_i)_i)$ be a standard financial structure satisfying Uniformity and Closedness. We consider the financial structure $\mathcal{F}_\pi$ which has the same payoff matrix as $\mathcal{F}$ and the portfolio sets $\mathrm{cl}\pi Z_i$ $(i \in \mathcal{I})$ where π is the orthogonal projection mapping[12] of $\mathbb{R}^J$ on the orthogonal space to $\mathcal{L}_{\mathcal{F}} := \mathcal{A}_{\mathcal{F}} \cap -\mathcal{A}_{\mathcal{F}}$. We recall that $Z_{\mathcal{F}} :=< \cup_i Z_i >$, $Z_{\mathcal{F}_\pi} :=< \cup_i \mathrm{cl}\pi Z_i >$ and the definition of $\mathcal{F}_\pi$ can be summarized by $\mathcal{F}_\pi = \left(V, (\mathrm{cl}\pi Z_i)_i\right)$, where

$$\pi = \mathrm{proj}_{(\mathcal{L}_{\mathcal{F}})^{\perp}},\ \mathcal{L}_{\mathcal{F}} := \mathcal{A}_{\mathcal{F}} \cap -\mathcal{A}_{\mathcal{F}},\ \text{and}\ \mathcal{A}_{\mathcal{F}} := \boldsymbol{A}\Big(\sum_{i \in \mathcal{I}} (Z_i \cap \{V(p) \geq 0\})\Big) \subset Z_{\mathcal{F}}.$$

Note that $\mathcal{L}_{\mathcal{F}} \subset \ker V(p)$ for all $p \in \mathbb{R}^L$. We will use extensively the following properties [13] of the linear mapping π: for all $(p, q, z) \in \mathbb{R}^L \times \mathbb{R}^J \times \mathbb{R}^J$,

$$q \cdot \pi z = \pi q \cdot \pi z = \pi q \cdot z,\ \ker \pi = \mathcal{L}_{\mathcal{F}},\ V(p)\pi z = V(p) z,\ \text{ hence }\ W(p, q)\pi z = W(p, \pi q) z. \quad (3.1)$$

Theorem 1 is a direct consequence of the following theorem.

Theorem 4. *Let $\mathcal{F} = \left(V, (Z_i)_i\right)$ be a standard financial structure satisfying the Uniformity and Closedness Assumptions. Then*

(a) The financial structure $\mathcal{F}_\pi$ is standard and reduced, and satisfies Uniformity.

(b) For every standard exchange economy $\mathcal{E}$, if $(\mathcal{E}, \mathcal{F}_\pi)$ has an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$, then there exists $z^* \in \prod_i Z_i$ such that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$.

(c) The financial structure $\mathcal{F}_\pi$ satisfies the following property **P**.

P *(i)* $\forall p \in \mathbb{R}^L, \mathrm{cl} Q_{\mathcal{F}'}(p) \cap Z_{\mathcal{F}'} \subset \mathrm{cl} Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}}$, *and*

P *(ii)* $\forall p \in \mathbb{R}^L, \forall q \in \mathrm{cl} Q_{\mathcal{F}'}(p) \cap Z_{\mathcal{F}'}, \forall i \in \mathcal{I}, \forall z_i \in Z_i, \exists\, z'_i \in Z'_i,\ q \cdot z_i = q \cdot z'_i$.

[12]When L is a subset of $\mathbb{R}^J$, we define the orthogonal set to L by $L^{\perp} := \left\{z \in \mathbb{R}^J : z \cdot \xi = 0 \text{ for all } \xi \in L\right\}$. When L is a linear space and $\varphi \in \mathbb{R}^J$, we denote by $\mathrm{proj}_L \varphi$ (resp. $\mathrm{proj}_{L^{\perp}} \varphi$) the orthogonal projection of φ on L (resp. on $L^{\perp}$), that is, the unique $\alpha \in L$ (resp. $\beta \in L^{\perp}$) such that $\varphi - \alpha \in L^{\perp}$ (resp. $\varphi - \beta \in L$).

[13]The first equality comes from the fact that $\pi q \cdot \pi z = \pi q \cdot z$, since $\pi q \in \mathrm{Im}\pi$ and $z - \pi z \in \ker \pi = (\mathrm{Im}\pi)^{\perp}$ since π is an orthogonal projection mapping; then by symmetry $q \cdot \pi z = \pi q \cdot \pi z = \pi q \cdot z$. The second one holds since $z - \pi z \in \ker \pi = \mathcal{L}_{\mathcal{F}}$ and $\mathcal{L}_{\mathcal{F}} := \mathcal{A}_{\mathcal{F}} \cap -\mathcal{A}_{\mathcal{F}} \subset \{V(p) \geq 0\} \cap -\{V(p) \geq 0\} = \ker V(p)$.

The proof of Theorem 4 is given in Section 3.3. The proof is provided under two assumptions that are weaker than the Closedness Assumption. We consider the following two conditions:

F2α. For all $p \in \mathbb{R}^L$, $\mathrm{cl}G(\Phi_p) \cap \left(\mathbb{R}^{S\mathcal{I}} \times \{0_{\mathbb{R}^J}\}\right) \subset G(\Phi_p)$.

F2a. For all $p \in \mathbb{R}^L$, and for all $v = (v_i)_{i\in I} \in (\mathbb{R}^S)^I$ such that $\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\}) \neq \emptyset$, the space $\boldsymbol{A}\left(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\right)$ does not depend on v.

By Section 2.6.2 the Closedness Assumption is equivalent to $G(\Phi_p)$ being closed for every $p \in \mathbb{R}^L$. Hence, Assumption **F2α** is clearly a consequence of Closedness. The following proposition whose proof is given in Section 6.6 shows that **F2a** is as well a consequence of the Closedness Assumption.

Proposition 5. *Let $\mathcal{F} = \left(V, (Z_i)_i\right)$ be a standard financial structure. Under the Closedness Assumption, for every $v = (v_i)_{i\in I} \in (\mathbb{R}^S)^I$ such that $\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\}) \neq \emptyset$, the space $\mathcal{L}_{\mathcal{F}}(p, v) := \boldsymbol{A}\left(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\right) \cap -\boldsymbol{A}\left(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\right)$ does not depend on v.*

In the next section we state and prove two lemmas which will be useful in the sequel.

3.2. Preliminary lemmas

Lemma 3.1. *Let $\mathcal{F} = \left(V, (Z_i)_i\right)$ be standard and satisfies Uniformity and* **F2a**. *For all $p \in \mathbb{R}^L$ and for all $(\hat{y}_i)_{i\in I} \in \prod_i \mathrm{cl}\pi Z_i$ such that $\sum_{i\in\mathcal{I}}\hat{y}_i = 0$, one has*

$$\sum_{i\in\mathcal{I}}\left(\mathrm{cl}\pi Z_i \cap \{V(p) \geq V(p)\hat{y}_i\}\right) \subset \mathrm{cl}\sum_{i\in\mathcal{I}}\left(Z_i \cap \{V(p) \geq V(p)\hat{y}_i - \varepsilon\}\right), \forall \varepsilon \gg 0. \textit{ Hence } \mathcal{L}_{\mathcal{F}_\pi} \subset \mathcal{L}_{\mathcal{F}}.$$

We prepare the proof by a claim.

Claim 3.1. *Let $\mathcal{F} = (V, (Z_i)_i)$ be standard and satisfies Uniformity and* **F2a**. *For all $p \in \mathbb{R}^L$ and for all $(v_i)_{i\in I} \in (\mathbb{R}^S)^I$, one has*

$$\sum_{i\in\mathcal{I}}\mathrm{cl}\left(\pi Z_i \cap \{V(p) \geq v_i\}\right) \subset \mathrm{cl}\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\}).$$

Proof. If $\sum_{i\in\mathcal{I}}(\pi Z_i \cap \{V(p) \geq v_i\}) = \emptyset$ then there is nothing to prove. Otherwise, we show that

$$\begin{aligned}
\sum_{i\in\mathcal{I}}(\pi Z_i \cap \{V(p) \geq v_i\}) &= \sum_{i\in\mathcal{I}}\pi(Z_i \cap \{V(p) \geq v_i\}) &(3.2)\\
&\subset \ker\pi + \sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\}) &(3.3)\\
&\subset \mathrm{cl}\Big(\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\Big). &(3.4)
\end{aligned}$$

To prove the equality (3.2), it suffices to notice that for every $i \in \mathcal{I}$, $\pi Z_i \cap \{V(p) \geq v_i\} = \pi(Z_i \cap \{V(p) \geq v_i\})$. Indeed, let $y_i \in \pi Z_i \cap \{V(p) \geq v_i\}$, then there exists $z_i \in Z_i$ such that $y_i = \pi z_i$, and $V(p)y_i \geq v_i$. But $V(p)z_i = V(p)y_i + V(p)(z_i - \pi z_i) = V(p)y_i$ (since $z_i - \pi z_i \in \ker \pi \subset \mathcal{L}_{\mathcal{F}}$ and obviously $\mathcal{L}_{\mathcal{F}} \subset \ker V(p)$). Then $z_i \in Z_i \cap \{V(p) \geq v_i\}$ and Consequently $y_i \in \pi(Z_i \cap \{V(p) \geq v_i\})$. The proof of the converse inclusion is similar.

To prove the inclusion (3.3), let $y = \sum_{i\in\mathcal{I}} \pi z_i$ with $z_i \in Z_i \cap \{V(p) \geq v_i\}$. Then $y = \pi z = (\pi z - z) + z$ with $\pi z - z \in \ker \pi$ and $z = \sum_{i\in\mathcal{I}} z_i \in \sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\})$. This ends the proof of the inclusion of (3.3).

The second inclusion (3.4) comes from the fact that

$$\ker \pi \subset \mathcal{L}_{\mathcal{F}} = \boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\})\Big) \cap -\boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\})\Big) \subset \boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\})\Big),$$

where the first inclusion holds by assumption, the equality comes from Assumption **F2a** and Uniformity, and the last inclusion is immediate. Consequently,

$$\begin{aligned} \ker \pi + \sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\}) \quad &\subset \quad \boldsymbol{A}\Big(\sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\})\Big) + \sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\}) \\ &\subset \quad \mathrm{cl} \sum_{i\in\mathcal{I}} (Z_i \cap \{V(p) \geq v_i\}). \end{aligned}$$

Using the above result (3.4) and recalling that for a finite family of sets $A_i \subset \mathbb{R}^k, (i \in \mathcal{I})$, one always has $\sum_{i\in\mathcal{I}} \mathrm{cl} A_i \subset \mathrm{cl}(\sum_{i\in\mathcal{I}} A_i)$, we get

$$\sum_{i\in\mathcal{I}} \mathrm{cl}\Big(\pi Z_i \cap \{V(p) \geq v_i\}\Big) \subset \mathrm{cl} \sum_{i\in\mathcal{I}} \Big(\pi Z_i \cap \{V(p) \geq v_i\}\Big) \subset \mathrm{cl} \sum_{i\in\mathcal{I}} \Big(Z_i \cap \{V(p) \geq v_i\}\Big).$$

This ends the proof of the claim. ■

Proof of Lemma 3.1

Let $y_i \in (\mathrm{cl}\pi Z_i) \cap \{V(p) \geq V(p)\hat{y}_i\}$ $(i \in \mathcal{I})$. Fix $\varepsilon \gg 0$. Take $v_i^n \uparrow V(p)\hat{y}_i$ such that $V(p)\hat{y}_i \gg v_i^n$ for every n and note that for n large enough $v_i^n \gg V(p)\hat{y}_i - \varepsilon$. Pick $\bar{y}_i \in \mathrm{ri}\pi Z_i$ and consider $y_i^n = (1 - \lambda^n)y_i + \lambda^n \bar{y}_i$ with $0 < \lambda^n < \frac{1}{n}$ small enough so that $V(p)y_i^n \gg v_i^n$. Then $y_i^n \in [\bar{y}_i, y_i) \subset \mathrm{ri}\pi Z_i$ since $y_i \in \mathrm{cl}\pi Z_i$ and $\bar{y}_i \in \mathrm{ri}\pi Z_i$ (Theorem 6.1 page 45 in Rockafellar (1997)). Thus $y_i^n \in \pi Z_i \cap \Big\{V(p) \geq v_i^n\Big\}$, so $y_i^n \in \pi Z_i \cap \{V(p) \geq V(p)\hat{y}_i - \varepsilon\}$ for n large enough. Hence $y_i \in \mathrm{cl}\Big(\pi Z_i \cap \{V(p) \geq V(p)\hat{y}_i - \varepsilon\}\Big)$. That is, $\sum_{i\in\mathcal{I}} y_i \in \sum_{i\in\mathcal{I}} \mathrm{cl}\Big(\pi Z_i \cap \{V(p) \geq V(p)\hat{y}_i - \varepsilon\}\Big)$. Therefore, by Claim 3.1,

$$\sum_{i\in\mathcal{I}} y_i \in \mathrm{cl} \sum_{i\in\mathcal{I}} \Big(Z_i \cap \{V(p) \geq V(p)\hat{y}_i - \varepsilon\}\Big).$$

This completes the proof of the first part of the lemma. Taking $\hat{y}_i = 0$ for each $i \in \mathcal{I}$, one has for every $\varepsilon \gg 0$, $\sum_{i\in\mathcal{I}}\Big((\mathrm{cl}\pi Z_i) \cap \{V(p) \geq 0\}\Big) \subset \mathrm{cl}\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq -\varepsilon\})$. Hence, $\mathcal{L}_{\mathcal{F}_\pi} \subset \boldsymbol{A}\Big(\mathrm{cl}\sum_{i\in\mathcal{I}}Z_i \cap \{V(p) \geq -\varepsilon\}\Big) \cap -\boldsymbol{A}\Big(\mathrm{cl}\sum_{i\in\mathcal{I}}Z_i \cap \{V(p) \geq -\varepsilon\}\Big) = \mathcal{L}_{\mathcal{F}}$ (by Uniformity and Assumption **F2a**). This ends the proof of the lemma. ∎

Given the financial structure $\mathcal{F}$ and given $p \in \mathbb{R}^L$, we denote

$$W_{\mathcal{F}}(p) := \left\{\Big(W(p,q)z_1, \cdots, W(p,q)z_I\Big) : (z_i)_i \in \prod_i Z_i, \sum_{i\in\mathcal{I}} z_i = 0, \text{ and } (q,z) \text{ is arbitrage-free at } p\right\}.$$

Lemma 3.2. *Let $\mathcal{F} = \Big(V, (Z_i)_i\Big)$ be standard and satisfies Uniformity,* **F2a**, *and* **F2α**. *The financial structure $\mathcal{F}_\pi$ satisfies $W_{\mathcal{F}_\pi}(p) \subset W_{\mathcal{F}}(p)$ for all $p \in \mathbb{R}^L$.*

Proof. We show that if (q, y) is arbitrage-free at p in $\mathcal{F}_\pi$ and $\sum_{i\in\mathcal{I}} y_i = 0$, then there exists a mutually compatible portfolio allocation $z^* \in \prod_i Z_i$ such that $W(p,q)y_i = W(p,\pi q)z_i^*$ for every $i \in \mathcal{I}$ and $(\pi q, z^*)$ is arbitrage-free at p. Let $\Big(q, (y_i)_i\Big) \in \mathbb{R}^J \times (\prod_i \mathrm{cl}\pi Z_i)$ be arbitrage-free in $\mathcal{F}_\pi$ and such that $\sum_{i\in\mathcal{I}} y_i = 0$. Then, by Lemma 3.1,

$$0 = \sum_{i\in\mathcal{I}} y_i \in \sum_{i\in\mathcal{I}}\Big((\mathrm{cl}\pi Z_i) \cap \{V(p) \geq V(p)y_i\}\Big) \subset \bigcap_{\varepsilon\gg 0} \mathrm{cl}\sum_{i\in\mathcal{I}}\Big(Z_i \cap \{V(p) \geq V(p)y_i - \varepsilon\}\Big).$$

Then for every $n \geq 1$ and for every $i \in \mathcal{I}$, there exists $z_i^n \in Z_i, V(p)z_i^n \geq V(p)y_i - \frac{1}{n}\mathbf{1}$, and $\sum_{i\in\mathcal{I}} z_i^n = 0$. For each $i \in \mathcal{I}$, the sequence $(V(p)z_i^n)_n$ is bounded below, moreover $\sum_{i\in\mathcal{I}} V(p)z_i^n = 0$, hence for each $i \in \mathcal{I}$, the sequence $(V(p)z_i^n)_n$ is bounded. We can therefore assume that for each $i \in \mathcal{I}$, the sequence $(V(p)z_i^n)_n$ converges (use subsequences if necessary) to $w_i \in \mathbb{R}^S$ satisfying $w_i \geq V(p)y_i$. Then $(w_1, \cdots, w_I, 0) = \lim_n(V(p)z_1^n, \cdots, V(p)z_I^n, \sum_{i\in\mathcal{I}} z_i^n) \in \mathrm{cl}G(\Phi_p) \cap \Big((\mathbb{R}^S)^I \times \{0_{\mathbb{R}^J}\}\Big)$. By **F2$\alpha$**, $(w_1, \cdots, w_I, 0) \in G(\Phi_p) = \mathcal{G}_{\mathcal{F}}(p)$. Therefore, for each $i \in \mathcal{I}$, there exists $z_i^* \in Z_i$ such that $\sum_{i\in\mathcal{I}} z_i^* = 0$ and $V(p)z_i^* = w_i \geq V(p)y_i$. Since $V(p)z_i^* = w_i \geq V(p)y_i$ for every i, and $\sum_{i\in\mathcal{I}} V(p)(z_i^* - y_i) = 0$ (because $\sum_{i\in\mathcal{I}}(z_i^* - y_i) = 0$), we conclude that $V(p)z_i^* = V(p)y_i$ for every i.

Now, we show that, for every i, $\pi q \cdot z_i^* = \pi q \cdot y_i$ (which, by (3.1), is equal to $q \cdot \pi y_i$). Let us first note that it suffices to show that for every $i \in \mathcal{I}$, $-\pi q \cdot z_i^* \leq -\pi q \cdot y_i$. In this case $-\pi q \cdot (\sum_{i\in\mathcal{I}} z_i^* - \sum_{i\in\mathcal{I}} y_i) = 0$ implies $\pi q \cdot z_i^* = \pi q \cdot y_i$, for every i. Suppose, that for some i, $-\pi q \cdot z_i^* > -\pi q \cdot y_i$. Since $V(p)z_i^* = V(p)y_i$, one has $W(p,\pi q)z_i^* > W(p,\pi q)y_i$. Then from (3.1) $W(p,q)\pi z_i^* > W(p,q)\pi y_i$. Moreover $\pi z_i^* \in \pi Z_i \subset \mathrm{cl}\pi Z_i$. It thus suffices to show that $\pi y_i = y_i$ to get $W(p,q)\pi z_i^* > W(p,q)y_i$ which would contradict the assumption that (q, y) is arbitrage-free in $\mathcal{F}_\pi$. Since $y_i \in \mathrm{cl}\pi Z_i$, one has $y_i = \lim_n \pi y_i^n$ with $y_i^n \in Z_i$. Then $\pi y_i = \pi \lim_n \pi y_i^n = \lim_n \pi(\pi y_i^n) = \lim_n \pi y_i^n = y_i$.

Finally, we show that $(\pi q, z^*)$ is arbitrage-free at p in $\mathcal{F}$. Assume that for some $i \in \mathcal{I}$, there exists $\bar{z}_i \in Z_i$ such that $W(p, \pi q)\bar{z}_i > W(p, \pi q)z_i^*$. Then from (3.1) $W(p, q)\pi\bar{z}_i > W(p, q)y_i$. A contradiction to the fact that (q, y) is arbitrage-free at p in $\mathcal{F}_\pi$. ∎

3.3. Proof of Theorem 4

3.3.1. *Proof of Part (a) of Theorem 4*

$\mathcal{F}_\pi$ is obviously standard.

$\mathcal{F}_\pi$ satisfies Uniformity: We show that for all $p \in \mathbb{R}^L$, $\pi(\mathcal{A}_{\mathcal{F}}(p)) = \mathcal{A}_{\mathcal{F}_\pi}(p)$ and the desired result follows from the fact that $\mathcal{F}$ satisfies Uniformity. For the first inclusion: we have

$$\pi\Big(\mathcal{A}_{\mathcal{F}}(p)\Big) \subset \boldsymbol{A}\pi\sum_{i\in\mathcal{I}}(Z_i \cap \{V(p) \geq 0\}) = \boldsymbol{A}\sum_{i\in\mathcal{I}}\pi Z_i \cap \{V(p \geq 0)\} \subset \mathcal{A}_{\mathcal{F}_\pi}(p).$$

The first inclusion follows from $\pi(\boldsymbol{A}C) \subset \boldsymbol{A}(\pi C)$ (see Rockafellar (1997)). The equality comes from (3.2). The last inclusion comes from "$C_1 \subset C_2 \Rightarrow \boldsymbol{A}C_1 \subset \boldsymbol{A}C_2$".

For the converse inclusion: From Lemma 3.1, taking $\hat{y}_i = 0$ for all $i \in \mathcal{I}$, and then the asymptotic cones of both sides of the inclusion we get $\mathcal{A}_{\mathcal{F}_\pi}(p) \subset \mathcal{A}_{\mathcal{F}}(p)$. Thus $\pi\Big(\mathcal{A}_{\mathcal{F}_\pi}(p)\Big) \subset \pi\Big(\mathcal{A}_{\mathcal{F}}(p)\Big)$, and it suffices to notice that $\mathcal{A}_{\mathcal{F}_\pi}(p) \subset \mathrm{Im}\pi$ to conclude that $\mathcal{A}_{\mathcal{F}_\pi}(p) \subset \pi\Big(\mathcal{A}_{\mathcal{F}}(p)\Big)$.

$\mathcal{F}_\pi$ is reduced: First, we claim that $\mathcal{L}_{\mathcal{F}_\pi} \subset \mathcal{L}_{\mathcal{F}} \cap \mathrm{Im}\pi$. Indeed, we clearly have $\mathcal{L}_{\mathcal{F}_\pi} \subset \mathrm{Im}\pi$ since $\sum_{i\in\mathcal{I}}(\mathrm{cl}\pi Z_i \cap \{V(p) \geq 0\}) \subset \mathrm{Im}\pi$, and by Lemma 3.1, $\mathcal{L}_{\mathcal{F}_\pi} \subset \mathcal{L}_{\mathcal{F}}$. This ends the proof of the claim. Since $\mathcal{L}_{\mathcal{F}} = \ker\pi$, then from the above claim, we get $\mathcal{L}_{\mathcal{F}_\pi} \subset \mathcal{L}_{\mathcal{F}} \cap \mathrm{Im}\pi = \ker\pi \cap \mathrm{Im}\pi = \{0\}$. This ends the proof of Part (a) of Theorem 4. ∎

3.3.2. *Proof of Part (b) of Theorem 4*

We show that if $(\mathcal{E}, \mathcal{F}_\pi)$ has an equilibrium $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$, then there exists $z^* \in \prod_i Z_i$ such that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$. Let $(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ be an equilibrium in $(\mathcal{E}, \mathcal{F}_\pi)$. Then by Proposition 1, $(\bar{q}, \bar{y})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}_\pi$, and by Lemma 3.2, for each i, there exists $z_i^* \in Z_i$ such that $W(\bar{p}, \pi\bar{q})z_i^* = W(\bar{p}, \bar{q})\bar{y}_i$, $\sum_{i\in\mathcal{I}} z_i^* = 0$, and $(\pi\bar{q}, z^*)$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$. We show that $(\bar{p}, \pi\bar{q}, \bar{x}, z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$. First, from $W(\bar{p}, \pi\bar{q})z_i^* = W(\bar{p}, \bar{q})\bar{y}_i$ for each $i \in \mathcal{I}$, we conclude that $(\bar{x}_i, z_i^*) \in B^i_{\mathcal{F}}(\bar{p}, \pi\bar{q})$ since $(\bar{x}_i, \bar{y}_i) \in B^i_{\mathcal{F}_\pi}(\bar{p}, \bar{q})$. To complete the proof, we need only show that for each $i \in \mathcal{I}$,

$$B^i_{\mathcal{F}}(\bar{p}, \pi\bar{q}) \cap \Big(P_i(\bar{x}) \times Z_i\Big) = \emptyset.$$

Since $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$, we have

$$B^i_{\mathcal{F}_\pi}(\bar{p}, \bar{q}) \cap \left(P_i(\bar{x}) \times \mathrm{cl}\pi Z_i\right) = \emptyset.$$

In view of the above, the proof will be completed if we show that if $(x_i, z_i) \in B^i_{\mathcal{F}}(\bar{p}, \pi\bar{q})$, then $(x_i, \pi z_i) \in B^i_{\mathcal{F}_\pi}(\bar{p}, \bar{q})$. This is true since $W(\bar{p}, \pi\bar{q})z_i = W(\bar{p}, \bar{q})\pi z_i$ by (3.1). ■

3.3.3. Proof of Part (c) of Theorem 4

We claim that, for every $p \in \mathbb{R}^L$,

$$Q_{\mathcal{F}_\pi}(p) \cap Z(\mathcal{F}_\pi) \subset Q_{\mathcal{F}}(p) \cap Z_{\mathcal{F}} \tag{3.5}$$

First, we show that $Q_{\mathcal{F}_\pi}(p) \cap Z(\mathcal{F}_\pi) \subset Q_{\mathcal{F}_\pi}(p) \cap \mathrm{Im}\pi \subset Q_{\mathcal{F}}(p)$. The first inclusion is a consequence of the fact that $Z(\mathcal{F}_\pi) \subset \mathrm{Im}\pi$. We prove the second inclusion by contradiction. Assume that there is some $q \in Q_{\mathcal{F}_\pi}(p) \cap \mathrm{Im}\pi$ such that $q \notin Q_{\mathcal{F}}(p)$. Then there exists $i \in \mathcal{I}$ and $\zeta_i \in \boldsymbol{A}Z_i$ such that $W(p, q)\zeta_i > 0$. But $\pi\zeta_i \in \pi(\boldsymbol{A}Z_i) \subset \boldsymbol{A}(\pi Z_i)$ (from Rockafellar (1997)) and (since $q \in \mathrm{Im}\pi$ implies $q = \pi q$), from (3.1) $W(p, q)(\pi\zeta_i) = W(p, q)\zeta_i > 0$, which contradicts the fact that $q \in Q_{\mathcal{F}_\pi}(p)$. This ends the proof of the two inclusions. We end the proof of (3.5) by showing that $Z(\mathcal{F}_\pi) \subset Z_{\mathcal{F}}$. Indeed, let $y \in Z(\mathcal{F}_\pi)$, then $y = \pi z$ for some $z \in Z_{\mathcal{F}}$ and $y = \pi z = \pi z - z + z \in \ker\pi + Z_{\mathcal{F}} \subset Z_{\mathcal{F}}$ since $\ker\pi \subset \mathcal{L}_{\mathcal{F}} \subset Z_{\mathcal{F}}$.

We turn now to the proof of Part (c) of Theorem 4. The inclusion (3.5) implies that $\mathcal{F}_\pi$ satisfies property **P**(i). We need only show that $\mathcal{F}_\pi$ satisfies property **P**(ii). Let $q \in \mathrm{cl}Q_{\mathcal{F}_\pi}(p) \cap Z_{\mathcal{F}_\pi}$, $i \in \mathcal{I}$, and $z_i \in Z_i$. Then $\pi z_i \in \mathrm{cl}\pi Z_i$ and $q \cdot \pi z_i = q \cdot z_i$ by (3.1) and using the fact that $\pi q = q$ because $q \in Q_{\mathcal{F}_\pi}(p) \cap Z_{\mathcal{F}_\pi} \subset \mathrm{Im}\pi$. This ends the proof of Theorem 4. ■

4. Proof of Theorem 2

Let $\mathcal{F} = (V, (Z_i)_i)$ be a standard financial structure satisfying assumptions **F0**, Uniformity and Closedness, and let π be an orthogonal projection mapping of $\mathbb{R}^J$ on the orthogonal space to $\mathcal{L}_{\mathcal{F}}$. Denote $\mathcal{F}_\pi := \left(V, (\mathrm{cl}\pi Z_i)_i\right)$. We show that $\mathcal{F}$ and $\mathcal{F}_\pi$ are equivalent. By Theorem 4, for every exchange economy $\mathcal{E}$ satisfying Assumption **C**, every consumption equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$ is a consumption equilibrium of $(\mathcal{E}, \mathcal{F})$. To end the proof of Theorem 2, we show that for every exchange economy $\mathcal{E}$ satisfying Assumption **C**, if $(\mathcal{E}, \mathcal{F})$ has an equilibrium (p^*, q^*, x^*, y^*), then $(p^*, \pi q^*, x^*, \pi z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$.

4.1. Preliminary lemmas

We need two Lemmas.

Lemma 4.1. *Let $\mathcal{F} = (V, (Z_i)_i)$ be a standard financial structure satisfying* **F0** *and Uniformity. If $(\bar{q}, \bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$ then $\bar{q} \in -\Big(\boldsymbol{A}\sum_{i\in\mathcal{I}}(Z_i \cap \{V(\bar{p}) \geq 0\})\Big)^o$. Hence $\bar{q} \in \mathcal{L}_{\mathcal{F}}^{\perp}$.*

Proof. By contraposition. Let $(\bar{q}, \bar{z})$ be arbitrage-free at $\bar{p}$ in $\mathcal{F}$ and suppose that $\bar{q} \notin -\Big(\boldsymbol{A}\sum_{i\in\mathcal{I}}(Z_i \cap \{V(\bar{p}) \geq 0\})\Big)^o$. Then there exists $\zeta \in \boldsymbol{A}\sum_{i\in\mathcal{I}}(Z_i \cap \{V(\bar{p}) \geq 0\})$ such that $-\bar{q}\cdot\zeta > 0$. Thus, for every $n \in \mathbb{N}$, $n^2\zeta = \sum_{i\in\mathcal{I}} z_i^n$ for some $z_i^n \in Z_i \cap \{V(\bar{p}) \geq 0\}$. Therefore $-\bar{q}\cdot\sum_{i\in\mathcal{I}}(z_i^n/n) = -n\bar{q}\cdot\zeta \underset{n\to\infty}{\longrightarrow} +\infty$. Hence, without any loss of generality, one can assume that for some agent, say $i = 1$, $-\bar{q}\cdot(z_1^n/n) \underset{n\to\infty}{\longrightarrow} +\infty$. By **F0**, there exists $\xi_1 \in \boldsymbol{A}Z_1$ such that $V(\bar{p})\xi_1 \gg 0$. Define

$$\zeta_1^n := \frac{1}{n}z_1^n + (1 - \frac{1}{n})(\bar{z}_1 + \xi_1).$$

We end the proof by showing that **(i)** $\zeta_1^n \in Z_1$, and **(ii)** for n large enough, $W(\bar{p}, \bar{q})\zeta_1^n > W(\bar{p}, \bar{q})\bar{z}_1$. In other words, we show that ζ_1^n is an arbitrage opportunity for agent 1 at $\bar{z}_1$ in $\mathcal{F}$, which is a contradiction to the fact that $(\bar{q}, \bar{z})$ is arbitrage-free at $\bar{p}$ in $\mathcal{F}$. First, since $\xi_1 \in \boldsymbol{A}Z_1$, one has $\bar{z}_1 + \xi_1 \in Z_1$, and since $z_1^n \in Z_1$ and $\bar{z}_1 + \xi_1 \in Z_1$, the convexity of Z_1 (since $\mathcal{F}$ is standard) allows to conclude that ζ_1^n belongs to Z_1. Second, since $-\bar{q}\cdot(z_1^n/n) \underset{n\to\infty}{\longrightarrow} +\infty$, one has, for n large enough $-\bar{q}\cdot\zeta_1^n = -\bar{q}\cdot\frac{1}{n}z_1^n + -\bar{q}\cdot(1-\frac{1}{n})(\bar{z}_1 + \xi_1) > -\bar{q}\cdot\bar{z}_1$.

Finally, since $z_1^n \in \{V(\bar{p}) \geq 0\}$ and $V(\bar{p})\xi_1 \gg 0$, one has, for n large enough

$$V(\bar{p})\zeta_1^n = V(\bar{p})\Big(\frac{1}{n}z_1^n + (1 - \frac{1}{n})(\bar{z}_1 + \xi_1)\Big) \geq (1 - \frac{1}{n})V(\bar{p})(\bar{z}_1 + \xi_1) \gg V(\bar{p})\bar{z}_1.$$

Hence, for n large enough, $W(\bar{p}, \bar{q})\zeta_1^n > W(\bar{p}, \bar{q})\bar{z}_1$. This ends the proof of the claim. ∎

Lemma 4.2. *Assume that for all $s \in \bar{\mathcal{S}}$, $p(s) \neq 0$ and for all $i \in \mathcal{I}$, $e_i \in \mathrm{int}X_i$, then*

$$B^i_{\mathcal{F}_\pi}(p, q) = \mathrm{cl}\,\{(x, y) \in X_i \times \pi Z_i : p \,\Box\, (x - e_i) \ll W(p, q)y\}\,.$$

Proof. We first claim that there exists $\delta = (\delta(s))_{s\in\bar{\mathcal{S}}} \in \mathbb{R}^L$ such that **(i)** $e_i - \delta \in X_i$ and **(ii)** $p(s)\cdot\delta(s) > 0$ for every $s \in \bar{\mathcal{S}}$. Indeed, take $\delta = \lambda p$ for $\lambda > 0$ small enough so that $e_i - \delta \in X_i$, using the fact that $e_i \in \mathrm{int}X_i$. Then, for all $s \in \bar{\mathcal{S}}$, $p(s)\cdot\delta(s) = \lambda p(s)\cdot p(s) > 0$, since $p(s) \neq 0$. Let $(x_i, y_i) \in B^i_{\mathcal{F}_\pi}(p, q)$. Let $\alpha \in (0, 1)$. Then $x_i^\alpha := \alpha x_i + (1 - \alpha)(e_i - \delta) \in X_i$ since $x_i \in X_i, e_i - \delta \in X_i$ and X_i is convex, and $\alpha y_i \in \mathrm{cl}\pi Z_i$ since $0 \in \mathrm{cl}\pi Z_i$, $y_i \in \mathrm{cl}\pi Z_i$, and $\mathrm{cl}\pi Z_i$ is convex. We claim that,

$$p \,\Box\, (x_i^\alpha - e_i) - W(p, q)(\alpha y_i) \ll 0.$$

Indeed, $p \,\square\, (x_i^\alpha - e_i) - W(p,q)(\alpha y_i) = \alpha\Big(p \,\square\, (x_i - e_i) - W(p,q)y_i\Big) - (1-\alpha)p \,\square\, \delta$. Since $(x_i, y_i) \in B^i_{\mathcal{F}_\pi}(p,q)$, i.e., $p \,\square\, (x_i - e_i) - W(p,q)y_i \leq 0$, and $\alpha > 0$, the first term is nonpositive. Since $p \,\square\, \delta \gg 0$ (from above) and $\alpha < 1$, the second term satisfies $-(1-\alpha)p \,\square\, \delta \ll 0$. This ends the proof of the claim. Consequently, there exists $y_i^\alpha \in \pi Z_i$ such that $\|y_i^\alpha - y_i\| \leq (1-\alpha)\|y_i\|$ and

$$p \,\square\, (x_i^\alpha - e_i) - W(p,q)y_i^\alpha \ll 0.$$

Noticing that, $(x_i^\alpha, y_i^\alpha) \to (x_i, y_i)$ when $\alpha \to 1$, we get the desired result. ■

4.2. Proof of Theorem 2

We show that if $(\mathcal{E}, \mathcal{F})$ has an equilibrium (p^*, q^*, x^*, z^*), then $(p^*, \pi q^*, x^*, \pi z^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F}_\pi)$. Let (p^*, q^*, x^*, z^*) be an equilibrium in $(\mathcal{E}, \mathcal{F})$. The asset market clearing condition in $(\mathcal{E}, \mathcal{F}_\pi)$: $\sum_{i \in \mathcal{I}} \pi z_i^* = 0$ is a direct consequence of $\sum_{i \in \mathcal{I}} z_i^* = 0$. First, we show that for each $i \in \mathcal{I}$, $(x_i^*, \pi z_i^*) \in B^i_{\mathcal{F}_\pi}(p^*, \pi q^*)$. It suffices to show that $W(p^*, \pi q^*)\pi z_i^* = W(p^*, q^*)z_i^*$. From (3.1) we have $W(p^*, \pi q^*)\pi z_i^* = W(p^*, q^*)\pi z_i^*$. Recall that by Proposition 1, (q^*, z^*) is arbitrage-free at p^* in $\mathcal{F}$. Thus by Lemma 4.1, $q^* \in \mathcal{L}_{\mathcal{F}}^\perp$. Hence $q^* \in (\ker \pi)^\perp$ since $\ker \pi \subset \mathcal{L}_{\mathcal{F}}$. Therefore $q^* \cdot \pi z_i^* = q^* \cdot z_i^*$ and $W(p^*, q^*)\pi z_i^* = W(p^*, q^*)z_i^*$, recalling that $V(p^*)\pi z_i^* = V(p^*)z_i^*$ from (3.1). We now show that for each $i \in \mathcal{I}$, $(x_i^*, \pi z_i^*)$ solves agent i's problem in $(\mathcal{E}, \mathcal{F}_\pi)$. Suppose on the contrary that for some agent, say $i = 1$, there exists $(x_1, z_1) \in B^1_{\mathcal{F}_\pi}(p^*, \pi q^*)$ such that $x_1 \in P_1(x^*)$. Recall that by **LNS** one has $p^*(s) \neq 0$ for all $s \in \bar{\mathbb{S}}$. From the above Lemma 4.2, $(x_1, z_1) = \lim_n (x_1^n, \pi z_1^n)$ for some sequences $(x_1^n)_n \subset X_1$ and $(z_1^n)_n \subset Z_1$ such that

$$p^* \,\square\, (x_1^n - e_1) - W(p^*, \pi q^*)(\pi z_1^n) \leq 0.$$

We have $W(p^*, \pi q^*)(\pi z_1^n) = W(p^*, q^*)(\pi z_1^n) = W(p^*, q^*)(z_1^n)$ (the first equality comes from (3.1) and the second equality is a consequence of the fact that under Assumption **F0**, $q^* \in \mathcal{L}_{\mathcal{F}}^\perp$ by Lemma 4.1 and therefore $q^* \in (\ker \pi)^\perp$ since $\ker \pi \subset \mathcal{L}_{\mathcal{F}}$). Consequently (from above),

$$p^* \,\square\, (x_1^n - e_1) - W(p^*, q^*)z_1^n = p^* \,\square\, (x_1^n - e_1) - W(p^*, \pi q^*)(\pi z_1^n) \leq 0.$$

Hence $(x_1^n, z_1^n) \in B^1_{\mathcal{F}}(p^*, q^*)$. Recalling that $x_1 \in P_1(x^*)$, $x_1 = \lim_n x_1^n$ and using the fact that $P_1(x^*)$ is open, we deduce that for n large enough $x_1^n \in P_1(x^*)$. The two assertions $(x_1^n, z_1^n) \in B^1_{\mathcal{F}}(p^*, q^*)$ and $x_1 \in P_1(x^*)$ contradict the optimality of (x_1^*, z_1^*) in $(\mathcal{E}, \mathcal{F})$. ■

5. Final Remark

In Aouani and Cornet (2009), we show that when portfolio sets are polyhedra, projecting on a supplementary space to $\boldsymbol{L}_{\mathcal{F}}$ yields a reduced financial structure, that is, if π is a linear projection satisfying $\ker \pi = \boldsymbol{L}_{\mathcal{F}}$ then $\boldsymbol{L}_{\mathcal{F}_\pi} = \{0\}$. A legitimate question is to ask whether this result holds with "general" portfolio sets, and if not under which conditions − other than polyhedral portfolio sets − does it hold? The following proposition provides an answer to this question.

Proposition 6. *Let π be a linear projection of $\mathbb{R}^J$ such that $\ker \pi \subset \mathcal{L}_{\mathcal{F}}$, and consider the following assertions.*

(i) $\ker \pi = \mathcal{L}_{\mathcal{F}}$.

(i') $\ker \pi = \boldsymbol{L}_{\mathcal{F}}$.

(ii) $\mathcal{L}_{\mathcal{F}_\pi} = \{0\}$.

(ii') $\boldsymbol{L}_{\mathcal{F}_\pi} = \{0\}$.

(iii) $\forall i \in \mathcal{I}, \boldsymbol{A}\pi Z_i \cap -\boldsymbol{A}\pi Z_i \cap \ker V = \{0\}$.

Then the following hold:

(a) $(i) \Rightarrow (ii) \iff (ii') \Rightarrow (iii)$.

(b) $(i') \not\Rightarrow (ii)$.

(c) If the cones $\boldsymbol{A}Z_i \cap \ker V$ satisfy **WPSI** *then* $(i) \iff (i') \iff (ii) \iff (ii') \iff (iii)$.

Proof. (a) $[(i) \Rightarrow (ii)]$. This is Part (d) of Theorem 4.

$[(ii) \iff (ii')]$. This equivalence holds not only for $\mathcal{F}_\pi$ but also for all financial structures $\mathcal{F}$ satisfying Assumption **F1**. The implication $(ii) \Rightarrow (ii')$ is immediate since $\boldsymbol{L}_{\mathcal{F}} \subset \mathcal{L}_{\mathcal{F}}$. We show $(ii') \Rightarrow (ii)$, that is, "$\boldsymbol{L}_{\mathcal{F}} = \{0\} \Rightarrow \mathcal{L}_{\mathcal{F}} = \{0\}$". Assume $\boldsymbol{L}_{\mathcal{F}} = \{0\}$ and let $\zeta \in \mathcal{L}_{\mathcal{F}}$, then for every integer n, $n\zeta = \sum_{i \in \mathcal{I}} z_i^n$ for some $z_i^n \in Z_i \cap \{V \geq 0\}$, or equivalently $\zeta = \sum_{i \in \mathcal{I}} z_i^n/n$, and we notice that $z_i^n/n \in Z_i \cap \{V \geq 0\}$ (since Z_i is convex and contains 0). Consider now the set

$$K := \left\{(z_1, \ldots, z_I) \in \prod_{i \in \mathcal{I}} Z_i : \forall i \in \mathcal{I}, V z_i \geq 0, \sum_{i \in \mathcal{I}} z_i = \zeta\right\}.$$

We claim that the set K is compact. Indeed, K is obviously closed and we only need to show that it is bounded. To this end, we show that the asymptotic cone $\boldsymbol{A}K$ of K is equal to $\{0\}$ (see Rockafellar

(1997)). We have

$$\boldsymbol{A}K := \left\{(\xi_1, \dots, \xi_I) \in \prod_{i\in\mathcal{I}} \boldsymbol{A}Z_i : \forall i, V\xi_i \geq 0, \sum_{i\in\mathcal{I}} \xi_i = 0\right\}.$$

Hence, if $(\xi_1, \dots, \xi_I) \in \boldsymbol{A}K$, then from $V\xi_i \geq 0$ for every $i \in \mathcal{I}$ and $\sum_{i\in\mathcal{I}}\xi_i = 0$ we deduce that $\xi_1 = -\sum_{i\neq 1}\xi_i \in \sum_{i\in\mathcal{I}}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -\sum_{i\in\mathcal{I}}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \boldsymbol{L}_{\mathcal{F}} = \{0\}$. Therefore $\xi_1 = 0$ and similarly, $\xi_i = 0$ for every $i \in \mathcal{I}$. That is $\boldsymbol{A}K = \{0\}$. This ends the proof of the claim.

From the compactness of K one deduces that, without any loss of generality each sequence (z_i^n/n) converges to some $\zeta_i \in \boldsymbol{A}Z_i \cap \{V \geq 0\}$. Hence $\zeta = \sum_{i\in\mathcal{I}}\zeta_i \in \sum_{i\in\mathcal{I}}\boldsymbol{A}Z_i \cap \{V \geq 0\}$. Similarly we prove that $-\zeta \in \sum_{i\in\mathcal{I}}\boldsymbol{A}Z_i \cap \{V \geq 0\}$. Therefore $\zeta = 0$.

[$(ii') \Rightarrow (iii)$]. This is obvious since, for each $i \in \mathcal{I}$, we have $\boldsymbol{A}\pi Z_i \cap -\boldsymbol{A}\pi Z_i \cap \ker V \subset \boldsymbol{L}_{\mathcal{F}_\pi} = \{0\}$.

(b) [$(i') \not\Rightarrow (ii')$]. The following is an example of a financial structure where

$$\ker\pi = \boldsymbol{L}_{\mathcal{F}} \not\Rightarrow \boldsymbol{L}_{\mathcal{F}_\pi} = \{0\}.$$

Let $V = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, $I = 2$, and

$$\begin{aligned} Z_1 &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3 \in \mathbb{R} \text{ or } z_1 \leq 0, z_2 \geq z_1^2, z_3 \in \mathbb{R}\right\}, \\ Z_2 &= \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \leq 0, z_3 \in \mathbb{R} \text{ or } z_1 \leq 0, z_2 \leq -z_1^2, z_3 \in \mathbb{R}\right\}. \end{aligned}$$

It is easy to check that $\{V \geq 0\} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+$, $\boldsymbol{A}Z_1 = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$, $\boldsymbol{A}Z_2 = \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}$. Thus $\boldsymbol{A}Z_1 \cap \{V \geq 0\} = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$, $\boldsymbol{A}Z_2 \cap \{V \geq 0\} = \mathbb{R}_+ \times \mathbb{R}_- \times \mathbb{R}_+$, and $\sum_{i\in\mathcal{I}}(\boldsymbol{A}Z_i \cap \{V \geq 0\}) = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$. So, if $\ker\pi = \boldsymbol{L}_{\mathcal{F}} = \{0\} \times \mathbb{R} \times \{0\}$, then $\mathrm{Im}\pi = (\ker\pi)^\perp = \mathbb{R} \times \{0\} \times \mathbb{R}$ so that $\boldsymbol{A}\pi Z_1 = \boldsymbol{A}\pi Z_2 = \mathbb{R} \times \{0\} \times \mathbb{R}$, and $\boldsymbol{A}\pi Z_1 \cap \{V \geq 0\} = \boldsymbol{A}\pi Z_2 \cap \{V \geq 0\} = \mathbb{R} \times \{0\} \times \mathbb{R}_+$. Hence $\boldsymbol{L}_{\mathcal{F}_\pi} = \mathbb{R} \times \{0\} \times \{0\} \neq \{0\}$.

(c) We need only show that, under **WPSI**, (i) is equivalent to (i') and (iii) implies (i).

[$(i) \iff (i')$]. Follows from Theorem 9.1 page 73 in Rockafellar (1997).

[$(iii) \Rightarrow (i)$ When the cones $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ satisfy **WPSI**]. Since $\ker\pi \subset \mathcal{L}_{\mathcal{F}}$, we only need to show the reverse inclusion. Let $\zeta \in \mathcal{L}_{\mathcal{F}}$. By Theorem 9.1 page 73 in Rockafellar (1997), if the cones $\boldsymbol{A}Z_i \cap \{V \geq 0\}$ are weakly positively semi-independent, then $\mathcal{L}_{\mathcal{F}} = \sum_{i\in\mathcal{I}}\Big((\boldsymbol{A}Z_i \cap \{V \geq 0\}) \cap -(\boldsymbol{A}Z_i \cap \{V \geq 0\})\Big)$. Hence $\zeta = \sum_{i\in\mathcal{I}}\zeta_i$ with $\zeta_i \in \boldsymbol{A}Z_i \cap -\boldsymbol{A}Z_i \cap \ker V$ for each $i \in \mathcal{I}$. Thus $\pi\zeta = \sum_{i\in\mathcal{I}}\pi\zeta_i$ and for all $i \in \mathcal{I}$,

$$\pi\zeta_i \in \pi(\boldsymbol{A}Z_i \cap -\boldsymbol{A}Z_i \cap \ker V) \subset \boldsymbol{A}\pi Z_i \cap -\boldsymbol{A}\pi Z_i \cap \ker V.$$

Note that for the inclusion we used the following fact: **(i)** $\pi(\ker V) \subset \ker V$, and **(ii)** $\pi \boldsymbol{A} Z_i \subset \boldsymbol{A}\pi Z_i$. Recall that, by assumption (iii), $\boldsymbol{A}\pi Z_i \cap -\boldsymbol{A}\pi Z_i \cap \ker V = \{0\}$ for every $i \in \mathcal{I}$. Hence $\pi\zeta_i = 0$ for all i and consequently $\pi\zeta = 0$, that is, $\zeta \in \ker \pi$. ■

6. Appendix

6.1. Proof of Proposition 1

(i) By contradiction, assume that for some $i \in \mathcal{I}$, there exists $z_i \in Z_i$ such that $W(\bar{p}, \bar{q})z_i > W(\bar{p}, \bar{q})\bar{z}_i$, namely $[W(\bar{p}, \bar{q})z_i](s) \geq [W(\bar{p}, \bar{q})\bar{z}_i](s)$, for every $s \in \bar{\mathcal{S}}$, with at least one strict inequality, say for $\bar{s} \in \bar{S}$. Then, since $\sum_{i \in \mathcal{I}}(\bar{x}_i - e_i) = 0$, from Assumption **LNS**, there exists $x \in \prod_{i \in I} X_i$ such that $x_i(-\bar{s}) = \bar{x}_i(-\bar{s})$ and $x_i \in P_i(\bar{x})$. Consider $\lambda \in (0, 1)$ and define $x_i^\lambda := \lambda x_i + (1 - \lambda)\bar{x}_i$. Then, by Assumption **LNS**, $x_i^\lambda \in (x_i, \bar{x}_i) \subset P_i(\bar{x})$. Now, we claim that for $\lambda > 0$ small enough, $(x_i^\lambda, z_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, which contradicts the fact that $\big(P_i(\bar{x}) \times Z_i\big) \cap B^i_{\mathcal{F}}(\bar{p}, \bar{q}) = \emptyset$ (since $(\bar{p}, \bar{q}, \bar{x}, \bar{z})$ is an equilibrium). Indeed, since $(\bar{x}_i, \bar{z}_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$, and $x_i^\lambda(-\bar{s}) = \bar{x}_i(-\bar{s})$ we have: for all $s \neq \bar{s}$,

$$\bar{p}(s) \cdot [x_i^\lambda(s) - e_i(s)] = \bar{p}(s) \cdot [\bar{x}_i(s) - e_i(s)] \leq [W(\bar{p}, \bar{q})\bar{z}_i](s) \leq [W(\bar{p}, \bar{q})z_i](s).$$

Now, for $s = \bar{s}$, we have

$$\bar{p}(\bar{s}) \cdot [\bar{x}_i(\bar{s}) - e_i(\bar{s})] \leq [W(\bar{p}, \bar{q})\bar{z}_i](\bar{s}) < [W(\bar{p}, \bar{q})z_i](\bar{s}).$$

Note that $x_i^\lambda \to \bar{x}_i$ when $\lambda \to 0$. Hence for $\lambda > 0$ small enough we have $\bar{p}(\bar{s}) \cdot [x_i^\lambda(\bar{s}) - e_i(\bar{s})] < [W(\bar{p}, \bar{q})z_i](\bar{s})$. Consequently, $(x_i^\lambda, z_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$.

(ii) Suppose that for some $i \in \mathcal{I}$, there exists a portfolio $\zeta_i \in \boldsymbol{A}Z_i$ such that $W(\bar{p}, \bar{q})\zeta_i > 0$, namely $[W(\bar{p}, \bar{q})\zeta_i](s) \geq 0$, for every $s \in \bar{\mathcal{S}}$, with at least one strict inequality, say for $\bar{s} \in \bar{\mathcal{S}}$. From Assumption **LNS**, there exists $x_i \in P_i(\bar{x})$ such that $x_i(-\bar{s}) = \bar{x}_i(-\bar{s})$. For $t > 0$ large enough, $\bar{p} \,\square\, (x_i - e_i) \leq W(\bar{p}, \bar{q})(\bar{z}_i + t\ \zeta_i)$. Since $\bar{z}_i + t\ \zeta_i \in Z_i$, we get $(x_i, \bar{z}_i + t\ \zeta_i) \in B^i_{\mathcal{F}}(\bar{p}, \bar{q})$ but since $x_i \in P_i(\bar{x})$, this contradicts the optimality of $(\bar{x}_i, \bar{z}_i)$ in $B^i_{\mathcal{F}}(\bar{p}, \bar{q})$. ■

6.2. Proof of Proposition 2

Note that assertions (a)-(e) are special cases of (f). Hence, we will prove only (f). First, we prove the result when for every $i \in \mathcal{I}$, $K_i = \{0\}$, i.e. when Z_i is polyhedral for every i. Let

$$f : \mathbb{R}^{JI} \to \mathbb{R}^{SI} \times \mathbb{R}^J, (z_1, \cdots, z_I) \mapsto (V(p)z_1, \cdots, V(p)z_I, \sum_{i \in \mathcal{I}} z_i).$$

Then f is linear and one has $\mathcal{G}_{\mathcal{F}}(p) = f\Big(\prod_i Z_i\Big)$. Since $\prod_i Z_i$ is polyhedral, Theorem 19.3 page 174 in Rockafellar (1997) allows to conclude that $\mathcal{G}_{\mathcal{F}}(p)$ is polyhedral, hence closed.

Now, we show the result in the general case. Let $(V(p)z_1^n, \cdots, V(p)z_I^n, \sum_{i\in\mathcal{I}} z_i^n)$ be a sequence in the set $\mathcal{G}_{\mathcal{F}}(p)$ such that $(V(p)z_1^n, \cdots, V(p)z_I^n, \sum_{i\in\mathcal{I}} z_i^n) \underset{n\to\infty}{\longrightarrow} (v_1, \cdots, v_I, y)$, where $(z_i^n)_n \subset Z_i$ for every $i \in \mathcal{I}$. By assumption, for all i and for all n, $z_i^n = k_i^n + \beta_i^n$ where $k_i^n \in K_i$ and $\beta_i^n \in P_i$. Since the K_i's are compact, we can assume $k_i^n \underset{n\to\infty}{\longrightarrow} k_i \in K_i$ for all $i \in \mathcal{I}$. The sequence $\Big((V(p)z_i^n - V(p)k_i^n)_i, \sum_{i\in\mathcal{I}} z_i^n - \sum_{i\in\mathcal{I}} k_i^n\Big)_n$ is in the set $\mathcal{G}_{\mathcal{F}'}(p)$, where $\mathcal{F}' = \Big(V, (P_i)_i\Big)$. Since, by the first part of the proof, $\mathcal{G}_{\mathcal{F}'}(p)$ is closed, one has

$$\Big((v_i - V(p)k_i)_i, y - \sum_{i\in\mathcal{I}} k_i\Big) = \lim_n \Big((V(p)z_i^n - V(p)k_i^n)_i, \sum_{i\in\mathcal{I}} z_i^n - \sum_{i\in\mathcal{I}} k_i^n\Big) \in \mathcal{G}_{\mathcal{F}'}(p).$$

Hence, for all i there exists $\beta_i \in P_i$ such that $V(p)\beta_i = v_i - V(p)k_i$ and $y - \sum_{i\in\mathcal{I}} k_i = \sum_{i\in\mathcal{I}} \beta_i$. Therefore $\Big((v_i)_i, y\Big) = \Big((V(p)(k_i + \beta_i))_i, \sum_{i\in\mathcal{I}}(k_i + \beta_i)\Big)$ with $k_i + \beta_i \in Z_i$ for each i. That is, $\Big((v_i)_i, y\Big) \in \mathcal{G}_{\mathcal{F}}(p)$. ∎

6.3. Proof of Proposition 3

Note that assertions (g)-$(k1)$ are special cases of $(k2)$. Hence, we will prove only $(k2)$. We show that if the sets $\boldsymbol{A}Z_i \cap \ker V(p)$ are **WPSI** then the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed. We have

$$\mathcal{G}_{\mathcal{F}}(p) = \left\{(V(p)z_1, \cdots, V(p)z_I, \sum_{i\in\mathcal{I}} z_i) : \forall i, z_i \in Z_i\right\} = \sum_{i\in\mathcal{I}} X_i$$

with

$$X_i = \{(0, \cdots, 0, V(p)z_i, 0, \cdots, 0, z_i) : z_i \in Z_i\}.$$

Then $\boldsymbol{A}X_i = \{(0, \cdots, 0, V(p)\zeta_i, 0, \cdots, 0, \zeta_i) : \zeta_i \in \boldsymbol{A}Z_i\}$. Now we show that the sets $\boldsymbol{A}X_i$ $(i \in \mathcal{I})$ are weakly positively semi-independent. This will end the proof (see Theorem 9.1 page 73 in Rockafellar (1997)). If $\sum_{i\in\mathcal{I}} w_i = \sum_{i\in\mathcal{I}}(0, \cdots, 0, V(p)\zeta_i, 0, \cdots, 0, \zeta_i) = 0$ with $\zeta_i \in \boldsymbol{A}Z_i$, then for every i, $V(p)\zeta_i = 0, \zeta_i \in \boldsymbol{A}Z_i$, and $\sum_{i\in\mathcal{I}} \zeta_i = 0$. Hence for each i, $\zeta_i \in \boldsymbol{A}Z_i \cap \ker V(p)$ and $\sum_{i\in\mathcal{I}} \zeta_i = 0$. By WPSI of the sets $\boldsymbol{A}Z_i \cap \ker V(p)$, we get $\zeta_i \in \boldsymbol{A}Z_i \cap -\boldsymbol{A}Z_i$ for each i. Hence $w_i \in \boldsymbol{A}X_i \cap -\boldsymbol{A}X_i$ for each $i \in \mathcal{I}$. ∎

6.4. The example of Section 2.6.1

Clearly $\boldsymbol{A}Z_2 = Z_2$ and it is easy to check that $\boldsymbol{A}Z_1 = \Big\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \geq 0, z_2 \geq 0, z_3^2 \leq z_1 z_2\Big\}$ and $\{V \geq 0\} = \{0\} \times \mathbb{R} \times \mathbb{R}$. Hence $\boldsymbol{A}Z_1 \cap \{V \geq 0\} = \{0\} \times \mathbb{R}_+ \times \{0\}$, and $\boldsymbol{A}Z_2 \cap \{V \geq 0\} = \{0\} \times \mathbb{R}_- \times$

$\{0\}$. Then the collection $\{\boldsymbol{A}Z_i \cap \{V \geq 0\} : i \in 1,2\}$ is not **WPSI**. We show that $\mathcal{F} = \Big(V, (Z_i)_{i=1,2}\Big)$ satisfies the Closedness Assumption. Let $(z_i^n)_n$ be a sequence in Z_i ($i = 1,2$), and assume that $(Vz_1^n, Vz_2^n, z_1^n + z_2^n) \underset{n\to\infty}{\longrightarrow} (v_1, v_2, z)$. We need to show that there exist $(z_1, z_2) \in Z_1 \times Z_2$ such that $(v_1, v_2, z) = (Vz_1, Vz_2, z_1 + z_2)$. Write $z_1^n = (\alpha_1^n, \beta_1^n, \gamma_1^n)$ and $z_2^n = (\alpha_2^n, \beta_2^n, \gamma_2^n)$ with

$$\begin{cases} \alpha_1^n \geq 0,\ \beta_1^n \geq 0,\ (\gamma_1^n)^2 \leq (\alpha_1^n + 1)\beta_1^n, \\ \alpha_2^n \geq 0,\ \beta_2^n \leq 0,\ \text{and } \gamma_2^n = 0. \end{cases}$$

Then $Vz_1^n = (\alpha_1^n, -\alpha_1^n)$, $Vz_2^n = (\alpha_2^n, -\alpha_2^n)$, and $z_1^n + z_2^n = (\alpha_1^n + \alpha_2^n, \beta_1^n + \beta_2^n, \gamma_1^n + \gamma_2^n)$. Since the sequences $(Vz_i^n)_n$ ($i = 1,2$) converge, we have $\alpha_1^n \underset{n\to\infty}{\longrightarrow} \alpha_1 \geq 0$, $\alpha_2^n \underset{n\to\infty}{\longrightarrow} \alpha_2 \geq 0$, and from the convergence of the sequence $(\gamma_1^n + \gamma_2^n)_n$ and the fact that $\gamma_2^n = 0$ we conclude that $\gamma_1^n \underset{n\to\infty}{\longrightarrow} \gamma_1$. Denote $s = \lim_n(\beta_1^n + \beta_2^n)$. Choose $\beta_1 \geq \max\Big(s, (\gamma_1)^2/(\alpha_1 + 1)\Big)$, and let $\beta_2 = s - \beta_1$. Then one can easily check that $z_1 := (\alpha_1, \beta_1, \gamma_1) \in Z_1$, $z_2 := (\alpha_2, \beta_2, 0) \in Z_2$, and $v_1 = Vz_1, v_2 = Vz_2, z = z_1 + z_2$. ∎

6.5. Proof of Proposition 4

Assume $\mathcal{G}'_{\mathcal{F}}(p)$ is closed and let $(w^n)_n$ be a sequence in $\mathcal{G}_{\mathcal{F}}(p)$ which converges to some $w \in (\mathbb{R}^S)^I \times \mathbb{R}^J$ i.e. $w^n = (V(p)z_1^n, \cdots, V(p)z_I^n, \sum_{i\in\mathcal{I}} z_i^n) \underset{n\to\infty}{\longrightarrow} w = (v_1, \cdots, v_I, z)$, with $z_i^n \in Z_i$ for each $i \in \mathcal{I}$ and for every $n \in \mathbb{N}$. Then $w^n \in \mathcal{G}'_{\mathcal{F}}(p)$ for every n, and since $\mathcal{G}'_{\mathcal{F}}(p)$ is closed, we have $w \in \mathcal{G}'_{\mathcal{F}}(p)$. That is $z = \sum_{i\in\mathcal{I}} z_i$ with $z_i \in Z_i$ and $V(p)z_i \geq v_i$ for every $i \in \mathcal{I}$. But $\sum_{i\in\mathcal{I}} v_i = \sum_{i\in\mathcal{I}} \lim_n V(p)z_i^n = \lim_n V(p)(\sum_{i\in\mathcal{I}} z_i^n) = V(p)z = V(p)(\sum_{i\in\mathcal{I}} z_i) = \sum_{i\in\mathcal{I}} V(p)z_i$, hence $v_i = V(p)z_i$ for each $i \in I$, and consequently, $w = (V(p)z_1, \cdots, V(p)z_i, \sum_{i\in\mathcal{I}} z_i) \in \mathcal{G}_{\mathcal{F}}(p)$.

Conversely, assume $\mathcal{G}_{\mathcal{F}}(p)$ closed and let $(w'^n)_n$ be a sequence in $\mathcal{G}'_{\mathcal{F}}(p)$ which converges to some $w' \in (\mathbb{R}^S)^I \times \mathbb{R}^J$ i.e. $w'^n = (v_1'^n, \cdots, v_I'^n, \sum_{i\in\mathcal{I}} z_i^n) \underset{n\to\infty}{\longrightarrow} w' = (v_1', \cdots, v_I', z)$, with $z_i^n \in Z_i$ and $V(p)z_i^n \geq v_i'^n$ for each $i \in \mathcal{I}$ and for every $n \in \mathbb{N}$. For each $i \in \mathcal{I}$, the sequence $(v_i'^n)_n$ converges hence is bounded, therefore the sequence $(V(p)z_i^n)_n$ is bounded below (since $V(p)z_i^n \geq v_i'^n$ for every n). Moreover the sequence $(\sum_{i\in\mathcal{I}} V(p)z_i^n)_n$ converges (towards $V(p)z$), hence for each $i \in \mathcal{I}$, the sequence $(V(p)z_i^n)_n$ is bounded. We can therefore assume that for each $i \in \mathcal{I}$, the sequence $(V(p)z_i^n)_n$ converges (use subsequences if necessary) to $v_i \in \mathbb{R}^S$ satisfying $v_i \geq v_i'$. Now we consider the sequence $(w^n)_n \subset \mathcal{G}_{\mathcal{F}}(p)$ where $w^n = (V(p)z_1^n, \cdots, V(p)z_I^n, \sum_{i\in\mathcal{I}} z_i^n)$. Then from above, $w^n \underset{n\to\infty}{\longrightarrow} w = (v_1, \cdots, v_I, z) \in \mathcal{G}_{\mathcal{F}}(p)$ (since $\mathcal{G}_{\mathcal{F}}(p)$ is assumed to be closed). Hence z can be written as $z = \sum_{i\in\mathcal{I}} z_i$ with $z_i \in Z_i$ and $V(p)z_i = v_i$ for each $i \in \mathcal{I}$. Recall that $V(p)z_i = v_i \geq v_i'$ for each $i \in \mathcal{I}$ and that $w' = w' = (v_1', \cdots, v_I', z) = (v_1', \cdots, v_I', \sum_{i\in\mathcal{I}} z_i)$, hence $w' \in \mathcal{G}'_{\mathcal{F}}(p)$. ∎

6.6. Proof of Proposition 5

We show that for all $p \in \mathbb{R}^L$ and for all $v = (v_i)_{i \in I} \in (\mathbb{R}^S)^I$ and $w = (w_i)_{i \in I} \in (\mathbb{R}^S)^I$ such that the sets $\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})$ and $\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq w_i\})$ are not empty, we have

$\boldsymbol{A}\Big(\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\Big) \subset \boldsymbol{A}\Big(\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq w_i\})\Big)$.

The result of the proposition is obviously a direct consequence of the above inclusion.

Let $\zeta \in \boldsymbol{A}\Big(\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq v_i\})\Big)$, then $\zeta = \lim_{n \to \infty} \lambda_n \sum_{i \in \mathcal{I}} z_i^n$ for some $z_i^n \in Z_i \cap \{V(p) \geq v_i\}$, $\lambda_n > 0$, and $\lambda_n \downarrow 0$. We need to show that $\zeta \in \boldsymbol{A}\Big(\sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq w_i\})\Big)$, that is, for $z_i \in Z_i$ ($i \in \mathcal{I}$) such that $V(p)z_i \geq w_i$, we have

$$\zeta + \sum_{i \in \mathcal{I}} z_i \in \sum_{i \in \mathcal{I}}(Z_i \cap \{V(p) \geq w_i\}).$$

From above,

$$\zeta + \sum_{i \in \mathcal{I}} z_i = \lim_{n \to \infty} \sum_{i \in \mathcal{I}} \Big(\lambda_n z_i^n + (1 - \lambda_n) z_i\Big).$$

Notice that, for n large enough, $\lambda_n \in [0, 1]$. Hence $y_i^n := \lambda_n z_i^n + (1 - \lambda_n) z_i$ belongs to Z_i (because z_i^n and z_i are in Z_i, and Z_i is convex). Furthermore $V(p) y_i^n \geq \lambda_n v_i + (1 - \lambda_n) w_i$. Therefore the sequence $(V(p) y_i^n)_n$ is bounded below. Moreover the sequence $(\sum_{i \in \mathcal{I}} V(p) y_i^n)_n$ converges (towards $V(p)(\zeta + \sum_{i \in \mathcal{I}} z_i)$), hence for each $i \in \mathcal{I}$, the sequence $(V(p) y_i^n)_n$ is bounded. We can henceforth assume that for each $i \in \mathcal{I}$, the sequence $(V(p) y_i^n)_n$ converges (use subsequences if necessary) to $v_i' \in \mathbb{R}^S$ satisfying $v_i' \geq w_i$. Since

$$\Big(V(p) y_1^n, \cdots, V(p) y_I^n, \sum_{i \in \mathcal{I}} y_i^n\Big) \in \mathcal{G}_{\mathcal{F}}(p),$$

$\Big(V(p) y_1^n, \cdots, V(p) y_I^n, \sum_{i \in \mathcal{I}} y_i^n\Big) \underset{n \to \infty}{\longrightarrow} (v_1', \cdots, v_I', \zeta + \sum_{i \in \mathcal{I}} z_i)$, and the set $\mathcal{G}_{\mathcal{F}}(p)$ is closed (by the Closedness Assumption), we conclude that there exists $(y_i)_{i \in \mathcal{I}} \in \prod_i Z_i$ such that $(v_1', \cdots, v_I', \zeta + \sum_{i \in \mathcal{I}} z_i) = (V(p) y_1, \cdots, V(p) y_I, \sum_{i \in \mathcal{I}} y_i)$. Hence, recalling that from above, $v_i' \geq w_i$, we have $\zeta + \sum_{i \in \mathcal{I}} z_i = \sum_{i \in \mathcal{I}} y_i \in \sum_{i \in \mathcal{I}} Z_i \cap \{V(p) \geq w_i\}$. ■

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