Competitive Outcomes, Endogenous Firm Formation and the Aspiration Core

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Abstract

The paper shows that the aspiration core allocations of any TU-game coincides with the set of competitive wages of a corresponding production economy. The set of firms that are active in the market is endogenously determined in equilibrium and it coincides with the generating collection of the corresponding aspiration core allocation.

Keywords: Endogenous firm formation, Aspiration core, Lottery equilibrium

1 Introduction

One of the main results regarding the decentralization of core vectors via Walrasian prices is given in Shapley and Shubik (1975). Starting from an arbitrary TU-game, the authors construct a particular pure-exchange economy, which they call a "direct market" (Shapley and Shubik 1969) and prove that its Walrasian equilibrium allocations coincide with the core allocations of the original game. This paper generalizes Shapley and Shubik's (1975) findings to arbitrary TU-games. We show that vectors in the aspiration core

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(Bennett 1983), a non-empty core extension, coincide with the equilibrium wages of a modified direct *production* economy.

An interesting feature of our model is that the price mechanism determines, endogenously not only the allocations but also the set of active coalitions. To make the coalition formation process explicit, we define direct production economies in which agents can form "coalitional" firms, following the description of Shapley and Shubik (1969).[1] We construct two such economies. In each of them agents are endowed with one unit of productive time which they sell in exchange for consumption. In the first production economy time is divisible and agents can spend it working for various firms. This corresponds to situations in which production activities need not take place at the same time. In the second model time is indivisible. Due to the inherent non-convexity introduced by indivisibilities, such economies do not always have a Walrasian equilibrium. We show, however that if agents and firms are allowed to trade lottery contracts that specify a positive probability of unemployment, a Walrasian equilibrium always exists and the total welfare is improved. The equilibrium allocations of these direct production economies are in a one-to-one and onto relation with the aspiration core vectors, while the set of active firms coincides with the family of coalitions that make such vectors feasible (i.e., coalitions that lie in their generating collection).

Most cooperative solution concepts do not simultaneously address the allocation and coalition formation problems. The core, for example, exogenously dictates that the grand coalition forms. Zhou (1994) moves a step forward by defining a new type of bargaining set that does not assume any particular coalition structure. On the down side, Zhou's (1994) bargaining set cannot be decentralized using a market economy (Anderson, Trockel, and Zhou 1997). The aspiration core, proposed by Bennett (1983) extends the core to arbitrary TU-games and does not assume the formation of the grand coalition. Bejan and Gómez (2009) propose a new interpretation of the aspiration core as the set of stable allocations that can be supported by an *overlapping* coalition structure, in which a player can be part of several non-disjoint coalitions. This paper provides the aspiration core with the link to competitive equilibrium that Zhou's (1994) bargaining set is missing, and uses competitive production economies to endogenously explain coalition formation.

Sun, Trockel, and Yang (2008) also tackle endogenous allocation and coalition formation using competitive outcomes of an associated coalition

[1] See section 6.1, "A Production Model."

production economy. The TU-game solution concept they use is the c-core (i.e., the core of the super-additive completion of the original game). Their coalition production economy is similar to our indivisible-labor economy, but they do not allow for trade in lotteries. Clearly, a degenerate lottery equilibrium in our framework, when it exists, implements the c-core allocations and thus our results extend theirs to arbitrary games. Garratt and Qin (1997) construct a direct pure-exchange lottery economy whose equilibrium allocations, when they exits, implement the core of the original game. As opposed to their results, our lottery production economy always has an equilibrium and the equilibrium implements the core of the corresponding game whenever that is non-empty, and the aspiration core otherwise.

The paper is organized as follows. After introducing the notation and definitions in Section 2, Section 3 shows how to construct a direct production economy with divisible labor associated to any TU-game. Our main results are in Sections 4 and 5, where we show how the aspiration core allocations can be supported as competitive equilibrium outcomes of both the divisible-good economy and the lottery economy. Section 6 concludes.

2 Definitions and Notation

Let N be a finite set of players, and $\mathcal{N}$ the collection of all non-empty subsets of N. Let Δ_N (respectively $\Delta_{\mathcal{N}}$) be the unit simplex in $\mathbb{R}^N$ (respectively $\mathbb{R}^{\mathcal{N}}$), and $e_i \in \Delta_N$ (resp. $e_S \in \Delta_{\mathcal{N}}$) the vertex corresponding to $i \in N$ (resp. $S \in \mathcal{N}$). For every $S \in \mathcal{N}$, let $\mathbf{1}_S \in \{0,1\}^N$ denote the indicator function of S. A *TU-game* (or simply a *game*) on N is a mapping $v : 2^N \to \mathbb{R}_+$ such that $v(\emptyset) = 0$. For any $S \subseteq N$, $v(S)$ is called the *worth of coalition* S. The restriction of a game v to $S \subseteq N$, is the game $v_{|S}$ on S with $v_{|S}(T) := v(T)$ for all $T \subseteq S$.

Given a game v on N, a possible outcome is represented by a *payoff vector* $x \in \mathbb{R}^N$ that assigns to every $i \in N$ a payoff x_i. Given $x \in \mathbb{R}^N$ and $S \subseteq N$, let $x(S) := \sum_{i \in S} x_i$, with the agreement that $x(\emptyset) = 0$. A payoff vector $x \in \mathbb{R}^N$ is *feasible* for coalition S if $x(S) \leq v(S)$. It is *individually feasible* if for every $i \in N$, there exists $S \subseteq N$ with $i \in S$ such that x is feasible for S. We say that coalition S is able to *improve upon* the outcome $x \in \mathbb{R}^N$ if $x(S) < v(S)$. A vector $x \in \mathbb{R}^N$ is *stable* if it cannot be improved upon by any coalition. The *core* of v, denoted $\mathcal{C}(v)$, is the set of stable outcomes that are feasible for N, i.e.,

$$\mathcal{C}(v) := \{x \in \mathbb{R}^N \mid x(S) \geq v(S)\ \forall S \subseteq N,\ x(N) = v(N)\}.$$

A stable payoff vector $x \in \mathbb{R}^N$ that is individually feasible is called an *aspiration*. We denote by $\mathcal{A}sp(v)$ the set of aspirations of game v. It is known that for any game v, $\mathcal{A}sp(v)$ is a non-empty, compact and connected set (Bennett and Zame 1988). The *generating collection* of an aspiration x is the family of coalitions S that can attain x, i.e.,

$$\mathcal{GC}(x) := \{S \in \mathcal{N} \mid x(S) = v(S)\}.$$

A collection of coalitions $\mathcal{B} \subseteq \mathcal{N}$ is called *balanced* if there exist non-negative numbers $(\lambda_S)_{S\in\mathcal{B}}$ such that for every $i \in N$, $\sum_{S\ni i} \lambda_S = 1$. The numbers λ_S are called *balancing weights*. A game v on N is called *balanced* if $\sum_{S\in\mathcal{B}} \lambda_S v(S) \leq v(N)$ for every balanced family $\mathcal{B}$ with balancing weights $(\lambda_S)_{S\in\mathcal{B}}$. (Bondareva 1963) and (Shapley 1953) showed that v is balanced if and only if $\mathcal{C}(v) \neq \emptyset$. A game v is called *totally balanced* if $v_{|S}$ is balanced for every $S \subseteq N$. For every game v, let $\bar{v}$ denote the least totally balanced set function that is greater or equal to v. The game $\bar{v}$ is called the *totally balanced cover* of v.

The *aspiration core* (Bennett 1983) of a game v, denoted $\mathcal{AC}(v)$, is the set of those aspiration $x \in \mathcal{A}sp(v)$ for which $\mathcal{GC}(x)$ is balanced. It is known that $\mathcal{AC}(v) = \mathcal{C}(v)$ if and only if v is balanced and $\mathcal{AC}(v) = \mathcal{C}(\bar{v})$ for every game v.

3 Production Economy Representation of a Game

Let L be a finite set of goods, I a finite set of consumers and J a finite set of firms. A *private-ownership production economy* or simply an *economy* is

$$E := ((X_i, u_i, \omega_i)_{i\in I}, (Y^j)_{j\in J}, (\theta_i^j)_{i\in I, j\in J})$$

where for every $i \in I$, $X_i \subseteq \mathbb{R}_+^L$ denotes agent i's consumption set, $u_i : X_i \to \mathbb{R}$ his utility function and $\omega_i \in X_i$ his endowments of goods, and for every $j \in J$, $Y^j \subseteq \mathbb{R}^L$ denotes firm j's production set and $\theta^j \in \mathbb{R}_+^I$ its distribution of shares across consumers, with $\sum_{i\in I} \theta_i^j = 1$.

We restrict attention to production economies in which consumers' utilities are continuous, quasi-linear (in the same good) and quasi-concave, and production sets are closed and convex cones. In the spirit of (Shapley and Shubik 1969), we are going to associate a TU-game to every private-ownership production economy and reciprocally, construct a private-ownership production economy (called a direct production economy) from any TU-game.

A *consumption allocation* for E is any $x \in \prod_{i\in I} X_i$. It is called *feasible for coalition* S if for every firm j there exists $y^j \in Y^j$ such that

$$\sum_{i\in S} x_i = \sum_{i\in S} \omega_i + \sum_{j\in J} \left(\sum_{i\in S} \theta_i^j \right) y^j. \tag{1}$$

We denote by $\mathcal{F}(S)$ the set of feasible consumption allocations for coalition S.[2]

Definition 3.1 *Given an economy E, define the TU-game $\mathcal{V}(E)$ on I by*

$$\mathcal{V}(E)(S) := \max \left\{ \sum_{i\in S} u_i(x_i) \mid x \in \mathcal{F}(S) \right\}, \tag{2}$$

for every $S \subseteq I$. A TU-game v is called a production market game if there exists a private-ownership production economy E such that $v = \mathcal{V}(E)$.

Note that, as in the case of (Shapley and Shubik 1969), there could be many economies E that generate the same production market game v.

Definition 3.2 *Given a game v on N, we define its direct production economy as the private-ownership production economy*

$$\mathcal{E}(v) = \left(\{L_i \mid i \in N\} \cup \{C\}, N, \mathcal{N}, (X_i, u_i, \omega_i)_{i\in N}, (Y^S)_{S\in\mathcal{N}}, (\theta_i^S)_{(i,S)} \right), \tag{3}$$

where each consumer $i \in I$ has a consumption set $X_i = \mathbb{R}_+^{|N|+1}$, a utility function such that $u_i(l_1, \ldots, l_N, c) = c$, and an endowment $\omega_i = (e_i, 0) \in \mathbb{R}_+^{|N|+1}$. Each firm $S \in \mathcal{N}$ has a production set

$$Y^S := \left\{ (l_1, ..., l_N, c) \in -\mathbb{R}_+^N \times \mathbb{R}_+ \mid l_i = 0 \text{ if } i \notin S, c \leq \min_{i\in S} |l_i| \cdot v(S) \right\}$$

and shares $\theta_i^S = \frac{1}{|S|} \cdot \mathbf{1}_S(i)$.

Thus, the economy $\mathcal{E}(v)$ has $|N|$ consumers, $2^{|N|} - 1$ firms and $|N| + 1$ commodities. The last commodity, denoted C, is a consumption good; the other $|N|$ commodities, denoted $L_1, ..., L_{|N|}$, represent agent-specific human capital (or skilled labor). Each consumer i cares only about the amount of good C he consumes and is endowed with one unit of human capital L_i.

[2] A similar construction is used by Hildenbrand (1974, page 228) to transform a private-ownership economy into a coalition production economy.

Firms are indexed by $S \subseteq N$ and each firm S uses human capital (skilled labor) $(L_i)_{i\in S}$ to produce the consumption good. A similar construction is used by Sun, Trockel, and Yang (2008) to associate to every TU-game a *coalition* production economy.

The following propositions are analogous to Shapley and Shubik's (1969) results within our production economy framework.

Proposition 3.3 *For any game v, $\mathcal{V}(\mathcal{E}(v)) = \bar{v}$.*

Proof. Fix a game v on N. By definition, for any $S \subseteq N$,

$$\mathcal{V}(\mathcal{E}(v))(S) = \max \sum_{i\in S} c_i$$

subject to the existence of production plans $(l^T, c^T) \in Y^T$, one for every firm T, satisfying

$$\sum_{i\in S}(0, c_i) = \sum_{i\in S}(e_i, 0) + \sum_{T\in\mathcal{N}} \frac{|T \cap S|}{|T|} \left(l_1^T, \ldots, l_N^T, (\min_{i\in T} |l_i^T|) \cdot v(T) \right).$$

To be active, a firm T requires a positive input of every good $(L_i)_{i\in T}$. Since coalition S does not have an endowment of goods L_i with $i \notin S$, coalition S can only operate firms corresponding to $T \subseteq S$. Without loss of generality we can assume $l_i^T = l_k^T = \overline{l^T}$ for any $i, k \in T$. Thus, the feasibility condition for the consumption commodity is reduced to

$$\sum_{i\in S} c_i = \sum_{T\subseteq S} \overline{l^T} \cdot v(T),$$

and thus

$$\mathcal{V}(\mathcal{E}(v))(S) = \max \left\{ \sum_{T\subseteq S} \overline{l^T} \cdot v(T) \mid \sum_{T\ni i} \overline{l^T} = 1, \forall i \in S \right\},$$

which is precisely $\bar{v}(S)$, as desired. ■

Corollary 3.4 *Any totally balanced game v is a production market game.*

Proof. The balanced cover of a totally balanced game is the game itself, so $v = \bar{v}$ and $\mathcal{V}(\mathcal{E}(v)) = \bar{v}$, according to Proposition 3.3. ■

Proposition 3.5 *A game v is totally balanced if and only if it is a production market game.*

Proof. Given the previous corollary, it is left to show that any production market game is totally balanced. Let $v = \mathcal{V}(E)$. Since the economy E is convex, its core is non-empty and thus v is balanced. For any $S \subseteq N$, the subgame $v_{|S}$ is also a production market game, as it can be generated by the restriction of E to S. As any subgame of v is balanced, we conclude that v is totally balanced. ■

4 Competitive Equilibria and the Aspiration Core

In this section we analyze competitive equilibria of a production economy in relation to the aspiration core allocations of the associated TU-game.

Definition 4.1 *Let v be a game and $\mathcal{E}(v)$ its direct production economy. A competitive (or Walrasian) equilibrium for $\mathcal{E}(v)$ is a vector*

$$[(\bar{w}, 1) \in \mathbb{R}_{+}^{|N|+1}, \bar{c} \in \mathbb{R}_{+}^{|N|}, (-\overline{l^S} \cdot \mathbf{1}_S, \overline{l^S} v(S))_{S \in \mathcal{N}}]$$

of (normalized) prices, allocations, and production plans such that:

1. $\pi^S = \overline{l^S}(v(S) - \bar{w}(S)) = \max_{l_S} l_S(v(S) - \bar{w}(S))$, *for every $S \in \mathcal{N}$*
2. $\bar{c}_i = \bar{w}_i + \sum_{S \ni i} \frac{1}{|S|} \pi^S$ *for every $i \in N$*
3. $\sum_{S \ni i} \overline{l^S} = 1$ *for all $i \in N$*
4. $\sum_{i \in N} \bar{c}_i = \sum_{S \in \mathcal{N}} \overline{l^S} v(S)$

The consumption vector $\bar{c} \in \mathbb{R}_{+}^{|N|}$ is called a Walrasian allocation. Let $\mathcal{W}(\mathcal{E}(v))$ denote the set of all Walrasian allocations of $\mathcal{E}(v)$.

Given a vector of relative wages $(\bar{w}_i)_{i \in N}$, each consumer i chooses an affordable consumption plan that maximizes her utility and each firm S selects an optimal production plan to maximize its profit. Given the production sets, each firm S's demand for labor must be of the form $\bar{l^S} \cdot \mathbf{1}_S$, with $l^S \in \mathbb{R}_{+}$.

Shapley and Shubik (1975) show that every payoff vector in the core of a game v is a Walrasian allocation of the corresponding direct market. Here we generalize their result in two ways. First, we make it applicable for general TU-games as opposed to balanced (or c-balanced, as in the case of Sun, Trockel, and Yang's (2008)), games. Second, not only allocations, but formed coalitions (in game v) and productive firms (in economy $\mathcal{E}(v)$), coincide. The following theorem shows that there is a bijection between $\mathcal{AC}(v)$ and $\mathcal{W}(\mathcal{E}(v))$. Moreover, firms that are active in equilibrium are in a one-to-one and onto correspondence with those elements of the generating collection that have a positive weight.

Theorem 4.2 *Let $\bar{x} \in \mathcal{AC}(v)$ and $(\bar{\lambda}_S)_{S\in\mathcal{N}}$ be a system of balancing weights associated with $\mathcal{GC}(\bar{x})$ such that $\bar{\lambda}_S = 0$ if $S \notin \mathcal{GC}(\bar{x})$. Then, $[(\bar{x}, 1), \bar{x}, (-\bar{\lambda}_S \cdot \mathbf{1}_S, \bar{\lambda}_S v(S))_S]$ is a competitive equilibrium for $\mathcal{E}(v)$.*

Reciprocally, if $[(\bar{w}, 1), \bar{c}, (-\overline{l^S} \cdot \mathbf{1}_S, \overline{l^S} v(S))_{S\in\mathcal{N}}]$ is a competitive equilibrium in $\mathcal{E}(v)$, then $\bar{c} \in \mathcal{AC}(v)$ and $S \in \mathcal{GC}(\bar{c})$ whenever $\overline{l^S} > 0$.

Proof. Let $\bar{x} \in \mathcal{AC}(v)$ and $(\bar{\lambda}_S)_S$ be a system of balancing weights associated with $\mathcal{GC}(\bar{x})$. For every firm $S \notin \mathcal{GC}(\bar{x})$ it is optimal to choose $\overline{l^S} = 0$ at prices $(\bar{x}, 1)$ and thus remain inactive. Every firm $S \in \mathcal{GC}(\bar{x})$ is indifferent over the choice of $l^S \in \mathbb{R}_+$ at prices $(\bar{x}, 1)$ and, in particular, it can choose $\bar{l}^S = \bar{\lambda}_S$. Since $\mathcal{GC}(\bar{x})$ is balanced, all labor markets clear. Finally, feasibility in the consumption good holds as $\bar{x}(N) = \sum_{S\in\mathcal{GC}(\bar{x})} \bar{\lambda}_S \bar{x}(S) = \sum_{S\in\mathcal{GC}(\bar{x})} \bar{\lambda}_S v(S) = \sum_{S\in\mathcal{N}} \bar{\lambda}_S v(S)$. Thus, $[(\bar{x}, 1), \bar{x}, (-\bar{\lambda}_S \cdot \mathbf{1}_S, \bar{\lambda}_S v(S))_S]$ is a competitive equilibrium for $\mathcal{E}(v)$.

Let now $[(\bar{w}, 1), \bar{c}, (-\overline{l^S} \cdot \mathbf{1}_S, \overline{l^S} v(S))_{S\in\mathcal{N}}]$ be a competitive equilibrium in $\mathcal{E}(v)$. Since production sets exhibit constant returns to scale, profits equal zero for all $S \in \mathcal{N}$. Then $\bar{c}_i = \bar{w}_i$ for every $i \in N$ and $v(S) \leq \bar{w}(S) = \bar{c}(S)$ for every $S \in \mathcal{N}$. Therefore $\bar{c}$ is stable. Moreover, $\overline{l^S} = 0$ for every S for which $v(S) < \bar{w}(S)$. The market clearing condition implies then that $\mathcal{GC}(\bar{c})$ is balanced, with balancing weights $(\overline{l^S})_S$, and thus $\bar{c} \in \mathcal{AC}(v)$. ■

Note that to prove the second part of the theorem, one might have used the known result that any competitive equilibrium allocation of an economy E is the in core of the game $\mathcal{V}(E)$) (Hildenbrand 1974). If $E = \mathcal{E}(v)$ then, according to Proposition 3.3, $\mathcal{V}(\mathcal{E}(v)) = \bar{v}$, and thus every competitive allocation of $\mathcal{E}(v)$ is in the aspiration core of v.

5 Indivisibilities, lotteries and the aspiration core

Decentralization of aspiration core allocations via the direct production economy presented in the previous section relied on perfect divisibility of players' resources. However, many TU-games describe economies with indivisibilities and the previous market implementation of the aspiration core is not satisfactory in those cases. We show in the following that aspiration core allocations can be decentralized via a market economy with indivisible goods and trade in lotteries.

We modify the analysis of the previous sections by assuming that "time" is indivisible. That is, we assume that goods $L_1, ...L_n$ are indivisible, while the consumption good, C remains perfectly divisible.

Given a game v on N, we define its *indivisible direct production market* as

$$\mathcal{E}^{indiv}(v) = \left[\{L_i \mid i \in N\} \cup \{C\}, N, \mathcal{N}, (X_i, u_i, \omega_i)_{i\in N}, (Y^S)_{S\in\mathcal{N}}, (\theta_i^S)_{(i,S)}\right],$$

where $X_i = \mathbb{R}_+^{|N|+1}$, $u_i(l_1, \ldots, l_N, c) = c$, and $\omega_i = (e_i, 0)$ for each consumer i. Each firm S has a production set $Y_S := \{k \cdot (-\mathbf{1}_S, v(S)) \mid k \in \mathbb{N}\}$ and shares $\theta_i^S = \frac{1}{|S|} \cdot \mathbf{1}_S(i)$. Notice that, with the exception of the production sets, the indivisible direct production market of v remains as described in (3).

An immediate consequence of Theorem 4.2 is that $x \in \mathcal{C}(v)$ if and only if x is a competitive allocation for $\mathcal{E}^{indiv}(v)$. Moreover, as proved by Sun, Trockel, and Yang (2008), the indivisible direct production market does not have an equilibrium unless the *super-additive completion* of v, is balanced.[3] We will show in the sequel that if firms and consumers are allowed to sign employment contracts contingent on the outcome of a lottery, an equilibrium of the modified *direct lottery market* constructed below always exist.

An *individual lottery* for agent i is a vector $p_i \in \Delta_{\mathcal{N}}$ such that $\sum_{S\ni i} p_i^S = 1$. Thus, p_i^S specifies the probability with which agent i chooses to work for firm S. We denote by P_i the set of agent i's individual lotteries.

Given a wage level w_i, agent i chooses a probability distribution over the firms $S \ni i$. We assume that consumers are risk-neutral and thus they are indifferent among the individual lotteries in their feasible sets.

An *employment contract* or *lottery* for firm S specifies a probability, $l_S \in [0, 1]$ of receiving employment from that firm. Alternatively, one can interpret l_S as the probability that S operates/stays in business. Each firm S chooses a probability of being active, $l_S \in [0, 1]$ and, contingent on being active, an operating level/labor force size $k_S \in \mathbb{N}$. Thus, firm S solves:

$$\max \{l_S \cdot k_S (v(S) - w(S)) \mid l_S \in [0, 1], k_S \in \mathbb{N}\}. \quad (4)$$

In order to establish when an allocation of individual and employment lotteries is feasible, we need to define first feasible firm-formation arrangements. A vector $x \in \{0, 1\}^{\mathcal{N}}$ is a *feasible firm-formation arrangement* if

[3]The *super-additive completion* of v is defined as a game $\tilde{v}$ such that $\tilde{v}(S) = v(S)$ if $S \neq N$ and $\tilde{v}(N) = \max_{\mathcal{B}\in\mathcal{P}} \sum_{S\in\mathcal{B}} v(S)$, where $\mathcal{P}$ denotes the set of all partitions of N.

$$[x_S = x_{S'} = 1] \Rightarrow \left[S \cap S' = \emptyset\right].$$

Note that a feasible firm-formation arrangement only requires that there is no excess demand for labor/human capital. It does *not* require that the labor market clear. For every feasible firm-formation arrangement x, define $T(x) := \bigcup\{S \mid x_S = 1\}$ as the set of employed agents. At a feasible firm-formation arrangement $T(x)$ may be a strict subset of N. Denote the set of all feasible firm-formation arrangements by $\mathcal{X}$.

Given γ, a given lottery on the elements of $\mathcal{X}$, the probability that firm S is active is $\sum_{\{x|x_S=1\}} \gamma(x)$. The probability that consumer i is employed is $\gamma_i := \sum_{\{x|T(x)\ni i\}} \gamma(x)$. Let Γ be the set of lotteries on $\mathcal{X}$ that assign to every consumer a positive probability of being employed, i.e., $\gamma_i > 0$ for all $i \in N$.

Definition 5.1 *An allocation of individual lotteries $(p_i)_{i\in N}$ and employment contracts $(l_S)_{S\in\mathcal{N}}$ is feasible if*

1. *there exists $\gamma \in \Gamma$ such that $l_S = \sum_{\{x|x_S=1\}} \gamma(x)$, for all $S \in \mathcal{N}$,*

2. *$p_i^S = \frac{l_S}{\sum_{T\ni i} l_T}$, for every $S \subseteq N$ and every $i \in S$, and $p_i^S = p_j^S$ if $i, j \in S$.*

We denote the set of feasible individual and firm lottery allocations by $\mathcal{F}$.

The first condition is a compatibility condition for firms' lotteries. It requires that each firm's lottery is the marginal of a joint probability distribution over the set of feasible firm-formation arrangements. The second condition requires compatibility between firms' and consumers' individual lotteries (or market clearing for labor). It states that the probability that agent i assigns to working for firm S is exactly the probability of firm S operating, conditional on i being employed. Note that the two conditions imply that $\sum_{S\ni i} l_S = \gamma_i > 0$ and $\gamma_i = \gamma_j$ for all $i, j \in N$.

Definition 5.2 *An equilibrium for the direct lottery market is a list of vectors*

$$\left[(\bar{w}_i)_i, (\bar{p}_i)_i, (\bar{l}_S)_S, (\bar{k}_S)_S\right]$$

such that

1. *$\bar{p}_i \in P_i$ for every $i \in N$*

2. *$(\bar{l}_S, \bar{k}_S)$ solves (4) for every $S \subseteq N$*

3. $k_S = 1$, *for all* $S \subseteq N$ *and* $((\bar{p}_i)_i, (\bar{l}_S)_S) \in \mathcal{F}$.

The following is an analogue of Theorem 4.2 to this lottery markets setting.

Theorem 5.3 *If* $\bar{w} \in \mathcal{AC}(v)$ *and* $(\lambda_S)_S$ *is a system of balancing weights associated with* $\mathcal{GC}(\bar{w})$*, then*

$$\left[(\bar{w}_i)_{i\in N}, ((\lambda_S)_{S\ni i})_{i\in N}, \left(\frac{\lambda_S}{\Lambda}\right)_{S\in\mathcal{N}}, (k_S = 1)_{S\in\mathcal{N}}\right]$$

is a competitive equilibrium for the direct lottery market, where $\Lambda := \sum_{S\in\mathcal{N}} \lambda_S$.

Reciprocally, if $[(\bar{w}_i)_{i\in N}, (\bar{p}_i)_{i\in N}, (\bar{l}_S)_{S\subseteq N}, (\bar{k}_S)_{S\subseteq N}]$ *is a competitive equilibrium for the direct lottery market, then* $\bar{w} \in \mathcal{AC}(v)$ *and* $S \in \mathcal{GC}(\bar{w})$ *whenever* $\bar{l}_S > 0$.

Proof. Let $\bar{w} \in \mathcal{AC}(v)$ and $(\lambda_S)_S$ a system of balancing weights associated with $\mathcal{GC}(\bar{w})$. Define $\bar{p}_i^S := \lambda_S$, $\bar{l}_S := \frac{\lambda_S}{\Lambda}$ and $\gamma \in \Gamma$ such that $\gamma(x) = \bar{l}_S$ if $x = e_S$ for some $S \in \mathcal{N}$ and $\gamma(x) = 0$ otherwise. Then $((\bar{p}_i)_{i\in N}, (\bar{l}_S)_{S\in\mathcal{N}})$ is a feasible lottery allocation, supported by the joint lottery $\gamma \in \Gamma$. Moreover, since $\bar{w} \in \mathcal{AC}(v)$, $\bar{w}(S) \geq v(S)$ and thus $(\bar{l}_S, 1)$ is an optimal choice for firm S, which generates an expected profit of 0.

Reciprocally, if $[(\bar{w}_i)_{i\in N}, (\bar{p}_i)_{i\in N}, (\bar{l}_S)_{S\subseteq N}, (\bar{k}_S)_{S\subseteq N}]$ is an equilibrium for the direct lottery market, then $\bar{w}(S) \geq v(S)$, otherwise firm S would make infinite profits. Profit maximization also dictates that $\bar{l}_S > 0$ only if $\bar{w}(S) = v(S)$. On the other hand, feasibility of firm's lotteries implies that $\sum_{S\ni i} \bar{l}_S > 0$ and thus, for every $i \in N$ there exists $S \ni i$ such that $\bar{l}_S > 0$ and $\bar{w}(S) = v(S)$, which implies that $\bar{w}$ is an aspiration. In addition, $\lambda_S := \frac{l_S}{\sum_{T\ni i} l_T}$ does not depend on i and $\sum_{S\ni i} \lambda_S = 1$ for every $i \in N$. This proves that $\mathcal{GC}(\bar{w})$ is balanced and thus $\bar{w} \in \mathcal{AC}(v)$. ∎

Note that for a given $\bar{w} \in \mathcal{AC}(v)$, there can be multiple distributions on $\mathcal{X}$ that support a lottery equilibrium in which wages are $\bar{w}$. These equilibria differ in their allocations of firm's lotteries. In the proof above we constructed one of them, which suggests that only one coalition/firm from $\mathcal{GC}(\bar{w})$ forms at one time, but each such coalition forms with a positive probability. Alternatively, one can construct a probability distribution γ which allows for multiple, disjoint coalitions from $\mathcal{GC}(\bar{w})$ to form simultaneously. This can be done by constructing the set of maximal families of disjoint coalitions from $\mathcal{GC}(\bar{w})$ and defining the probability γ such that it puts its entire mass on that set.

An immediate consequence of Theorem 5.3 is that the core of a game v is non-empty if and only if the direct lottery market has a degenerate equilibrium in which $p_i^N = 1$ for every $i \in N$, $l_N = 1$ and $l_S = 0$ for every $S \subsetneq N$. On the other hand, the game v has a non-empty c-core if and only if the direct lottery market has an equilibrium for which $\gamma_i = 1$ for all $i \in N$. If the c-core of the game is empty then each player faces a positive probability of being left out of the realized coalition structure and thus, in every realization of the joint lottery, the labor market is in excess supply.

6 Final Remarks

An analog of Theorems 4.2 and 5.3 can be obtained in the context of pure-exchange economies, extending thus the results of Shapley and Shubik (1975) and Garratt and Qin (1997). Shapley and Shubik (1975) showed that there is a one-to-one and onto mapping between the core allocations of a totally balanced game and the competitive allocations of its direct market (pure-exchange economy). Indeed, if $\mathcal{E}^0(v)$ is the pure-exchange direct economy associated to an arbitrary game v, then $v(\mathcal{E}^0(v))$ is the totally balanced cover of v. But every competitive equilibrium of $\mathcal{E}^0(v)$ is in the core of $v(\mathcal{E}^0(v))$ and thus in the aspiration core of v. Hence, every competitive equilibrium of $\mathcal{E}^0(v)$ is in the aspiration core of v. Applying Shapley and Shubik's (1975) results to the totally balanced cover of v we also get the converse.

Garratt and Qin (1997) construct a direct lottery market (also pure-exchange) from any TU-game and show that utilities generated by the equilibrium allocation of the direct lottery market coincide with the core allocations of the game. Direct lottery markets associated to non-balanced games do not have an equilibrium, but they do have a *free-disposal* equilibrium.[4] It can be shown that utilities generated by free-disposal equilibrium allocations of the direct lottery market coincide with the aspiration core of the game.

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