Constrained Inefficiency and Optimal Taxation with Uninsurable Risks*

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March 1, 2010

Abstract

When individuals' labor and capital income are subject to uninsurable idiosyncratic risks, should capital and labor be taxed, and if so how? In a two period general equilibrium model with production, we first show that reducing investment is welfare improving if households are homogeneous enough ex ante. On the other hand, when the degree of heterogeneity is sufficiently high a welfare improvement is achieved by increasing investment, even if the investment level is already higher than at the efficient allocation obtained when full insurance markets were available. Consequently, the optimal capital tax rate might be negative. We derive a decomposition formula of the effects of the tax which allow us to determine how the sign of optimal tax on capital and labor depends both on the nature of the shocks and the degree of heterogeneity among consumers as well as on the way in which the tax revenue is allocated. (JEL codes: D52, H21. Keywords: optimal tax, incomplete markets, constrained efficiency)

1 Introduction

The main objective of this paper is to investigate the role and optimal form of taxation of investment and labor income in a dynamic production economy with uninsurable background risk. More precisely, we investigate whether the introduction of linear, distortionary taxes on labor income and/or on the returns from savings are welfare improving and what is then the optimal sign of such taxes. This amounts to studying the solution of a *Ramsey problem* in a general equilibrium set-up. We depart however from most of the literature on the subject[1] for the fact that we consider an environment with no public expenditure, where there is then no need to raise taxes. Still, optimal taxes are typically nonzero as we will show. The reason is that even distortionary taxes can improve the allocation of risk in the face of incomplete markets. The issue then arises of what should be taxed, and what economic properties determine the signs of the optimal taxes.

A possible answer to this question may come from the following consequence of the agents' precautionary motive for saving: under uninsurable risk, this motive implies that savings and hence capital accumulation will be higher compared to the situation where markets are complete. This point was made in an influential paper by Aiyagari (1995). The result may appear to imply, and in

*We thank Naoki Yoshihara for his valuable comments. Remaining errors and omissions are of course ours. Kajii acknowledges financial supports by the Inamori Foundation and the Grant-in-Aid for Scientific Research. Nakajima gratefully acknowledges financial supports from the Murata Science Foundation, and the Japanese Ministry of Education, Culture, Sports, Science and Technology.

[1] See Chari and Kehoe (1999) for a survey.

fact various papers thereafter suggested it implicitly and explicitly, that with incomplete markets, the *precautionary saving motive* leads to *over-accumulation of capital* and hence that a *positive tax on capital is welfare improving.*[2]

We argue however that this implication is unwarranted: the comparison between the level of capital accumulation with and without complete markets has no clear welfare implication. If there were a policy tool which could allow to attain the complete market allocation, there would be little doubt for the policy maker to adopt such a policy as far as attaining efficiency is concerned. Since a tax and subsidy scheme of the kind mentioned will not complete the markets, the aforementioned comparison tells little about the effectiveness of taxation, not to mention whether or not capital should be taxed.

To properly assess whether or not positive taxes on capital are welfare improving when markets are incomplete, one should rather compare the competitive equilibria with and without taxes, keeping the other parts of the market structure, and in particular the set of available financial assets, fixed. This "second best" exercise is what we do formally in this paper. We consider explicitly market equilibria with taxation, for various tax-subsidy schemes. We say capital should be taxed (resp. subsidized) if there is an equilibrium with a positive (resp. negative) tax on capital where consumers' welfare is higher than in an equilibrium without tax. Similarly for labor.

Moreover, our main interest here is not on the effect of taxes in the long run, on steady state allocations, but rather on their immediate effects, in the short run (in contrast to Judd (1985), Chamley (1986) and the literature which followed). To this end we shall consider a two period economy. This is primarily for simplicity and will allow us to identify more clearly and evaluate the various effects of taxes.

The reader may still wonder if the optimal capital tax should ever be negative in the sense above when the equilibrium stock of capital is higher than when markets are complete. We show that indeed subsidizing capital may be welfare improving in such a case. This finding does not rely on the presence of upward sloping demand curves, so that subsidizing capital further increases its level but nevertheless raises consumers' welfare. To give some intuition for this, let us first describe the model more explicitly to outline our results.

We consider a two period economy with production, where the savings of each consumer can be invested to obtain capital, which is then used as input in the production process the next period. In addition, the consumer has to choose how much to work, and the productivity of his work is subject to idiosyncratic shocks. The amount of capital obtained per unit invested by a consumer may also be subject to idiosyncratic shocks. This is all the uncertainty in the model, idiosyncratic shocks are independent and there is a continuum of consumers so that there is no aggregate uncertainty. Consumers may differ in terms of their initial income as well as of their preferences. Capital and labor are exchanged between consumers and firms in competitive markets. Since the consumers' investment is the only instrument allowing them to transfer income over time, markets are clearly incomplete. Linear, uniform taxes on wage income as well as on the investment income may be introduced and the net revenue from these taxes is redistributed to consumers via lump sum transfers. Such taxes and transfers affect individual savings and labor supply decisions and thus induce a change in equilibrium prices, and of course in consumers' welfare.

With incomplete markets these price changes affect the risk allocation among agents, a pecuniary externality which was first noticed by Hart (1975) and Stiglitz (1982). In our setup, the pecuniary externality consists of two effects. First, a price change has an *insurance effect*: if the price of a

[2]See for instance, Ljungqvist and Sargent (2004, Chapter 15, pp. 535-536), Aiyagari (1995, page 1160), and Mankiew, Weinzierl and Yagan (2009, pages 167-8).

risky factor of production goes down, the risk in the agent's future income is reduced, the more so the riskier the factor, thus partly offsetting the consequences of the missing markets. Secondly, a price change also has an *indirect distribution effect* as it does with complete markets: since agents are endowed differently in the factors of production, a change in the relative prices of factors will induce transfers of income across different types of consumers. In addition, depending on the way in which the revenue of the tax is redistributed to consumers, there may be some *direct distribution effect* as well: for instance, if the lump sum transfer is constant and common to every consumer, it will directly provide some income smoothing across consumers.

Even in this relatively simple environment to determine the actual level of the optimal tax rates is not an easy task. We shall therefore primarily focus our analysis on the effects of introducing taxes at an infinitesimal level. The first order effects of such taxes can be obtained by differentiating the equilibrium system, at an equilibrium with no taxes, whenever such equilibrium is regular. We will decompose them into insurance and distribution effects and investigate their properties. We are then able to identify sets of conditions (concerning characteristics both of the economy and of the tax schemes considered) under which a welfare improvement can be obtained with a positive, or a negative, tax on capital, and similarly for labor. Moreover, these findings allow to shed some light on the sign of the optimal tax rate on capital and labor, which tends to be the same as the sign of the optimal infinitesimal tax at zero.

In Section 3, as a preliminary step, we consider a situation where consumers' savings or labor supply can be directly modified, which we shall refer to as the constrained optimality exercise. This problem was also investigated by Davila *et al* (2005) in a similar setup, but with an exogenous labor supply and with idiosyncratic shocks only affecting labor productivity. We find that the insurance effect operates in favor of a decrease in consumers' investment, or alternatively of an increase in labor supply, when the shocks affect primarily labor rather than capital (and vice versa in the opposite case). With significant heterogeneity in the pattern of consumers' initial income (and possibly preferences), the distribution effect also becomes important. This effect has a different sign across consumers: while the relatively poor consumers benefit even more from a decrease in the investment level, the relatively rich ones might actually loose from it. As a consequence, when heterogeneity is large a change (in investment or labor supply) improving the utility of all consumers cannot be found. Therefore we investigate which changes improve consumers' ex ante welfare (under the veil of ignorance), that is before a type is assigned to any agent.

We find that the distribution effect works for ex ante welfare in the opposite direction as the insurance effect. Hence, again in the case where shocks affect primarily labor, ex ante welfare increases if investment is reduced (that is, there is *over* investment in equilibrium) when the heterogeneity is small, and if investment is increased when the heterogeneity is large; vice versa for labor supply. These findings are independent of the level of the equilibrium interest rate and of the presence of a precautionary motive. They show that, to determine whether or not there is over investment in equilibrium, a primary role is played by the degree of heterogeneity among consumers and by the nature of the shocks. For instance, in an economy where consumers are ex ante identical and exhibit a precautionary saving motive, capital should be *subsidized* when the shocks affect primarily capital, even though the level of capital accumulation is higher than the efficient level, at the equilibrium with complete markets.

We turn then to the analysis of optimal taxes in Section 4. With taxes, savings and labor supply can only be controlled indirectly and the welfare effects of taxes depend then on the way they affect savings and labor supply.

When the tax revenue is rebated to consumers without inducing any reallocation of income across consumers or states, the effects of taxes are analogous to those of directly controlling investment or labor supply (Section 4.1). We then find that the optimal tax on capital is positive (and the optimal tax on labor negative) exactly when there is over investment in the constrained optimality exercise. Therefore when the degree of heterogeneity among consumers' income is sufficiently limited, capital should be taxed. The reverse conclusion holds instead when the heterogeneity is large (or the shocks affect primarily the returns on savings).

In Sections 4.2 and 4.3 we turn our attention to the case where the tax revenue is no longer redistributed to each consumer exactly in proportion to the consumer's tax payments in each state, so that the tax scheme also provides some insurance and/or some income redistribution among consumers. In this situation the basic trade off is as follows: the provision of insurance strengthens the case for a positive tax, especially for the factor whose income is more affected by the shocks (hence labor in the main case considered). In contrast, the provision of redistribution tends to strengthen the case for taxing capital and weakens that for taxing labor, since it is typically the case that the main source of income is capital for wealthy consumers and labor for poor consumers. Thus the sign of the optimal tax depends on the relative importance of these two elements.

We also consider (in Section 4.4) the case where lump sum transfers are not available, so that the revenue of the tax on one factor is redistributed to consumers via a subsidy on the other factor. Surprisingly enough, we obtain that it is optimal to tax capital whenever there is *under* investment in equilibrium. This is exactly the opposite of what we found when lump sum transfers are possible (without redistribution).

The analyses thus far are local around a given equilibrium. The characterization of the level of the optimal tax rate requires a global analysis of the equilibria with taxes and it is then difficult to obtain general results for this. In Section 5 we consider a numerical example of an economy exhibiting standard properties, for which the optimal tax rates are derived for the different tax schemes considered. The numerical results also show that the sign of the optimal tax rates are typically in accord with our findings from the local analysis and illustrate once again how the level of the optimal tax on capital and labor depends on the degree of heterogeneity among consumers.

2 The Economy

We consider a two period competitive market economy as follows. The economic agents consist of one representative firm and I types of consumers, where there is a continuum of consumers of size one for each type.

The firm has a constant returns to scale technology described by a smooth homogeneous concave production function $F(K, L)$ per capita (that is, relative to the size of each type), where K is the amount of capital input per capita and L is the amount of labor input per capita, both measured in efficiency units (as made clearer in what follows). The firm maximizes profits taking prices as given: writing r for the per efficiency unit cost of capital and w for the wage in efficiency unit, K and L will be chosen so that $F_K(K, L) = r$ and $F_L(K, L) = w$. The firm operates in the second period, when both the production activity and the purchases of inputs take place, although other interpretations are possible.

Consumers of the same type are identical ex ante and make the same choices in the first period. Each consumer of type i is endowed with $e_i > 0$ units of consumption good in the first period, which may be consumed or invested. If invested, it will yield some amount of the capital good next period (which can also be interpreted as human capital), to be sold to the firm at price r. Denote by k_i the amount invested by type i, thus $e_i - k_i$ is the consumption in the first period.

The output of the investment in the technology yielding units of the capital good is subject to idiosyncratic risks. For each i, denote by (Θ_i, P_i) the probability space which describes the shock affecting type i consumers. We assume that the shock is independently and identically distributed across the consumers of type i, and independently distributed across different types. In state $\theta_i \in \Theta_i$, an investment of k_i units in the first period by a type i consumer yields

$$K_i^{\theta_i} := \rho_i^K(\theta_i)\, k_i \tag{1}$$

in efficiency units of capital the following period, where ρ_i^K is a random variable on the state space (Θ_i, P_i). We further assume that the i.i.d. assumption of the shocks implies that the aggregate supply of capital K_i from type i consumers in efficiency units is equal to k_i times the expected value of the returns obtained from the consumers' investment, $\gamma_i := \mathbf{E}\left[\rho_i^K(\theta_i)\right]$,[3] that is $K_i = \gamma_i k_i$. By definition the aggregate per-capita supply of capital is given by $K = \frac{1}{I}\sum_i \gamma_i k_i$.

In the second period, a type i consumer is endowed with $\bar{H}_i$ units of labor hour ($\bar{H}_i > 0$). The labor efficiency is also subject to the idiosyncratic risk affecting consumers, and the level of the labor supply is chosen after $\theta_i \in \Theta_i$ is realized: writing $h_i^{\theta_i}$ for the labor hours supplied after the consumer observed θ_i, the labor supply in efficiency units $L_i^{\theta_i}$ is defined by

$$L_i^{\theta_i} := \rho_i^L(\theta_i)\, h_i^{\theta_i}, \tag{2}$$

where ρ_i^L is another random variable on (Θ_i, P_i). We normalize units so that $\mathbf{E}\left[\rho_i^L(\theta_i)\right] = 1$ for every i. Again we assume that the aggregate supply of labor of type i consumers in efficiency units, L_i, is equal to the expected level of the labor supply. That is, if we write L_i for the total supply of labor in efficiency units by the consumers of type i, $L_i := \mathbf{E}\left[L_i^{\theta_i}\right]$ holds, and then by definition the aggregate per-capita labor supply is given by $L = \frac{1}{I}\sum_i \mathbf{E}\left[L_i^{\theta_i}\right]$. In the special case of inelastically supplied labor, $h_i^{\theta_i} = \bar{H}_i$ at every θ_i. In such a case, $L_i^{\theta_i} = \rho_i^L(\theta_i)\bar{H}_i$, and $L_i = \bar{H}_i$.

The structure of the uncertainty thus allows both for idiosyncratic labor income risk, as in Aiyagari (1994) and idiosyncratic capital income risk, as in Angeletos (2007).

To ensure that the model is well defined, we assume throughout our analysis that both the individual labor endowment and the gross return on savings are always positive: that is, $\rho_i^K(\theta_i) > 0$ and $\rho_i^L(\theta_i) > 0$ occur with probability one. We shall also assume that the two random variables ρ_i^L and ρ_i^K are comonotonic, i.e., $\left(\rho_i^L(\theta_i) - \rho_i^L(\theta_i')\right)\left(\rho_i^K(\theta_i) - \rho_i^K(\theta_i')\right) \geq 0$ for any pair of states θ_i and θ_i'. That is, if the labor endowment of a type i household is relatively large, the productivity of capital tends to be high as well. We shall therefore use the convention that the household is (relatively) rich at state θ_i if the corresponding $\rho_i^L(\theta_i)$ is (relatively) large. This assumption will help us in getting some clear-cut results on the form of the inefficiency and then the signs for optimal tax rates. It is however not essential for most of the analysis; our main decomposition result which identifies the origins of welfare effects does not depend on it. Notice also that the assumption holds automatically if there is no shock for the capital, i.e., ρ_i^K is constant, which we regard as a benchmark case; in applications, it is common to consider shocks on the *aggregate* productivity of capital, but not at the level of individual households.

A type i consumer's risk preferences are represented by a time additively separable utility function: the first period utility is given by a function v_i of the first period consumption of the good, and the second period utility is given by a function u_i of the consumption of the good and leisure. So when a type i individual chooses to invest k_i and supply $L_i^{\theta_i}$ efficiency units of labor at θ_i, he

[3] Since the shocks are independent, the meaning of the expectation will be clear and so we shall omit the reference to the underlying measure P_i.

consumes $e_i - k_i$ units of the good in the first period, and $wL_i^{\theta_i} + rK_i^{\theta_i}$ units of the good and $\bar{H}_i - h_i^{\theta_i}$ units of leisure in the second period in state θ_i. Thus his choice problem is given as follows:

$$\max_{k_i,\left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}} v_i\left(e_i-k_i\right)+\mathbf{E}\left[u_i\left(wL_i^{\theta_i}+rK_i^{\theta_i},\bar{H}_i-h_i^{\theta_i}\right)\right], \tag{3}$$

where $K_i^{\theta_i}$ and $L_i^{\theta_i}$ are defined and understood as functions of k_i and $h_i^{\theta_i}$ as in (1) and (2). We assume that both v_i and u_i are smooth and concave, strictly increasing in the consumption good and non-decreasing in leisure. We also assume that the random variables are well behaved so that the first order approach is valid: i.e., we assume that the following first order condition completely characterizes the solution to the consumer's choice problem:

$$-v_i'\left(e_i-k_i\right)+\mathbf{E}\left[u_{ic}\left(wL_i^{\theta_i}+rK_i^{\theta_i},\bar{H}_i-h_i^{\theta_i}\right)\cdot r\rho_i^K\left(\theta_i\right)\right]=0. \tag{4}$$

$$u_{ic}\left(wL_i^{\theta_i}+rK_i^{\theta_i},\bar{H}_i-h_i^{\theta_i}\right)\rho_i^L\left(\theta_i\right)w-u_{il}\left(wL_i^{\theta_i}+rK_i^{\theta_i},\bar{H}_i-h_i^{\theta_i}\right)=0,\text{ at every }\theta_i, \tag{5}$$

where u_{ic} and u_{il} stand for the partial derivatives with respect to consumption and leisure, respectively. This assumption is satisfied, for instance, if each state space is finite. Similar convention will be used throughout the paper, e.g., u_{icc} stands for the second derivative with respect to consumption, and u_{icl} stands for the cross derivative. Furthermore, for the special case of $u_{il} \equiv 0$, labor is inelastically supplied and condition (5) is replaced with

$$\rho_i^L\left(\theta_i\right)\bar{H}_i-L_i^{\theta_i}=0\text{ at every }\theta_i. \tag{6}$$

Note that, since all individuals of the same type solve the same problem and such problem is convex, their optimal decisions are also the same.

It can be readily verified that the consumption good markets clear when all the factor markets clear. So in this economy a competitive equilibrium occurs when the firm's profit maximization condition is satisfied at a level of the aggregate input variables that is equal to the one derived from the consumers' maximization problems. Formally,

Definition 1 *A collection* $\left(\hat{w},\hat{r},\left(\hat{k}_i,\left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^I\right)$ *constitutes a competitive equilibrium if, for each i,* $\left(\hat{k}_i,\left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)$ *is a solution to (3), and the profit maximization conditions,* $F_K\left(\hat{K},\hat{L}\right)=\hat{r}$ *and* $F_L\left(\hat{K},\hat{L}\right)=\hat{w}$*, hold for* $\hat{K}=\frac{1}{I}\sum_{i=1}^I\gamma_i\hat{k}_i$ *and* $\hat{L}=\frac{1}{I}\sum_{i=1}^I\mathbf{E}\left[\hat{L}_i^{\theta_i}\right]$.

By construction, equilibrium labor hours $\left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}$ are a function of the realization of θ_i, so are the capital in efficiency units, the second period consumption, and the labor supply in efficiency units. The specific form of the inefficiency and the sign of the optimal taxes depend on how these variables vary with respect to θ_i, $i = 1, .., I$. It is then convenient to identify a standard pattern for the behavior of such variables in equilibrium:

Definition 2 *A competitive equilibrium* $\left(\hat{w},\hat{r},\left(\hat{k}_i,\left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^I\right)$ *is said to exhibit a* ***standard response to shocks*** *(in short,* ***standard to shocks****) if, for every i:*
i) $u_{ic}\left(\hat{c}_i^{\theta_i},\hat{l}_i^{\theta_i}\right)$ *is decreasing in* $\rho_i^L\left(\theta_i\right)$ *where* $\hat{c}_i^{\theta_i}=\hat{w}\hat{L}_i^{\theta_i}+\hat{r}\hat{K}_i^{\theta_i}$*; i.e.,* $u_{ic}\left(\hat{c}_i^{\theta_i},\hat{l}_i^{\theta_i}\right)<u_{ic}\left(\hat{c}_i^{\theta_i'},\hat{l}_i^{\theta_i'}\right)$ *holds whenever* $\rho_i^L\left(\theta_i'\right)<\rho_i^L\left(\theta_i\right)$.
ii) $\hat{K}_i^{\theta_i}$ *is non-decreasing and* $\hat{L}_i^{\theta_i}$ *is increasing in* $\rho_i^L\left(\theta_i\right)$.

Under the general assumptions made so far, a competitive equilibrium is not necessarily standard to shocks. We can argue however that the two properties above are quite natural and hold in 'normal' cases. Intuitively speaking, a high realization of $\rho_i^L(\theta_i)$ implies that the consumer is rich ex post, and so consumption should be relatively high and hence the marginal utility from consumption relatively low. Also, the amount of labor in efficiency units should be relatively high. We provide below some sufficient conditions which guarantee that an equilibrium is standard to shocks. First, we have the following for condition i):

Lemma 1 *Assume that u_i is strictly concave ($u_{icc}u_{ill} - (u_{icl})^2 > 0$ everywhere) and consumption is a normal good ($u_{icc}u_{il} - u_{icl}u_{ic} < 0$ everywhere). Then in any competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$, $u_{ic}\left(\hat{c}_i^{\theta_i}, \hat{l}_i^{\theta_i}\right)$ is decreasing in $\rho_i^L(\theta_i)$ where $\hat{c}_i^{\theta_i} = \hat{w}\hat{L}_i^{\theta_i} + \hat{r}\hat{K}_i^{\theta_i}$, i.e., $u_{ic}\left(\hat{c}_i^{\theta_i}, \hat{l}_i^{\theta_i}\right) < u_{ic}\left(\hat{c}_i^{\theta_i'}, \hat{l}_i^{\theta_i'}\right)$ holds whenever $\rho_i^L(\theta_i') < \rho_i^L(\theta_i)$.*

This result can be proved by applying usual consumer theory and we supply a proof in the Appendix for completeness.

Condition ii) is slightly more tricky: labor supply $\hat{L}_i^{\theta_i} = \rho_i^L(\theta_i)\hat{h}_i^{\theta_i}$ is obviously increasing in $\rho_i^L(\theta_i)$ when u_i is constant in leisure (and hence the supply of labor hours is inelastic). But when u_i is increasing in leisure, it is not clear-cut whether or not $\hat{L}_i^{\theta_i}$ is still increasing in $\rho_i^L(\theta_i)$: a higher $\rho_i^L(\theta_i)$ means a higher effective wage which induces more labor, but it also generates a higher income from capital which induces more leisure. So $\hat{L}_i^{\theta_i}$ should be increasing, roughly speaking, when the income effect from the higher revenue from the capital investment is not excessively large. A formal sufficient condition is stated in the following (also proved in the Appendix):

Lemma 2 *Assume that $\rho_i^K(\theta_i)$ is constant, and that u_i is strictly concave and leisure is a normal good ($u_{ill}u_{ic} - u_{icl}u_{il} < 0$ everywhere). Then in any competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$, $\hat{L}_i^{\theta_i}$ is increasing in $\rho_i^L(\theta_i)$.*

By Lemmas 1 and 2 it thus follows that competitive equilibria are always standard to shocks when $\rho_i^K(\theta_i)$ is constant and the strict concavity and normality properties hold for both consumption and leisure. The same is true for the case of inelastic labor hour supply. Indeed in such case condition ii) is obviously satisfied, and i) holds since u_{icc} is negative and independent of $l_i^{\theta_i}$.[4]

The normality condition for consumption and labor is met for commonly used functional forms of the utility function u_i. The assumption of constant $\rho_i^K(\theta_i)$ means that only labor efficiency is subject to idiosyncratic shocks, which constitutes an economically sensible benchmark case anyway. Moreover, to emphasize that our results on the signs of the optimal taxes are not due to any general equilibrium pathology, it will be convenient to consider these cases. In the following analysis therefore we shall focus our attention on equilibria that are standard to shocks.

3 Constrained inefficiency of competitive equilibria

3.1 Feasible policies and allocations

As we discussed in the Introduction, it is important to specify the set of available policy tools to provide an economically meaningful definition of over or under investment. In this section, we shall

[4] Davila et al. (2005) consider the case of inelastic labor supply and $\rho_i^K(\theta_i)$ constant, thus an equilibrium in their set up is automatically standard to shocks.

consider policy tools consisting in the direct control of capital and labor. The use of such tools is clearly very demanding for the policy maker, and so these tools are not of practical value. But the analysis of such case will provide some quite useful insights, and be instrumental to the subsequent analysis of the case where the policy tools are only given by linear, anonymous taxes on labor and capital.

More specifically, suppose the social planner can directly control the amounts of investment, k_i, as well as of labor hours, $h_i^{\theta_i}$, for all consumers in every state θ_i. That is, a policy instrument available to the planner can be identified with a tuple $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$. The other non-policy variables are determined in competitive markets. Specifically, since the planner cannot control the firm's decisions, the prices r and w are endogenously determined in equilibrium in such a way that the firm's demand for inputs equal the aggregate supply of inputs set by the planner: $r = F_K(K, L)$ and $w = F_L(K, L)$ for $L = \frac{1}{I}\sum_i \mathbf{E}\left[\rho_i^L(\theta_i) h_i^{\theta_i}\right]$, $K = \frac{1}{I}\sum_{i=1}^{I} \gamma_i k_i$. We shall write $r(K, L)$ and $w(K, L)$ to indicate these maps associating the market clearing prices to the aggregate quantities, K and L.

The levels of consumption of each type i consumer in the two periods and his leisure in the second period depend both on the levels of k_i and $\left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$ chosen by the planner and the associated market clearing prices. Specifically, first period consumption for type i is $c_i^0 = e_i - k_i$ while in the second period in state θ_i his leisure is $l_i^{\theta_i} = \bar{H}_i - h_i^{\theta_i}$ and consumption is $c_i^{\theta_i} = w(K, L) L_i^{\theta_i} + r(K, L) K_i^{\theta_i}$. So the feasibility of a policy is naturally defined as follows:

Definition 3 *A policy* $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ *is said to be* ***feasible*** *if for every* i, $e_i - k_i \geq 0$, $\bar{H}_i \geq h_i^{\theta_i}$, *and* $w(K, L) L_i^{\theta_i} + r(K, L) K_i^{\theta_i} \geq 0$ *at every* θ_i, *where* $r(K, L)$ *and* $w(K, L)$ *are the market clearing prices for* $K = \frac{1}{I}\sum_{i=1}^{I} \gamma_i k_i$ *and* $L = \frac{1}{I}\sum_{i=1}^{I} \mathbf{E}\left[L_i^{\theta_i}\right]$.

Clearly, if $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$ is a competitive equilibrium, $\left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ is a feasible policy. By construction, the utility level of a type i consumer induced by a feasible policy $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ is given by:

$$U_i\left(\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right) := v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(w(K, L) L_i^{\theta_i} + r(K, L) K_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right], \quad (7)$$

where $K = \frac{1}{I}\sum_{i=1}^{I} \gamma_i k_i$ and $L = \frac{1}{I}\sum_{i=1}^{I} \mathbf{E}\left[\rho_i^L(\theta_i) h_i^{\theta_i}\right]$. Following the common idea of second best analysis, we can present then a constrained efficiency notion[5]:

Definition 4 *A feasible policy* $\left(k_i, \left\{h_i^{\theta_i} : \theta_i\right\}\right)_{i=1}^{I}$ *is said to be* ***constrained inefficient*** *if there is a feasible policy* $\left(\tilde{k}_i, \left(\tilde{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}$ *that is Pareto improving:* $U_i\left(\left(\tilde{k}_i, \left(\tilde{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right) \geq U_i\left(\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$ *for every type* i, *strictly for some* i, *where* U_i *is defined as in (7).*

We can similarly give a precise definition of *over investment* as characterizing situations where there exists a Pareto improving feasible policy such that $\sum_i \tilde{k}_i < \sum_i k_i$ (and symmetrically for *under investment*).

[5] An equivalent notion could be stated for allocations instead of policies after defining a constrained feasible allocation as a consumption-leisure allocation achievable with a feasible policy.

It might appear that the planner in this story is very powerful and one may wonder whether a Pareto improvement can always be implemented by some feasible policy. Notice however that the planner is still constrained by the fact that the agents' second period consumption must respect the budget constraint $c_i^{\theta_i} = r\rho^K(\theta_i)\, k_i + w\rho^L(\theta_i)\, h_i^{\theta_i}$ for every i and θ_i, with prices r and w set at the competitive equilibrium level. Therefore, the set of allocations attainable with feasible policies is smaller than the set of feasible allocations considered in the usual Pareto efficiency notion, and so although a competitive equilibrium tends to be Pareto inefficient in our model, it might still be constrained efficient in principle.

3.2 First Order effects and constrained inefficiency

From now on, fix a competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I}\right)$ which is standard to shocks. The question we intend to ask is whether or not $\left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I}$ is constrained efficient. If it is not, we want to see what kind of policies improve upon it and in particular whether or not there is over investment.

For this purpose, we shall study how the function U_i behaves around the equilibrium, by differentiating it and evaluating it at the equilibrium values $\left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I}$. Notice that a policy $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I}$ has two effects on the expected utility level of a type i consumer given in (7): the first is of course the direct effect of the change in the values of k_i and $\left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}$; the second is an indirect effect due to the change in the values of the equilibrium prices r and w. At a competitive equilibrium, however, the direct effect has no first order effect on welfare by the envelope property; in view of (3), the values $\hat{k}_i$ and $\left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}$ already maximize the utility of consumer i at the prices $(\hat{w}, \hat{r})$. Therefore, the only first order welfare effect of the policy change is the indirect effect, that is, only the *pecuniary externality* of the change in prices.

For this reason, *as long as we are concerned with the derivative evaluated at an equilibrium,* we can take U_i in (7) as a function of K and L only, taking $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I} = \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i\in\Theta_i}\right)_{i=1}^{I}$ as fixed constants. Let us calculate then its derivatives at the equilibrium values $(\hat{K}, \hat{L})$. Since the first period utility v_i does not depend on (K, L), we only need to differentiate the second period expected utility with respect to (K, L) taking $L_i^{\theta_i}$, $K_i^{\theta_i}$ and $h_i^{\theta_i}$ as fixed at $\hat{L}_i^{\theta_i}$ $(= \rho_i^L(\theta_i)\, \hat{h}_i^{\theta_i})$, $\hat{K}_i^{\theta_i}$ $(= \rho_i^K(\theta_i)\, \hat{k}_i)$ and $\hat{h}_i^{\theta_i}$, respectively. Therefore, we have:

$$\begin{aligned}
\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} &= \left.\frac{\partial}{\partial K}\mathbf{E}\left[u_i\left(w(K,L)\,\hat{L}_i^{\theta_i} + r(K,L)\,\hat{K}_i^{\theta_i}, \bar{H}_i - \hat{h}_i^{\theta_i}\right)\right]\right|_{(\hat{K},\hat{L})} \\
&= \mathbf{E}\left[u_{ic}\cdot \left.\frac{\partial}{\partial K}\left(w(K,L)\,\hat{L}_i^{\theta_i} + r(K,L)\,\hat{K}_i^{\theta_i}\right)\right|_{(\hat{K},\hat{L})}\right] \\
&= \mathbf{E}\left[u_{ic}\cdot\left(\frac{\partial w}{\partial K}\cdot \hat{L}_i^{\theta_i} + \frac{\partial r}{\partial K}\cdot \hat{K}_i^{\theta_i}\right)\right], \qquad (8)
\end{aligned}$$

where u_{ic} is evaluated at the equilibrium levels of leisure and consumption in the second period, respectively $\bar{H}_i - \hat{h}_i^{\theta_i}$ and $\hat{w}\hat{L}_i^{\theta_i} + \hat{r}\hat{K}_i^{\theta_i}$, and $\frac{\partial w}{\partial K}$ and $\frac{\partial r}{\partial K}$ are both evaluated at $\left(\hat{K}, \hat{L}\right)$. A similar convention will be used throughout the paper. By definition, there is over investment in equilibrium if $\frac{\partial U_i}{\partial K} < 0$ for every i at $\left(\hat{K}, \hat{L}\right)$.

Similarly for labor, we have:

$$\begin{aligned}
\left.\frac{\partial U_i}{\partial L}\right|_{(\hat{K},\hat{L})} &= \frac{\partial}{\partial L}\mathbf{E}\left[u_i\left(w(K,L)\,\hat{L}_i^{\theta_i} + r(K,L)\,\hat{K}_i^{\theta_i}, \bar{H}_i - \hat{h}_i^{\theta_i}\right)\right]\Bigg|_{(\hat{K},\hat{L})} \\
&= \mathbf{E}\left[u_{ic}\cdot\frac{\partial}{\partial L}\left(w(K,L)\,\hat{L}_i^{\theta_i} + r(K,L)\,\hat{K}_i^{\theta_i}\right)\Bigg|_{(\hat{K},\hat{L})}\right] \\
&= \mathbf{E}\left[u_{ic}\cdot\left(\frac{\partial w}{\partial L}\cdot\hat{L}_i^{\theta_i} + \frac{\partial r}{\partial L}\cdot\hat{K}_i^{\theta_i}\right)\right]. \qquad (9)
\end{aligned}$$

There is under supply of labor in equilibrium if $\frac{\partial U_i}{\partial L} > 0$ for every i at $\left(\hat{K},\hat{L}\right)$.

Expressions (8) and (9) can be re-written in a more informative way as follows. Recall that $F_K(K,L) = r(K,L)$ and $F_L(K,L) = w(K,L)$. Hence $\frac{\partial r}{\partial K} = F_{KK} < 0$ and $\frac{\partial w}{\partial K} = F_{KL} > 0$. Moreover from the Euler equation, $F_K(K,L)\,K + F_L(K,L)\,L = F(K,L)$, we obtain:

$$\frac{\partial r}{\partial K}\cdot K + \frac{\partial w}{\partial K}\cdot L = 0. \qquad (10)$$

Similarly, we have $\frac{\partial r}{\partial L} = F_{KL} > 0$ and $\frac{\partial w}{\partial L} = F_{LL} < 0$, and

$$\frac{\partial r}{\partial L}\cdot K + \frac{\partial w}{\partial L}\cdot L = 0. \qquad (11)$$

Coming back to the welfare change, taking (10) into account, we can decompose the marginal change in type i's utility (8) as follows:

$$\begin{aligned}
\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} &= \mathbf{E}\left\{u_{ic}\cdot\left[\left(\frac{\partial w}{\partial K}\hat{L}_i^{\theta_i} + \frac{\partial r}{\partial K}\hat{K}_i^{\theta_i}\right) - \left(\frac{\partial r}{\partial K}\hat{K} + \frac{\partial w}{\partial K}\hat{L}\right)\right]\right\}, \\
&= \left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right] + \mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i - \hat{K}\right)\right\}\frac{\partial r}{\partial K} \qquad (12) \\
&\quad + \left\{\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right] + \mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i - \hat{L}\right)\right\}\frac{\partial w}{\partial K},
\end{aligned}$$

where all the variables are evaluated at the equilibrium.

In this decomposition, the terms $\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right]$ and $\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right]$ describe the relationship between the agent's marginal utility and the idiosyncratic shocks. In what follows, we shall use a short-hand notation to refer to them:

$$I_i^K := \mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right], \qquad (13)$$

$$I_i^L := \mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right], \qquad (14)$$

where I stands for "insurance". The reason is that such terms capture the component of the welfare effect of the change in prices that depends on how individual risks affect the agent's consumption and leisure choices, that is on the extent by which such risks are insured. When such shocks are fully insured these terms are in fact zero.

Lemma 3 *At an equilibrium which is standard to shocks, the insurance effects I_i^K and I_i^L defined in (13) and (14) are both negative.*

Proof. By the definition of the standard response to shocks, u_{ic} and $\hat{K}_i^{\theta_i}, \hat{L}_i^{\theta_i}$ move in the opposite direction when θ_i varies, hence these variables are negatively correlated. Recall that for two random variables X and Y, we have $\mathbf{E}(XY) = \mathbf{E}(X)\mathbf{E}(Y) + COV(X,Y)$. Here $\mathbf{E}\left(K_i^{\theta_i} - K_i\right) = 0 =$

$\mathbf{E}\left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)$ by construction, so the negative correlation implies that both $\mathbf{E}\left[u_{ic} \cdot \left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right] < 0$ and $\mathbf{E}\left[u_{ic} \cdot \left(\hat{L}_i^{\theta_i} - \hat{L}_i\right)\right] < 0$ hold. ■

The fact that I_i^K and I_i^L are both negative tells us that the insurance effect associated to either a decrease in r or in w is an increase in individual welfare. To gain some economic intuition for this, notice that labor and capital constitute two alternative, 'risky' ways to provide wealth for future consumption. An increase in the market price of labor or of capital thus increases such risk and is so detrimental, the more so the riskier the instrument is.

A change in K, however, has the opposite effect on factor prices w and r. Hence the insurance effect of, say, an increase in K is given by the first and the third term in (12), which have respectively a positive and a negative sign, since $\frac{\partial r}{\partial K} < 0$ and $\frac{\partial w}{\partial K} > 0$. To determine which one prevails, notice that each of these terms will be smaller in absolute value the less random is the variable, $\hat{K}_i^{\theta_i}$ or $\hat{L}_i^{\theta_i}$, appearing in it, that is the less volatile is the return from the instrument considered to transfer wealth to the future. In particular, if ρ_i^K is a constant (i.e., there is no shock to capital accumulation) then $I_i^K = 0$, i.e., the insurance effect from the investment choice is zero. Consequently, since $\frac{\partial w}{\partial K} > 0$, a marginal increase in aggregate investment will *reduce* the induced utility of *every* household. So as far as the insurance effect is concerned, the households unanimously prefer a *reduction* of capital.

The remaining terms in (12), $\mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i - \hat{K}\right)$ and $\mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i - \hat{L}\right)$, describe the relationship between the (expected) marginal utility of type i and the deviation of his average supply of capital and labor from the aggregate average supply. For future reference, we shall denote these terms as follows:

$$D_i^K := \mathbf{E}\left[u_{ic}\right]\left(\hat{K}_i - \hat{K}\right), \tag{15}$$

$$D_i^L := \mathbf{E}\left[u_{ic}\right]\left(\hat{L}_i - \hat{L}\right), \tag{16}$$

where D stands for "distribution". They capture the effect of the price change on type i's utility that is due to the relative size of his trades in the market with respect to those of the whole economy, that is to the 'relative position' of type i in the market. Evidently, when the economy consists of ex ante homogeneous types, these terms will be zero, hence their magnitude depends on the degree of heterogeneity among consumers in the economy at the equilibrium.

Summing up, we have the following decomposition result:

Proposition 4 *The first order effect on the welfare of type i consumers at a competitive equilibrium of a policy can be decomposed into an insurance effect and a distribution effect as follows:*

$$\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} = \left\{I_i^K + D_i^K\right\} F_{KK} + \left\{I_i^L + D_i^L\right\} F_{KL},$$

$$\left.\frac{\partial U_i}{\partial L}\right|_{(\hat{K},\hat{L})} = \left\{I_i^K + D_i^K\right\} F_{LK} + \left\{I_i^L + D_i^L\right\} F_{LL},$$

where all terms are evaluated at the equilibrium values.

Proof. The expression of the derivative with respect to a change in K is obtained from (12) by substituting $\frac{\partial r}{\partial K}$ and $\frac{\partial w}{\partial K}$ with F_{KK} and F_{KL} and using (13) - (16). The next expression, for the change in L, is analogously obtained, adding (11) to (9), collecting terms as in (12), using then (13) - (16) and replacing $\frac{\partial r}{\partial L}$ and $\frac{\partial w}{\partial L}$ with F_{LK} and F_{LL}. ■

Remark 1 *The terms I_i^K, I_i^L, D_i^K, and D_i^L describe the marginal effects on consumers' utility of a unit change in the respective prices r, w,* at the equilibrium we are considering*. The total marginal effect will be the sum of these terms multiplied by the marginal change in prices. In the present section, the marginal change in prices is induced by the direct control of K and L, and hence is given by the second derivatives of the production function F as we have seen in the decomposition result above. In the next sections various other policy tools are considered, leading to different marginal changes in prices. Because of this common structure, the expression describing the welfare effects of such policies will be analogous to the one in Proposition 4: the same I_i^K, I_i^L, D_i^K, and D_i^L appear with different terms describing the price change multiplying them.*

Proposition 4 shows that the first order effect on agents' welfare of a change in the aggregate supply of capital or labor consists of the weighted sum of the insurance and the distribution effects. The only difference between the effect of a change in capital and labor is in the value of these weights, which are the derivatives of the production function. This point is better appreciated if we re-write the above decomposition expression using the Euler equations (10) and (11) as follows:

$$\left.\frac{\partial U_i}{\partial K}\right|_{(\hat{K},\hat{L})} = \hat{K}F_{KK}\left\{\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right\}, \tag{17}$$

$$\left.\frac{\partial U_i}{\partial L}\right|_{(\hat{K},\hat{L})} = \hat{L}F_{LL}\left\{\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right\}. \tag{18}$$

This decomposition result offers some clear insights on the relevant welfare effects. We give a few simple but interesting corollaries here. First of all, we observe that those who favor a decrease in the stock of capital are exactly those who favor an increase in the amount of labor:

Corollary 5 *For any type i, $\frac{\partial U_i}{\partial K} \leq 0$ if and only if $\frac{\partial U_i}{\partial L} \geq 0$.*

Proof. Compare (17) and (18): both F_{KK} and F_{LL} are negative by assumption, and the terms multiplying them are identical but with opposite sign. ■

Secondly, when the (absolute) magnitude of the distribution effects is bigger than that of the insurance effects, a reduction of the stock of capital benefits those types who invest more and work less than the economy average; vice versa an increase of capital.

Corollary 6 *When $\left|I_i^K\right| < \left|D_i^K\right|$,*[6] *$\frac{\partial U_i}{\partial K} < 0$ holds if $\hat{K}_i - \hat{K} > 0$ and $\hat{L}_i - \hat{L} < 0$. When $\left|I_i^L\right| < \left|D_i^L\right|$, $\frac{\partial U_i}{\partial K} > 0$ if $\hat{K}_i - \hat{K} < 0$ and $\hat{L}_i - \hat{L} > 0$.*

Proof. Note first that $\hat{K}_i - \hat{K} > 0$ and $\hat{L}_i - \hat{L} < 0$ imply that $D_i^K > 0$ and $D_i^L < 0$. Hence by Proposition 4 $\frac{\partial U_i}{\partial K} < 0$ follows when $\left|I_i^K\right| < \left|D_i^K\right|$, since $F_{KK} < 0$ and $F_{KL} > 0$. The second claim is established by a symmetric argument: when $\hat{K}_i - \hat{K} < 0$ and $\hat{L}_i - \hat{L} > 0$ we have $D_i^K < 0$ and $D_i^L > 0$ and hence $\frac{\partial U_i}{\partial K} > 0$ follows from $\left|I_i^L\right| < \left|D_i^L\right|$. ■

Finally, we show that, when the productivity of the investment is deterministic and consumers are ex ante homogeneous ($I = 1$), at an equilibrium that is standard to shocks we always have over investment and under supply of labor.[7] In the absence of ex ante heterogeneity among consumers the distribution effects, whose signs are in general ambiguous, are zero. Although we do not establish it formally here, the result can be readily extended to economies with consumers who are almost ex ante identical.

[6] This condition is always satisfied when there is no shock to the productivity of the investment, in which case $I_i^K = 0$.

[7] This claim generalizes a result in Davila et al. (2005), who only consider the case of inelastically supplied labor.

Corollary 7 *Assume that* $I = 1$ *and* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$. *Then at an equilibrium which is standard to shocks, a reduction of the stock of capital (if* $\hat{K} > 0$*) and/or an increase in the amount of labor (if* $\hat{L} < \bar{H}_i$*) improve consumers' utility. Thus an equilibrium which is standard to shocks is constrained inefficient and exhibits over investment.*

Proof. When $I = 1$, $\hat{K}_i = \hat{K}$ and $\hat{L}_i = \hat{L}$ by construction, and so $D_i^K = D_i^L = 0$. Also $\rho_i^K(\theta_i) = \gamma_i$ for all θ_i implies that the insurance effect for capital vanishes as well: i.e., $I_i^K = 0$. So by Proposition 4, $\frac{\partial U_i}{\partial K} = I_i^L F_{KL}$. This is negative since $F_{KL} > 0$ by the assumed properties of the technology and $I_i^L < 0$ by Lemma 3. Similarly for labor, assuming an interior solution, Proposition 4 shows that $\frac{\partial U_i}{\partial L} = I_i^L F_{LL} > 0$. ■

Remark 2 *We have thus established that, in economies with ex ante identical agents, there is over investment in equilibrium when the idiosyncratic shocks only affect the productivity of labor. Even though this finding may appear in line with the one by Aiyagari (1995) the notion of over investment used here is quite different. Moreover the logic behind it is also different. To see this, notice that the result holds irrespectively of whether a precautionary motive is present or not, and in fact the level of the equilibrium interest rate plays no role in the above arguments. What is crucial is, on the other hand, the structure of the shocks: the result is completely overturned when the idiosyncratic shocks only affect the productivity of capital.*

3.3 Social constrained optimality

The analysis in the previous section shows that, in the absence of heterogeneity among consumers, (standard) competitive equilibria are constrained inefficient and exhibit over investment. When consumers are sufficiently heterogeneous on the other hand, we do not know whether a Pareto improvement can still be found only by modifying K. This is because for some type i consumer the distribution effect may have the opposite sign and overturn the insurance effect, so that $\frac{\partial U_i}{\partial K} > 0$. If we consider a change both in K and L, a welfare improvement exists if we can find weights σ^K and σ^L, $\left(\sigma^K, \sigma^L\right) \neq 0$, such that the terms $\sigma^K \frac{\partial U_i}{\partial K} + \sigma^L \frac{\partial U_i}{\partial L} \geq 0$ have the same sign for every i. Notice however that when $I > 2$ even this condition is not easily met. Indeed, we will see in our numerical example that the equilibrium can in fact be constrained efficient.

Remark 3 *This is altogether in accord with the general constrained inefficiency result of Citanna - Kajii - Villanacci (1998): they show that with incomplete markets competitive equilibria can be Pareto improved (in terms of first order effects) if the planner has at least as many policy tools as the number of households plus one. In our framework, the number of policy tools which can have first order effects is effectively two,* K *and* L*, independent of the number of households. So their analysis can be compared to ours only when* $I = 1$*, in which case we have indeed established constrained inefficiency (even with only one policy tool).*

The above discussion reveals the difficulty of establishing general efficiency properties in the present framework for economies with heterogenous agents, if we use the Pareto efficiency criterion. It is then useful to allow for some sort of welfare comparison across households, adopting a weighted sum of utility functions, $W(K, L) := \sum_{i=1}^{I} \lambda_i U_i(K, L)$, as the social welfare function, where each $\lambda_i > 0$ is fixed exogenously. The choice of a social welfare function is of course arguable and influences the design of optimal policies. We do not attempt to justify this particular form of the social welfare function, unless we take an "under the veil of ignorance perspective", before the type of an agent is determined: each agent is assigned to any type i with equal probability. In such case, the ex ante welfare level of any individual is given by $W(K,L)$, with $\lambda_i = 1/I$ for every i. Such specification is also consistent with the view of the two period economy as a section of a dynamic economy where types are generated as the result of past realizations of productivity shocks. In what follows, we shall therefore focus primarily on this case.

We shall refer to the problem of maximizing the social welfare function by changing K or L as the *constrained social (ex ante) optimality* problem, whereas constrained efficiency, defined in the previous section, refers to the maximization of the expected utility of each individual type separately.

The derivative of the social welfare function W, evaluated at a competitive equilibrium, is simply the weighted sum of the derivatives of the individual utility function found in Proposition 4 as well as (17) and (18):

$$\left.\frac{\partial W}{\partial K}\right|_{(\hat{K},\hat{L})} = \sum_i \frac{1}{I}\left[\left\{I_i^K + D_i^K\right\}F_{KK} + \left\{I_i^L + D_i^L\right\}F_{KL}\right], \tag{19}$$
$$= \hat{K}F_{KK}\sum_i \frac{1}{I}\left[\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right].$$

$$\left.\frac{\partial W}{\partial L}\right|_{(\hat{K},\hat{L})} = \sum_i \frac{1}{I}\left[\left\{I_i^K + D_i^K\right\}F_{LK} + \left\{I_i^L + D_i^L\right\}F_{LL}\right], \tag{20}$$
$$= \hat{L}F_{LL}\sum_i \frac{1}{I}\left[\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right].$$

Definition 5 *A competitive equilibrium exhibits (ex ante)* ***over investment*** *if* $\frac{\partial W}{\partial K} < 0$*, and under investment if* $\frac{\partial W}{\partial K} > 0$*. Similarly, there is (ex ante)* ***under supply*** *of labor if* $\frac{\partial W}{\partial L} > 0$ *and over supply of labor if* $\frac{\partial W}{\partial L} < 0$.

Although the sign of the distribution effect term may vary as noticed across types, the economy average $\sum_i \frac{1}{I}D_i^K$ and $\sum_i \frac{1}{I}D_i^L$ might still be signed. To see this, think of assigning a type i to an agent at random; $\hat{K}_i$, $\hat{L}_i$ and $\mathbf{E}[u_{ic}]$ can thus be regarded as random variables over states $i = 1,..,I$ which are equally likely. Since $\sum_i \left(\hat{K}_i - \hat{K}\right) = 0$ and $\sum_i \left(\hat{L}_i - \hat{L}\right) = 0$, we have $\sum_i \frac{1}{I}D_i^K = Cov\left[\hat{K}_i, \mathbf{E}(u_{ic})\right]$ and $\sum_i \frac{1}{I}D_i^L = Cov\left[\hat{L}_i, \mathbf{E}(u_{ic})\right]$ by construction. We should expect that at a competitive equilibrium the relatively "rich" type of households whose consumption level tends to be higher than the economy average, also tend to invest more than the average and work less than the average.[8] This property relies on some normality of consumers' demands and so we shall use again the term 'standard' to refer to it:

Definition 6 *A competitive equilibrium is said be* ***standard in distribution*** *if* $\mathbf{E}[u_{ic}]$ *is negatively correlated with* $\hat{K}_i$ *and positively correlated with* $\hat{L}_i$*. When the equilibrium is standard both to shocks and in distribution, we simply call it a* ***standard equilibrium****.*

An immediate implication of this property is as follows:

Lemma 8 *If* $I > 1$*, in an equilibrium standard in distribution the average distribution effect of a change in the price of capital* r *is negative and that of a change in the price of labor* w *is positive: i.e.,* $\sum_i \frac{1}{I}D_i^K < 0$ *and* $\sum_i \frac{1}{I}D_i^L > 0$.

Hence at an equilibrium that is standard in distribution, the average distribution effect of an increase in K has a positive sign in (19). On the other hand, if there are no investment productivity

[8]This property holds for instance in the example we discuss later.

shocks, the average insurance effect has a negative sign in (19) since $I_i^L F_{KL} < 0$ for all i by Lemma 3. There is a clear trade-off: whether there is under or over investment from a social welfare perspective depends on whether or not the average distribution effect prevails over the average insurance effect. Intuitively, the distribution effect gets magnified as the heterogeneity of income across households increases, whereas the insurance effect has no direct link to the heterogeneity. So we can expect that there is *under investment* in terms of ex ante welfare in an equilibrium with large income disparity, and indeed we will see this in numerical examples. We summarize this observation below:

Proposition 9 *Assume that $\rho_i^K(\theta_i) = \gamma_i$ for all $\theta_i \in \Theta_i$. A standard equilibrium exhibits (ex ante) under investment if the average distribution effect is larger than the average insurance effect in the sense that $\sum_i \left|\frac{I_i^L}{L}\right| < \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$.*

Proof. The fact that the productivity of the investment is not subject to idiosyncratic shocks implies that $I_i^K = 0$ for every i. Using (19), we obtain that $\frac{\partial W}{\partial K} > 0$ if and only if $\sum_i \left(-\frac{I_i^L}{L} + \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right) < 0$. Since $I_L^i < 0$ by Lemma 3 and $\sum_i D_i^K < 0$ and $\sum_i D_i^L > 0$ by Lemma 8, under the assumed properties of the equilibrium, the claim follows. ■

Remark 4 *This result together with Corollary 7 show that, to determine whether or not there is over investment at an equilibrium, we should primarily look at the distribution of wealth across households. At an equilibrium where the income disparity is large enough, and hence the distribution effect is also large, we should expect that subsidizing capital is welfare improving. As already noticed in Remark 2, one can then deduce little concerning the direction of desirable policies from the observation that the level of the equilibrium interest rate is lower than with complete markets, that is if agents were able to trade in a complete set of contingent markets at the initial date.*

Remark 5 *Notice that our local characterization results go through even when there are no shocks at all, so that markets are complete. In this special case, competitive equilibria are Pareto efficient and so a change in capital or labor will result in an efficiency loss. Society's welfare can be improved nonetheless, since the social welfare function we consider favors income smoothing across the agents. So another interpretation of our results is that they clarify the sources of the trade-off between the efficiency and the equity measured in income distribution.*

4 Optimal Taxation

We analyzed so far whether agents' welfare at a competitive equilibrium might be improved by suitably reducing/increasing the aggregate level of capital and labor when the available policy tools consist in the direct control of the levels of individual investment and labor supply. We turn now our attention to the case where such variables can only be indirectly controlled via anonymous, linear taxes on labor and capital. The net revenue of such taxes is then redistributed to consumers via lump sum taxes or transfers. In this case, consumers choose optimally the level of their investment and labor supply, while the firm still chooses the level of its inputs so as to maximize profits and prices r and w are set at a level such that markets clear.

Various scenarios can be considered concerning the specific definition of the taxes and subsidies. We shall study a few cases, which will clarify the essence of the optimal taxation problem in our context and its relationship with the decomposition found in Proposition 4.

Throughout this section, we shall fix a competitive equilibrium $\left(\hat{w}, \hat{r}, \left(\hat{k}_i, \left(\hat{h}_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}\right)$, and study the first order effects of introducing a tax and subsidy scheme. In particular, we shall investigate whether capital and/or labor should be taxed from the point of view of the social welfare function.

4.1 Taxes on returns

We first look at the following tax/subsidy scheme. Denote by τ_K the tax rate on the revenue from the investment in capital, and τ_L the tax rate on labor income. When consumer i invests k_i and chooses $h_i^{\theta_i}$ in each state θ_i, an amount $w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i}$ of his revenue next period in state θ_i must be paid in taxes. The consumer also receives a lump sum transfer $T_i(\theta_i)$ in state θ_i. The choice problem of a type i consumer is then modified as follows:

$$\max_{k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}} v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(w(1-\tau_L)L_i^{\theta_i} + r(1-\tau_K)K_i^{\theta_i} + T_i(\theta_i), \bar{H}_i - h_i^{\theta_i}\right)\right], \tag{21}$$

where $K_i^{\theta_i}$ and $L_i^{\theta_i}$ are still as defined in (1) and (2). The maximization problem (21) remains a concave problem, and so the following the first order conditions characterize the maximizers:

$$-v_i'(e_i - k_i) + \mathbf{E}\left[u_{ic} \cdot \rho_i^K(\theta_i) r(1-\tau_K)\right] = 0. \tag{22}$$

$$u_{ic} \cdot w(1-\tau_L)\rho_i^L(\theta_i) - u_{il} = 0, \text{ at every } \theta_i, \tag{23}$$

where the derivatives of u_i are evaluated at $\left(w(1-\tau_L)L_i^{\theta_i} + r(1-\tau_K)K_i^{\theta_i} + T_i(\theta_i), \bar{H}_i - h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}$.

In order to isolate the pure substitution effect of the tax, we shall consider first a tax-subsidy scheme which does not induce any redistribution of income across agents nor even across realizations of the idiosyncratic state θ_i. That is, the tax subsidy scheme must satisfy the following budget balance condition for every realization of θ_i:

$$w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i} = T_i(\theta_i), \tag{24}$$

for every i. In other words, whatever an agent pays in taxes in any state θ_i he also gets back as a lump sum transfer in that same state. The implementation of this tax scheme is is informationally rather demanding as it requires knowledge of individual trades and of the realization θ_i of the idiosyncratic shocks, which may be private information of the agent. This scheme is thus not very realistic, but it offers a useful theoretical benchmark for the subsequent analyses.

Definition 7 *An* equilibrium with taxes and no redistribution nor insurance *is a collection* $(w, r, (\tau_K, \tau_L), \left(T_i(\cdot), k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I})$ *such that: i) for each i,* $\left(k_i, \left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)$ *is a solution to (21), ii) profit maximization holds, i.e.,* $F_K(K, L) = r$, $F_L(K, L) = w$, *where* $K = \frac{1}{I}\sum_{i=1}^{I}\gamma_i k_i$ *and* $L = \frac{1}{I}\sum_{i=1}^{I} E\left[L_i^{\theta_i}\right]$, *and iii) the budget balance (24) holds, for each* θ_i, i.

An ex ante optimal tax scheme (τ_K, τ_L)[9] is such that consumers' ex ante welfare is maximized at the associated competitive equilibrium. Formally, such scheme solves the following problem:[10]

$$\max_{w, r, \tau_K, \tau_L, \left(k_i, T_i, \left(h_i^{\theta_i}, K_i^{\theta_i}, L_i^{\theta_i}\right)_{\theta_i \in \Theta_i}\right)_{i=1}^{I}} \sum_i \left\{v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(w(1-\tau_L)L_i^{\theta_i} + r(1-\tau_K)K_i^{\theta_i} + T_i(\theta_i), \bar{H}_i - h_i^{\theta_i}\right)\right]\right\},$$

[9] The level of the lump sum transfers $T_i(.)$ is then uniquely determined by the budget balance condition (24).

[10] The expression below should be multiplied by the constant $1/I$. Since this clearly plays no role in the analysis it is then omitted, both here and in what follows.

subject to the equilibrium conditions (22), (23), (24), conditions (1) and (2) defining K_i^θ and L_i^θ, and the profit maximization conditions $w = F_L(K,L)$ and $r = F_K(K,L)$ evaluated at $(K,L) = \left(\sum_i \gamma_i k_i/I, \sum_i \mathbf{E}\left[L_i^\theta\right]/I\right)$. Equivalently, using (24) the objective function of this problem can be simplified as follows:

$$\sum_{i=1}^{I} \left\{ v_i(e_i - k_i) + \mathbf{E}\left[u_i\left(wL_i^{\theta_i} + rK_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i} \right) \right] \right\}, \tag{25}$$

In what follows we shall denote this term as $W(\tau_K, \tau_L)$, to highlight the dependence of social welfare on the parameters describing the tax policy.

By construction, a competitive equilibrium is an equilibrium with taxes, where $\tau_K = \tau_L = 0$, and $T_i(\cdot) \equiv 0$ for every i. We intend to examine in particular whether $W(\tau_K, \tau_L)$ is increasing in τ_K and/or τ_L at $\tau_K = \tau_L = 0$. This will allow us to conclude that at least locally a positive tax on the realized return on capital/labor is welfare improving. To this end we assume the variables at an equilibrium with taxes are smooth functions of (τ_K, τ_L) around $(\tau_K, \tau_L) = 0$.[11] Hence, differentiating $W(\tau_K, \tau_L)$ and evaluating it at $\tau_K = \tau_L = 0$ we shall say that capital should be taxed if $\frac{\partial}{\partial \tau_K} W(\tau_K, \tau_L) > 0$ and it should be subsidized if $\frac{\partial}{\partial \tau_K} W(\tau_K, \tau_L) < 0$. A similar analysis can be done for labor.

Note that the envelope property from the individual optimization applies here again. Also note that the profit maximization condition must hold at any choice of tax rates, hence the Euler equation gives us relations analogous to (10) and (11):

$$\frac{\partial r}{\partial \tau_K} \cdot K + \frac{\partial w}{\partial \tau_K} \cdot L = 0, \tag{26}$$

$$\frac{\partial r}{\partial \tau_L} \cdot K + \frac{\partial w}{\partial \tau_L} \cdot L = 0. \tag{27}$$

We obtain so a decomposition result for the effects of the introduction of the taxes which resembles our findings in Proposition 4 and equation (19), except that the changes in equilibrium prices are induced by a change in τ_K and τ_L, not by a direct change of K and L:

Proposition 10 *The (first order) welfare effects of the introduction of taxes with no redistribution nor insurance at a competitive equilibrium can be decomposed as follows:*

$$\left.\frac{\partial W}{\partial \tau_K}\right|_{\tau=0} = \sum_i \left\{ \left(I_i^K + D_i^K\right) \frac{\partial r}{\partial \tau_K} + \left(I_i^L + D_i^L\right) \frac{\partial w}{\partial \tau_K} \right\}, \tag{28}$$

$$= \hat{K} \frac{\partial r}{\partial \tau_K} \sum_i \left\{ \left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}} \right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}} \right) \right\},$$

$$\left.\frac{\partial W}{\partial \tau_L}\right|_{\tau=0} = \sum_i \left\{ \left(I_i^K + D_i^K\right) \frac{\partial r}{\partial \tau_L} + \left(I_i^L + D_i^L\right) \frac{\partial w}{\partial \tau_L} \right\}, \tag{29}$$

$$= \hat{L} \frac{\partial w}{\partial \tau_L} \sum_i \left\{ \left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}} \right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}} \right) \right\},$$

where the terms I_i^K, I_i^L, D_i^K, and D_i^L are still as in (13), (14), (15), and (16).

Proof. Since individual income in every state is not affected by the tax scheme, by the envelope property the only first order effect of the scheme on consumers' utility is given by the change in equilibrium prices. Hence $L_i^{\theta_i}$, $K_i^{\theta_i}$, $h_i^{\theta_i}$ and k_i can be treated as constants and w, r as - differentiable

[11] This will be generically the case at least if the underlying state space is finite. The number of equilibrium variables exceeds in fact the number of equations defining an equilibrium by two.

by assumption - functions of (τ_K, τ_L). So we can do exactly the same operations as we did for the decomposition formula in Proposition 4 and (17) and (18), replacing $\frac{\partial r}{\partial K}$, $\frac{\partial r}{\partial L}$, $\frac{\partial w}{\partial K}$, and $\frac{\partial w}{\partial L}$ with $\frac{\partial r}{\partial \tau_K}$, $\frac{\partial r}{\partial \tau_L}$, $\frac{\partial w}{\partial \tau_K}$, and $\frac{\partial w}{\partial \tau_L}$, respectively. ■

This time we need first to identify the signs of the changes in equilibrium prices in order to determine the sign of the welfare effects in (28) and (29). In general, prices could move in any direction, depending on the signs of the derivatives of the excess demand functions for capital and labor. We shall assume so that prices change in the *natural direction*: at the margin, an increase in the tax on the revenue from the sale of an input increases the gross revenue (i.e., cum tax) of the input but reduces the net revenue (i.e., net of tax). That is, we assume the following signs:

$$\left.\frac{\partial r(\tau_K, \tau_L)}{\partial \tau_K}\right|_{\tau=0} > 0, \quad \left.\frac{\partial}{\partial \tau_K}\left[(1-\tau_K) r(\tau_K, \tau_L)\right]\right|_{\tau=0} < 0 \tag{30}$$

$$\left.\frac{\partial w(\tau_K, \tau_L)}{\partial \tau_L}\right|_{\tau=0} > 0, \quad \left.\frac{\partial}{\partial \tau_L}\left[(1-\tau_L) w(\tau_K, \tau_L)\right]\right|_{\tau=0} < 0 \tag{31}$$

We shall refer to (30) and (31) as the *natural signs* for the changes in equilibrium factor prices.

We obtain a corollary similar to Corollary 5, which says whenever it is good to tax capital, it should be good to subsidize labor as well.

Corollary 11 *Assume the natural signs as above. Then taxing capital and taxing labor has opposite effects on welfare:* $\frac{\partial W}{\partial \tau_K} \geq 0$ *if and only if* $\frac{\partial W}{\partial \tau_L} \leq 0$.

We can thus focus our attention on identifying the conditions under which capital should be taxed. The next result is an analogue of Proposition 9:

Proposition 12 *Assume the natural signs (30) and (31). Then capital should be taxed at a standard competitive equilibrium with taxes and no insurance nor redistribution if and only if there is over investment. Suppose, in addition, that* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$ *and the competitive equilibrium is standard. Then capital should be taxed if* $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$ *and subsidized if the reverse inequality holds; when* $I = 1$*, capital should always be taxed.*

Proof. Under (30), $\partial r / \partial \tau_K$ has always the opposite sign of $\partial r / \partial K$. Hence the same is true for the expression for $\frac{\partial W}{\partial \tau_K}$ in (28) and that for $\frac{\partial W}{\partial K}$ in (19), which establishes the first claim. Given this, the following claims are an immediate corollary of Proposition 9 and Corollary 7. ■

Remark 6 *Proposition 12 says that the idea of taxing an over used input is correct. And, as noticed in Remark 4, the determination of whether a positive tax is beneficial or not depends primarily on the comparison between insurance and distribution effects, not on the level of the equilibrium price of an input.*

4.2 Lump-sum rebate as insurance

We consider next the case where there is still a linear tax on labor and capital income but the lump sum rebate is deterministic. The tax paid by a type i consumer equals $w\tau_L L_i^{\theta_i} + r\tau_K K_i^{\theta_i}$ in each state $\theta_i \in \Theta_i$. By the i.i.d. assumption, the per capita tax paid by type i consumers is deterministic and equal to $w\tau_L L_i + r\tau_K K_i$, hence budget balance is still ensured with a deterministic lump sum rebate T_i satisfying:

$$w\tau_L L_i + r\tau_K K_i = T_i, \tag{32}$$

for every i. The rebate has in this case an insurance effect, as the difference between the tax paid and the rebate received is positive whenever the return on capital and labor exceeds its mean and negative otherwise. Although ex ante types need to be observable for this scheme, the informational requirement is less demanding than in the previous case since the rebate is determined independently of the realization θ_i of the individual shock.

The choice problem of a consumer of type i for this case is given as follows:

$$\max_{k_i,\left(h_i^{\theta_i}\right)_{\theta_i \in \Theta_i}} \quad v_i\left(e_i - k_i\right) + \mathbf{E}\left[u_i\left(w\left(1-\tau_L\right)L_i^{\theta_i} + r\left(1-\tau_K\right)K_i^{\theta_i} + T_i, \bar{H}_i - h_i^{\theta_i}\right)\right],$$

and an *equilibrium with taxes and insurance but no redistribution* can be defined, analogously to Definition 7, by suitably replacing the expression of the consumers' objective function in (21) with the one above and the budget balance condition (24) with (32). By proceeding in the same way as in the previous section, we find that the objective function of the optimal taxation problem has now the following form:

$$\sum_i \left\{v_i\left(e_i-k_i\right) + \mathbf{E}\left[u_i\left(wL_i^{\theta_i} + rK_i^{\theta_i} - \left\{w\tau_L\left(L_i^{\theta_i}-L_i\right) + r\tau_K\left(K_i^{\theta_i}-K_i\right)\right\}, \bar{H}_i - h_i^{\theta_i}\right)\right]\right\}, \tag{33}$$

and shall similarly denote it as $W^I\left(\tau_K,\tau_L\right)$, where the superscript I highlights the new, insurance component.

Assume again the competitive equilibrium we are considering is a regular equilibrium in (τ_K,τ_L), so that the equilibrium variables are smooth functions of (τ_K,τ_L) around $(\tau_K,\tau_L)=(0,0)$. We shall use again the superscript I - e.g. $r^I\left(\tau_K,\tau_L\right)$, $w^I\left(\tau_K,\tau_L\right)$ - to indicate that these functions are different from before as the equilibrium system is different. Differentiating $W^I\left(\tau_K,\tau_L\right)$ with respect to τ_K and τ_L and evaluating it at $\tau_K=\tau_L=0$ yields the following expression:

$$\left.\frac{\partial W^I}{\partial \tau_K}\right|_{\tau=0} = \sum_i \left\{\left(I_i^K + D_i^K\right)\frac{\partial r^I}{\partial \tau_K} + \left(I_i^L + D_i^L\right)\frac{\partial w^I}{\partial \tau_K} - \mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i} - \hat{K}_i\right)\right]\hat{r}\right\} \tag{34}$$

$$= \sum_i \left\{I_i^K\left(\frac{\partial r^I}{\partial \tau_K} - \hat{r}\right) + D_i^K\frac{\partial r^I}{\partial \tau_K} + \left(I_i^L + D_i^L\right)\frac{\partial w^I}{\partial \tau_K}\right\}$$

$$\left.\frac{\partial W^I}{\partial \tau_L}\right|_{\tau=0} = \sum_i \left\{\left(I_i^K + D_i^K\right)\frac{\partial r^I}{\partial \tau_L} + I_i^L\left(\frac{\partial w^I}{\partial \tau_L} - \hat{w}\right) + D_i^L\frac{\partial w^I}{\partial \tau_L}\right\}. \tag{35}$$

Comparing these expressions with the ones obtained in the previous section, (28) and (29), we see that there is an additional term in each of them. This is due to the fact that the expression of second period consumption in (33) differs from the one in (25) for the presence of the term $-\left\{w\tau_L\left(L_i^{\theta_i}-L_i\right) + r\tau_K\left(K_i^{\theta_i}-K_i\right)\right\}$. The derivative of this additional term with respect to prices, when evaluated at $\tau_L=\tau_K=0$, is zero - hence this term does not contribute to the price effect - but the derivative with respect to taxes is nonzero, $-w\left(L_i^{\theta_i}-L_i\right)$ and $-r\left(K_i^{\theta_i}-K_i\right)$ respectively. Hence the direct effects on agents' utility of a change in taxes, at the margin, do not vanish, and they are equal to $-\mathbf{E}\left[u_{ic}\cdot\left(\hat{L}_i^{\theta_i}-\hat{L}_i\right)\right]\hat{w}$ and $-\mathbf{E}\left[u_{ic}\cdot\left(\hat{K}_i^{\theta_i}-\hat{K}_i\right)\right]\hat{r}$, which are nothing but the insurance effects, I_i^L and I_i^K, multiplied by the opposite of the respective factor prices.

Since I_i^K and I_i^L are both negative by Lemma 3, we conclude that the additional term in both (34) and (35) is positive. Hence the claim in Corollary 11 is not valid in the present situation and it is possible that both the optimal tax on capital and that on labor are positive.

Somewhat in contrast to Proposition 12 we find that, under the same conditions, the optimal tax on labor is always positive when tax rebates have an insurance role. On the other hand, the properties of the sign of the optimal tax on capital are unchanged.

Proposition 13 *Assume the natural signs*[12] *(30) and (31), and that* $\rho_i^K(\theta_i) = \gamma_i$ *for all* $\theta_i \in \Theta_i$*. At a standard competitive equilibrium with taxes and insurance but no redistribution, labor should always be taxed while capital should be taxed whenever there is overinvestment, that is if* $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$*, and subsidized otherwise. If* $I = 1$*, capital should always be taxed.*

Proof. No productivity shock for capital implies $I_i^K = 0$ for every i. At a standard equilibrium, by Lemma 8 the average distribution effects are respectively negative and positive, $\sum_i D_i^K < 0$ and $\sum_i D_i^L > 0$, while by Lemma 3 the insurance effect I_i^L is negative for all i. The natural sign assumption means then $\hat{w} > \frac{\partial w^I}{\partial \tau_L} > 0$ and so also $\frac{\partial r^I}{\partial \tau_L} < 0$ by the profit maximization conditions $r = F_K$, $w = F_L$. Hence all the three terms in (35) are positive, which establishes the first claim.

Comparing (34) with (28), we see they differ only for the term multiplying I_i^K. Since by assumption $I_i^K = 0$ for all i, the derivative with respect to τ_K has the same form at an equilibrium without and with insurance. So the result follows from Proposition 12. ■

Remark 7 *One might wonder why labor should be taxed even when there is (ex ante) under supply of labor. The argument of the proof of Proposition 9 implies that, when* $\sum_i \left|\frac{I_i^L}{L}\right| > \left|\sum_i \left(\frac{D_i^K}{K} - \frac{D_i^L}{L}\right)\right|$ *there is ex ante over investment and hence also, by Corollary 5, under supply of labor. But in the present situation the lump sum tax rebate provides insurance against private idiosyncratic risks. This insurance kicks in only if labor is taxed since the idiosyncratic shocks only affect labor productivity if* $\rho_i^K(\theta_i) = \gamma_i$ *for all* θ_i*. The result in Proposition 13 shows that, under the natural sign assumption, the benefits from such direct insurance exceeds the welfare loss from further discouraging already under supplied labor.*

4.3 Lump-sum rebate as insurance and redistribution

Next, we shall consider the case where the lump sum rebate is not only deterministic but also the same for all types. Then the per capita rebate equals the average tax payment across types:

$$w\tau_L L + r\tau_K K = T. \tag{36}$$

In this case the rebate has not only an insurance role, with respect to the individual shocks, but also a role of redistributing wealth across different types. Notice that this scheme relies on neither private signals nor ex ante types, and hence it is completely anonymous.

An *equilibrium with taxes and insurance as well as redistribution* is then similarly defined, by suitably replacing T with $T_i(\theta_i)$ in (21) and the budget balance condition (24) with (36). The objective function of the optimal taxation problem becomes:

$$\sum_i \left\{ v_i\left(e_i - k_i\right) + \mathbf{E}\left[u_i\left(c_i^{\theta_i}, \bar{H}_i - h_i^{\theta_i}\right)\right]\right\}, \tag{37}$$

where for each i, the second period consumption level is given by

$$c_i^{\theta_i} = wL_i^{\theta_i} + rK_i^{\theta_i} - \left\{w\tau_L\left[\left(L_i^{\theta_i} - L_i\right) + (L_i - L)\right] + r\tau_K\left[\left(K_i^{\theta_i} - K_i\right) + (K_i - K)\right]\right\}, \tag{38}$$

and will be denoted by $W^{IR}(\tau_L, \tau_K)$ where R marks the new, redistribution element of the tax scheme.

[12] Strictly speaking, the natural sign assumption in the present framework should be stated by replacing $\frac{\partial r}{\partial \tau_K}$ and $\frac{\partial w}{\partial \tau_L}$ in (30) and (31) with $\frac{\partial r^I}{\partial \tau_K}$ and $\frac{\partial w^I}{\partial \tau_L}$ to reflect the fact that the equilibrum price maps are different. With a slight abuse of language we avoid to make this explicit, here and in what follows.

Assuming again that the equilibrium variables are smooth functions of (τ_L, τ_K) at $(0,0)$,[13] we study under what conditions $W^{IR}(\tau_L, \tau_K)$ is increasing in τ_K and/or τ_L at $\tau_K = \tau_L = 0$, to conclude that a positive tax on the realized return on capital and/or on labor is welfare improving. The expression of the derivatives of the ex ante welfare function is now:

$$\begin{aligned}\left.\frac{\partial W^{IR}}{\partial \tau_K}\right|_{\tau=0} &= \sum_i \left\{ \left(I_i^K + D_i^K\right) \frac{\partial r^{IR}}{\partial \tau_K} + \left(I_i^L + D_i^L\right) \frac{\partial w^{IR}}{\partial \tau_K} - \mathbf{E}\left[u_{ic}\left(K_i^{\theta_i} - K_i\right)\right]\hat{r} - \mathbf{E}\left[u_{ic}\right](K_i - K)\hat{r}\right\} \\ &= \sum_i \left\{ \left(I_i^K + D_i^K\right)\left(\frac{\partial r^{IR}}{\partial \tau_K} - \hat{r}\right) + \left(I_i^L + D_i^L\right)\frac{\partial w^{IR}}{\partial \tau_K}\right\} \\ &= \hat{K}\frac{\partial r}{\partial \tau_K}\sum_i\left\{\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right\} - \hat{r}\sum_i\left(I_i^K + D_i^K\right)\end{aligned} \tag{39}$$

$$\begin{aligned}\left.\frac{\partial W^{IR}}{\partial \tau_L}\right|_{\tau=0} &= \sum_i \left\{ \left(I_i^K + D_i^K\right) \frac{\partial r^{IR}}{\partial \tau_L} + \left(I_i^L + D_i^L\right)\left(\frac{\partial w^{IR}}{\partial \tau_L} - \hat{w}\right)\right\} \\ &= \hat{L}\frac{\partial w}{\partial \tau_L}\sum_i\left\{\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right\} - \hat{w}\sum_i\left(I_i^L + D_i^L\right).\end{aligned} \tag{40}$$

They only differ from the corresponding terms in the previous section, (34) and (35), for the presence of an additional term in each of them, respectively $-\sum_i \{\mathbf{E}[u_{ic}](K_i - K)\}\hat{r}$ and $-\sum_i \{\mathbf{E}[u_{ic}](L_i - L)\}\hat{w}$. This is due to the fact that the expression for $c_i^{\theta_i}$ in (38) also has two additional terms, $-w\tau_L(L_i - L) - r\tau_K(K_i - K)$, which describe the redistributive component of the tax rebate. Differentiating them with respect to taxes and evaluating the effect on agents' utility yields[14] $-\hat{r}\mathbf{E}[u_{ic}](K_i - K)$ and $-\hat{w}\mathbf{E}[u_{ic}](L_i - L)$, which are equal to $-\hat{r}D_i^K$ and $-\hat{w}D_i^L$. Using Lemma 8 we can then say that the average of these terms, constituting the new terms in (34) and (35), has respectively a positive and a negative sign. That is, the new redistributive effect of the tax scheme strengthens the case for taxing capital and weakens that for taxing labor.

Proposition 14 *Assume the natural signs (30), (31) and that $\rho_i^K(\theta_i) = \gamma_i$ for all $\theta_i \in \Theta_i$. At a standard competitive equilibrium with taxes and insurance as well as redistribution, both capital and labor should be taxed if $\left|\sum_i I_i^L\right| > \left|\sum_i D_i^L\right|$, i.e., the distribution effects of labor is larger than the insurance effects of labor. Moreover, capital should be taxed if there is over investment.*

Proof. Under the stated assumptions, for the same argument as in the proof of Proposition 13, we have again $I_i^K = 0$ and $I_i^L < 0$ for every i, and $\sum_i D_i^K < 0$ and $\sum_i D_i^L > 0$. Since the natural signs assumption implies $\left(\frac{\partial r^{IR}}{\partial \tau_K} - \hat{r}\right) < 0$ and $\frac{\partial w^{IR}}{\partial \tau_K} < 0$, from (39) we see that $\left.\frac{\partial W^{IR}}{\partial \tau_K}\right|_{\tau=0} > 0$ if $\sum_i\left(I_i^L + D_i^L\right) < 0$, i.e., $\left|\sum_i I_i^L\right| > \left|\sum_i D_i^L\right|$. Moreover, since the natural signs assumption also implies $\left(\frac{\partial w^{IR}}{\partial \tau_L} - \hat{w}\right) < 0$ and $\frac{\partial r^{IR}}{\partial \tau_L} < 0$, from (40) we see that $\left.\frac{\partial W^{IR}}{\partial \tau_L}\right|_{\tau=0} > 0$ follows from $\sum_i\left(I_i^L + D_i^L\right) < 0$. ■

Compare the condition, obtained from (39), for the positivity of the optimal tax on capital in the present framework, with the one for over investment we get from (19). Since both $F_{KK} > 0$ and $\frac{\partial r^{IR}}{\partial \tau_K} > 0$ we see the condition is now weaker, thus the optimal tax on capital is always positive when there is over investment (e.g., when $I = 1$), but may also be positive also when there is under investment (in contrast with Propositions 12 and 13).

Capital taxation, as we saw, is beneficial when the average insurance effect prevails over the average distribution effect. When the lump sum rebate is equal for all types, the tax has also a redistributive element, since wealthier consumers tend to have a higher income from capital. Hence the tax on capital effectively creates an income transfer from wealthier to poorer consumers, which is beneficial from the point of view of ex ante welfare.

[13] Equilibrium price maps are similarly denoted as $w^{IR}(\tau_L, \tau_K)$, $r^{IR}(\tau_L, \tau_K)$.

[14] This is only the direct effect of the change in τ_L, τ_K. For the same argument as in the previous section the price effect, when evaluated at $(\tau_L, \tau_K) = (0,0)$ is zero.

We showed in Proposition 13 that taxing labor is beneficial when the tax rebate has an insurance effect. But when the rebate is equal for all types, and hence has an additional redistributive effect, this works in the opposite direction since it effectively takes income away from the poor to the rich through the equal rebate. So if the distribution effect $\sum_i D_i^L$ is large enough, labor should rather be subsidized.

4.4 Taxing capital or labor?

In the previous analysis the effect of the tax on a factor's income was combined with the effect of the lump sum rebate and the latter played an important role in the results, especially when we allowed for an insurance and a redistribution role for the rebate. The following natural question then arises: if the government cannot generate such lump sum transfers, should we tax capital or rather labor? The budget balance condition, in the absence of lump sum rebates, becomes:

$$w\tau_L L + r\tau_K K = 0. \tag{41}$$

Therefore the two tax rates can no longer be independently set: if τ_K is positive τ_L has to be negative at the level needed to satisfy (41). The objective function of the optimal taxation problem is then directly obtained from the function $W^{IR}(.)$ considered in the previous section, simply by setting $\tau_L = -\tau_K \frac{rK}{wL}$: $W^{IR}\left(\tau_K, -\tau_K \frac{rK}{wL}\right)$. Its derivative with respect to τ_K is then $\frac{\partial W^{IR}}{\partial \tau_K} - \frac{\hat{r}\hat{K}}{\hat{w}\hat{L}} \cdot \frac{\partial W^{IR}}{\partial \tau_L}$, and its expression can be directly obtained from (39) and (40). Noting that

$$-\hat{r}\sum_i \left(I_i^K + D_i^K\right) + \left(\frac{\hat{r}\hat{K}}{\hat{w}\hat{L}}\right)\hat{w}\sum_i \left(I_i^L + D_i^L\right) = \hat{r}\hat{K}\sum_i \left\{\left(\frac{I_i^L}{\hat{L}} - \frac{I_i^K}{\hat{K}}\right) + \left(\frac{D_i^L}{\hat{L}} - \frac{D_i^K}{\hat{K}}\right)\right\},$$

we get[15]:

$$\left.\frac{dW^{IR}}{d\tau_K}\right|_{\tau=0} = \hat{K}\hat{r}\left(\frac{\partial r^{IR}}{\partial \tau_K}/\hat{r} + \frac{\partial w^{IR}}{\partial \tau_L}/\hat{w} - 1\right)\sum_i \left\{\left(\frac{I_i^K}{\hat{K}} - \frac{I_i^L}{\hat{L}}\right) + \left(\frac{D_i^K}{\hat{K}} - \frac{D_i^L}{\hat{L}}\right)\right\}. \tag{42}$$

Notice that the sum in the above expression is identical to the one appearing in (19) and has a negative sign if, and only if, we have ex ante under investment. The term premultiplying it, $\hat{r}\left(\frac{\partial r^{IR}}{\partial \tau_K}/\hat{r} + \frac{\partial w}{\partial \tau_L}/\hat{w} - 1\right)$, describes the effect of a marginal increase of the tax on capital on the net revenue $\hat{r}(1-\tau_K)$ from the sale of capital. This is because now, when τ_K varies, τ_L also varies, so as to satisfy (41). Hence, at the margin $\frac{d\tau_L}{d\tau_K} = -\left(\frac{\hat{r}\hat{K}}{\hat{w}\hat{L}}\right)$ and

$$\begin{aligned}\left.\frac{d\left[r(1-\tau_K)\right]}{d\tau_K}\right|_{\tau=0} &= \frac{\partial r^{IR}}{\partial \tau_K} - \frac{\partial r^{IR}}{\partial \tau_L}\left(\frac{\hat{r}\hat{K}}{\hat{w}\hat{L}}\right) - \hat{r} \\ &= \frac{\partial r^{IR}}{\partial \tau_K} + \frac{\partial w^{IR}}{\partial \tau_L}\left(\frac{\hat{r}}{\hat{w}}\right) - \hat{r},\end{aligned}$$

where the second equality follows from the Euler equation and the profit maximization conditions, which imply: $\frac{\partial r}{\partial \tau_L} = -\frac{\partial w}{\partial \tau_L}\frac{\hat{L}}{\hat{K}}$.

One of the conditions stated in the natural sign assumption ((30) and (31)) says that, when the tax revenue is rebated with a lump sum transfer, the net revenue from the sale of a factor decreases if the tax on the factor increases. The condition that $\left(\frac{\partial r^{IR}}{\partial \tau_K}/\hat{r} + \frac{\partial w}{\partial \tau_L}/\hat{w} - 1\right) < 0$ says the same property holds for the tax on capital in the present framework, where the revenue from such tax is rebated by a suitably defined subsidy on the sales of labor. The condition, a little stronger than

[15] In particular, from the last expressions in (39) and (40).

(30), holds if factor prices are not too sensitive to the introduction of marginal taxes, and one can expect $\frac{\partial r}{\partial \tau_K}/\hat{r} + \frac{\partial w}{\partial \tau_L}/\hat{w} - 1 < 0$ holds in typical examples.

The previous discussion establishes the following, somewhat surprising result:

Proposition 15 *Assume that* $d\left[r^{IR}(1-\tau_K)\right]/d\tau_K\big|_{\tau=0} < 0$. *If no lump sum transfer is allowed, capital should be* ***taxed*** *if and only if there is* ***under*** *investment.*

The sign of the optimal tax on capital is thus exactly the opposite of what we found in Proposition 12 and, partly, also in Proposition 13. So when there is no productivity shock for the investment in capital and $I = 1$, capital should be *subsidized* while we know there is over investment.

To gain some intuition for this result, recall first our previous finding that, when taxes have no insurance nor redistribution components, it is optimal to tax capital and to subsidize labor if and only if there is over investment. The presence of an insurance component in the tax scheme strengthens the case for taxing both capital and labor, the more so the riskier is the return of the factor. Finally, when a redistribution component is also present, this strengthens the case for taxing capital and weakens that for taxing labor. In the present environment, the tax scheme satisfying (41) still entails a change in consumers' disposable income, equal to $\hat{K}\hat{r}\left(L_i^{\theta_i}/\hat{L} - K_i^{\theta_i}/\hat{K}\right)d\tau_K$. This is equal to the sum of the changes in disposable income we have with a marginal increase in the tax on capital and corresponding decrease in the tax on labor, when the tax scheme has both an insurance and a redistribution component. The condition $\left(\frac{\partial r^{IR}}{\partial \tau_K}/\hat{r} + \frac{\partial w}{\partial \tau_L}/\hat{w} - 1\right) < 0$ says that the overall welfare effect is primarily determined by the income effect, over the effect of the price change. For instance, when $I = 1$ and there are no shocks to the investment in capital, there is obviously no redistribution component and the insurance component only strengthens the case for taxing labor. In such case in fact, the marginal change in disposable income for an increase in τ_K reduces to $\hat{K}\hat{r}\left(L^\theta/\hat{L} - 1\right)$ and its effect on consumers' utility equals $I_L\hat{K}\hat{r}/\hat{L} < 0$.

5 Numerical Example

In this section we consider a simple numerical example for which we derive the level of the optimal capital and labor tax rates for the different types of tax transfer schemes investigated in the previous section. In this framework we will also illustrate the findings of the local analysis carried out in the previous section, and compare them to the globally optimal tax rates. In principle, there is little reason to believe that the local information around the competitive equilibrium is sufficient to identify the properties of the optimal tax rates in the general set up we considered. But it will be seen that, under the functional forms and the specification of parameters which are commonly used in the literature, the results of the local analysis turn out to be useful to infer the properties of the global maximum.

There are two types of consumers and so $I = 2$. Consumers have the same preferences and second period endowments, they only differ for their initial endowments. We set $e_1 > e_2$ without loss of generality. So type 1 consumers are richer than type 2 consumers and we shall refer to the first ones as *rich* and the second ones as *poor*. There are two equally likely individual states, $\Theta = \{\theta_H, \theta_L\}$, and the common expected utility is

$$U\left(c^0, \left(c^\theta, h^\theta\right)_{\theta\in\Theta}\right) = \frac{1}{1-\sigma}\left(c^0\right) + \mathbf{E}\left[\frac{B}{1-\sigma}\left(c^\theta\right)^{1-\sigma} - \frac{\chi}{1+\varphi}\left(h^\theta\right)^{1+\varphi}\right]$$

where c^0 is consumption in the first period, and c^θ and h^θ are consumption and labor supply in the second period in state θ. There is no shock to the productivity of investments in capital and the shock to the productivity of labor is the identity map:

$$\rho_i^K(\theta) = 1, \qquad \text{and} \qquad \rho_i^L(\theta) = \theta \text{ for all } \theta \in \Theta$$

The production technology is Cobb-Douglas:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}.$$

The values of the parameters are set as follows: $\sigma = 3$, $\varphi = 1$, $B = 8$, $\chi = 2.5$, $A = 1$, $\alpha = 0.36$. We also fix the average value of the labor productivity shocks, $\bar{\theta}$, at unity, and the average initial endowment, $\bar{e} = (e_1 + e_2)/2$, at 3.7162,[16] while allowing for different values for the magnitude of the shocks, identified by θ_H, and the degree of heterogeneity, identified by $e_1/\bar{e}$. Note that under all parameter configurations considered, our example yields a standard equilibrium, and the sign conditions assumed in Propositions 12, 13, 14, and 15 are satisfied.

To start with, Figure 1 shows how market incompleteness affects capital accumulation and labor supply. There we fix distribution at $e_1/\bar{e} = 1.42$ and $e_2/\bar{e} = 0.58$, which implies a standard deviation of about 60 percent. Then we let θ_H vary from 1 to 1.4 with $\theta_L = 2 - \theta_H$. Thus the standard deviation of the idiosyncratic shock θ changes from zero to about 57 percent. The solid lines in the two panels of Figure 1 portray, respectively, the aggregate capital stock and the aggregate labor supply at the competitive equilibrium of the economy for the different values of the standard deviation. The dotted lines in the same panels depict also the aggregate capital and labor supply but at a competitive equilibrium where agents can trade in a complete market for contingent claims. We see that aggregate capital is greater with incomplete asset markets than with complete markets (due to the precautionary saving motive exhibited by the utility function considered). On the other hand, aggregate labor supply is lower, due to the income effect caused by the higher aggregate stock of capital. Moreover, the difference between the values with incomplete and complete markets is larger when the standard deviation of the idiosyncratic shocks is also larger, that the larger the uninsurable shocks.

In the subsequent figures we fix the values of the idiosyncratic shocks at $\theta_H = 1.2$ and $\theta_L = 0.8$, and let the standard deviation of the initial endowments, $e_i/\bar{e}$, vary from zero to 45 percent. Figure 2 plots $\partial W/\partial \tau_K$ and $\partial W/\partial \tau_L$ evaluated at $\tau_K = \tau_L = 0$, that is, the marginal effects on ex ante welfare of introducing taxation on capital and labor income at the competitive equilibrium, which are the objects of our local analyses. Figure 3 plots the optimal tax rates, that is, the tax rates that maximize the ex ante welfare $W(\tau_K, \tau_L)$. In both figures, we examine alternative specifications of the lump-sum transfers as considered in the previous section.

In each of the two figures the north-west panel corresponds to the tax scheme with no redistribution nor insurance discussed in Section 4.1 (see equation (24)). Under this tax scheme, as shown in Proposition 12, taxing capital or subsidizing labor is welfare enhancing (marginally at the competitive equilibrium), when the average distribution effect is relatively small. In the example here, the average distribution effect increases as the inequality in the initial endowments rises. This can be seen in the north-west panel of Figure 2: $\partial W/\partial \tau_K > 0$ and $\partial W/\partial \tau_L < 0$ when the inequality in initial distribution is sufficiently small, and vice versa when it is large. This local result is in accord with the optimal tax rates, shown in the north-west panel of Figure 3. When the standard deviation of income distribution is close to zero the optimal tax rate on capital is around 2 percent.

[16] These parameter values are mostly in line with those used in the macroeconomics literature. An exception may be the value for B. In the two-period economy this value needs to be larger than what is assumed in infinite-horizon models in order to make the capital-output ratio around three. The value of the average e is chosen so that the consumer chooses the same amount of consumption between the two periods when $e_1 = e_2$ and $\theta_H = \theta_L$.

The optimal tax rate then decreases monotonically as inequality increases, becomes negative at the point found in Figure 2 and equals approximately -3 percent when the standard deviation of income distribution is 45 percent. The reverse properties hold for the optimal tax on labor (whose value is a bit lower).

The second tax scheme we consider is the one with insurance but no redistribution examined in Section 4.2 (see equation (32)). The north-east panel of Figure 2 plots again the marginal effects of capital and labor taxes, illustrating the claim in Proposition 13. The sign of the marginal effect on ex ante welfare of capital taxation at the competitive equilibrium is the same as in the previous tax scheme.[17] This is in accord with this proposition, since in our example there is no idiosyncratic shock to the return on capital. Also, the marginal effect of labor taxation is now positive regardless of the degree of inequality in the initial endowment. The north-east panel of Figure 3 shows how the optimal tax rates vary with the inequality in the initial endowment. With no inequality the optimal tax rates on capital and labor are both positive, equal respectively to around 6 and 10 percent. As the degree of inequality increases, the optimal tax rate on capital decreases. Note however that it stays positive even when the marginal effect of capital taxation at an equilibrium with no taxes becomes negative. This is the region where the suggestions from the local analysis are misleading for the global optimal, but the size of the region appears to be small. The optimal tax rate on labor is positive and increases with the degree of inequality, in accord with what suggested by the local analysis, reaching a level of around 12 percent when the standard deviation of income distribution is 45 percent.

The third tax scheme we consider is the one with insurance as well as redistribution, discussed in Section 4.3 (see equation (36)). The south-west panel of Figure 2 plots the marginal effects of taxation for this case, which illustrate the claim in Proposition 14. The marginal welfare effect of capital taxation turns out to be always positive and, in contrast with the previous cases, to increase with the degree of initial inequality as we see in Figure 2. So in this example capital should be taxed even when there is under investment. We see in Figure 2 that the effect of labor taxation is also positive but falls as the degree of inequality increases, reflecting the fact that the distribution effect works against taxing labor. The south-west panel of Figure 3 shows that the optimal capital and labor tax rates are again in line with what the local analysis suggests: the optimal tax rate on capital income is positive and increases with the degree of inequality, reaching a level of around 35 percent when the standard deviation of income is the highest considered. The optimal tax rate on labor income is also positive though considerably lower and declines slightly with the degree of inequality. As argued in Section 4.3, the nature of the tax rebate in this case strengthens the case for taxing capital.

The last tax scheme we consider is the one without lump-sum transfers, analyzed in Section 4.4 (equation 41). The south-east panel of Figure 2 plots the marginal effects of taxation for this case. By construction, the tax rates on capital and labor income move in opposite directions to balance the government's budget. By comparing the south-east with the north-east panels we see that under this tax scheme the sign of the marginal effect of capital taxation is exactly the opposite to the one found for the tax schemes without insurance nor redistribution analyzed in Section 4.1, in accord with what shown in Proposition 15, Thus the marginal effect of capital taxation on ex ante welfare is negative with no inequality in the initial endowment, but increases as inequality rises and becomes positive for a sufficiently large degree of inequality. In the south-east panel of Figure 3 we confirm that the optimal tax rates on capital and labor behave as our local analysis suggests: the optimal tax on capital ranges from -10 to 10 percent as the standard deviation of income distribution varies

[17] The values are different since the derivatives of equilibrium price functions are different.

from 0 to 45 percent.

6 Concluding remarks

The reader may still wonder why, when the stock of capital is above the efficient, complete market level, capital should ever be subsidized, i.e., the tax rate on capital should ever be negative. Our results demonstrate that this indeed occurs in some cases we identify. Moreover this finding applies for equilibria exhibiting standard properties and hence does not rely on any pathological properties, as the existence of upward sloping demand curves. In these cases, by subsidizing capital and taxing labor the level of capital accumulation increases beyond the already "excessively high" level of the equilibrium with no taxes.

As we have pointed out, the determination of the optimal taxation level is a second best problem, constrained by the set of allocations attainable as suitably defined tax equilibria, and so a simple comparison to the "efficient level" in the complete market equilibrium does not provide a right intuition. But it is nevertheless useful to discuss where it goes wrong in depth.

A competitive equilibrium in our model is inefficient in general, and the source of inefficiency can be decomposed into two parts: the allocational inefficiency and the production inefficiency. The latter is caused by over/under use of capital/labor in our model. Thus if the level of capital/labor is adjusted toward the efficient level, an economic surplus should be generated, and it would make the agents better off if it is distributed among the agents appropriately.

This logic, which is fundamental in the partial equilibrium analysis, is correct. But with incomplete markets, the level of equilibrium prices affect the consumers' ability to hedge the risk they face and the introduction of taxes, by affecting such prices, may improve this ability, i.e., the allocational efficiency may be improved. It is indeed the insurance effect in our analysis what captures this intuition.

This is however not the only effect. As we saw taxes also have a distribution effect: taxation inevitably induces income redistribution, indirectly through price changes and directly through the distribution of the tax revenue. Such distribution effects will be clearly different for agents with different income levels and preferences. In particular, they will be negative for some agents if the economy is heterogeneous enough. That is, the standard taxation scheme we consider is not a perfect instrument to distribute the economic surplus from improved production efficiency among the agents, and hence it is impossible to improve upon everybody's welfare if the economy is heterogeneous enough.

Then one might think that even if a Pareto improvement cannot be achieved, the improved production efficiency ought to contribute positively to the utilitarian social welfare function we considered. But this is not the case in general, because the social welfare function favors equality in income. The social welfare can be improved by sacrificing the production efficiency, even when markets are complete.

After all, the answer to optimal taxation in incomplete markets is delicate, and one cannot deduce it from the partial equilibrium idea. It is our contribution to identify the sources and the determinants of the sign and magnitude of insurance and distribution effects of the tax, which gives useful information about the sign and magnitude of the optimal tax on capital and labor, in a general equilibrium setting.

Finally, we readily acknowledge that the simple set up we investigated in this paper, while it allows us to identify the effects of taxes in a clear manner, is hardly sufficient. As already mentioned, our results concern the effects of taxes in the short run and do not allow us to draw conclusions on the optimal value of taxes in the long run or on the effects of taxes on steady state allocations. In a companion paper we shall examine an infinite horizon environment: assuming that agents are identical ex ante, the income distribution at a particular time period can also be viewed as the result of past realizations of the uncertainty. We shall confirm that the main intuitions developed in this simple two period setup go through.

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Appendix

Proofs for Lemmas 1 and 2. Consider the second-period utility maximization problem for a type i consumer at θ_i:

$$\max_{c,l} u_i(c,l)$$

subject to

$$c + w\rho_i^L(\theta_i)l = r\rho_i^K(\theta_i)k_i + w\rho_i^L(\theta_i)\bar{H}_i,$$

where k_i has already been chosen in the first period. This problem can be restated as a standard consumer problem in general equilibrium:

$$\max_{c,l} u(c,l)$$
$$\text{subject to } c + pl = m + p\bar{H},$$

where $p = w\rho_i^L(\theta_i)$ and $m = r\rho_i^K(\theta_i)k_i$ are taken as given. Writing $c(p,m)$ and $l(p,m)$ for the derived demand functions for the consumption good and leisure respectively, and denoting by $\lambda(p,m)$ the Lagrange multiplier, the following first-order conditions characterize the demand functions:

$$\begin{aligned} u_c(c(p,m), l(p,m)) - \lambda(p,m) &= 0, \\ u_l(c(p,m), l(p,m)) - \lambda(p,m)\,p &= 0, \\ -(c(p,m) + pl(p,m)) &= -\left(m + p\bar{H}\right). \end{aligned}$$

Recall that ρ_i^L and ρ_i^K are comonotonic, and so are both p and m. Therefore, to establish Lemma 1, it suffices then to show that $\lambda(p,m)$ is decreasing in m and p. Since $\lambda(p,m)$ is the derivative of the indirect utility function with respect to income and the indirect utility function is concave in income for a concave utility, it readily follows that λ is decreasing in m.

We shall now show that $\lambda(p,m)$ is decreasing in p as well. To simplify the notation, we shall omit reference to (p,m) below. To find the derivatives of λ (as well as those for c and l), we follow the standard technique of differentiating the system of the first order conditions:

$$\begin{bmatrix} u_{cc} & u_{cl} & -1 \\ u_{cl} & u_{ll} & -p \\ -1 & -p & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial p}c \\ \frac{\partial}{\partial p}l \\ \frac{\partial}{\partial p}\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda \\ l - \bar{H} \end{bmatrix}.$$

The strict concavity assumption implies that the determinant of the square matrix above is positive:

$$\triangle := -u_{cc}p^2 + 2u_{cl}p - u_{ll} > 0,$$

and we have:

$$\begin{bmatrix} u_{cc} & u_{cl} & -1 \\ u_{cl} & u_{ll} & -p \\ -1 & -p & 0 \end{bmatrix}^{-1} = \frac{1}{\triangle}\begin{bmatrix} -p^2 & p & u_{ll} - pu_{cl} \\ p & -1 & u_{cc}p - u_{cl} \\ u_{ll} - pu_{cl} & u_{cc}p - u_{cl} & u_{cc}u_{ll} - (u_{cl})^2 \end{bmatrix}.$$

Thus we have:

$$\frac{\partial}{\partial p}\lambda = \frac{1}{\triangle}\left((u_{cc}p - u_{cl})\,\lambda + \left(u_{cc}u_{ll} - (u_{cl})^2\right)\left(l - \bar{H}\right)\right),$$

which is negative. Indeed, using the first order condition, $(u_{cc}p - u_{cl})\,\lambda = u_{cc}u_l - u_{cl}u_c < 0$ where the inequality holds by the normality of consumption good, and $\left(u_{cc}u_{ll} - (u_{cl})^2\right) > 0$ by concavity and $\left(l - \bar{H}\right) < 0$. Therefore, Lemma 1 has been established.

Notice that the labor supply in efficiency units corresponds to $\left(\bar{H} - l\right)p/w$ in the consumer problem above, so in order to establish Lemma 2 it suffices to show that $\left(\bar{H} - l\right)p$ is increasing in p. From the system of equations above,

$$\frac{\partial}{\partial p}l = \frac{1}{\triangle}\left(-\lambda + (u_{cc}p - u_{cl})\left(l - \bar{H}\right)\right),$$

and so

$$
\begin{aligned}
\frac{d}{dp}\left(\left(\bar{H}-l\right)p\right) &= \left(\bar{H}-l\right) - p\frac{\partial}{\partial p}l, \\
&= \left(\bar{H}-l\right) - \frac{p}{\triangle}\left(-\lambda + \left(u_{cc}p - u_{cl}\right)\left(l-\bar{H}\right)\right), \\
&= \left(\bar{H}-l\right)\left(1 + \frac{p}{\triangle}\left(u_{cc}p - u_{cl}\right)\right) + \frac{p\lambda}{\triangle}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
1 + \frac{1}{\triangle}\left(u_{cc}p^2 - pu_{cl}\right) &= \frac{1}{\triangle}\left(\left(-u_{cc}p^2 + 2u_{cl}p - u_{ll}\right) + \left(u_{cc}p^2 - pu_{cl}\right)\right), \\
&= \frac{1}{\lambda\triangle}\left(u_{cl}u_l - u_c u_{ll}\right), \\
&> 0,
\end{aligned}
$$

where the last inequality follows from the normality of leisure. This proves Lemma 2.

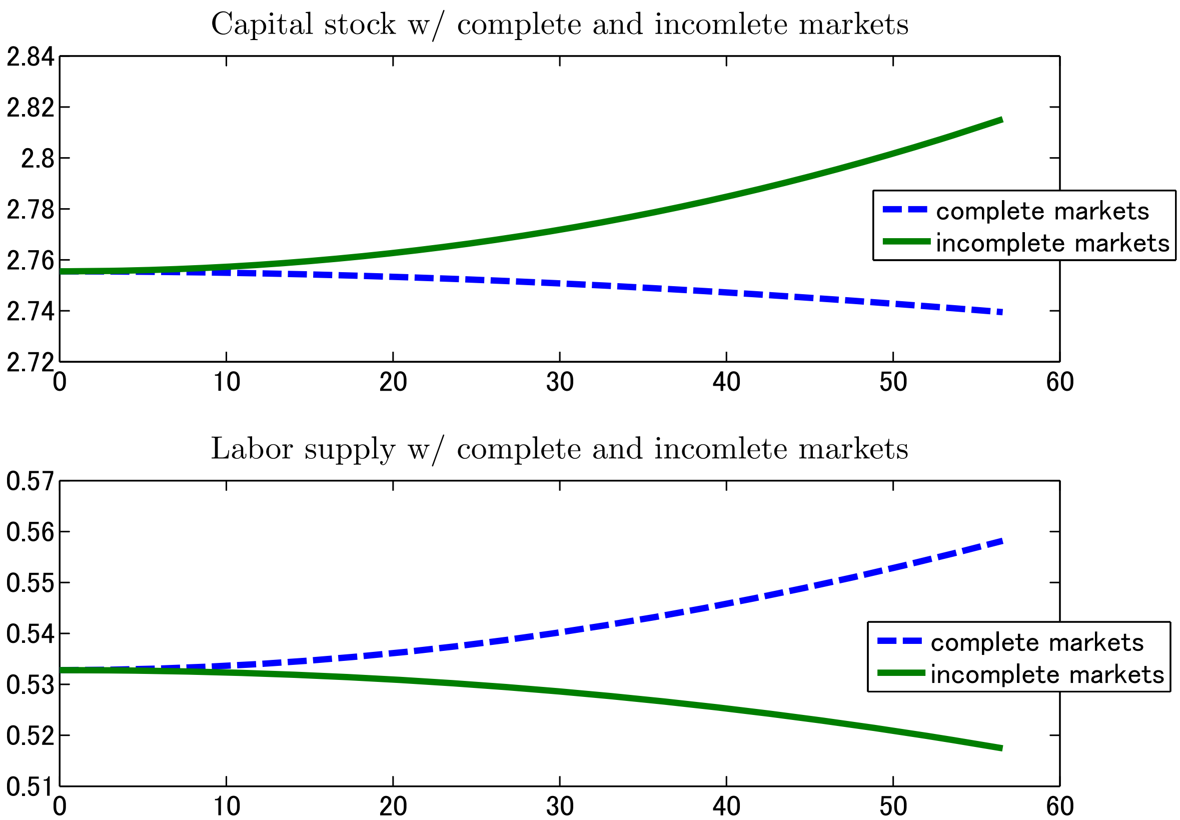

Figure 1: Capital stock and labor supply with complete and incomplete markets. The horizontal axis measures the standard deviation of the labor productivity shock (percent).

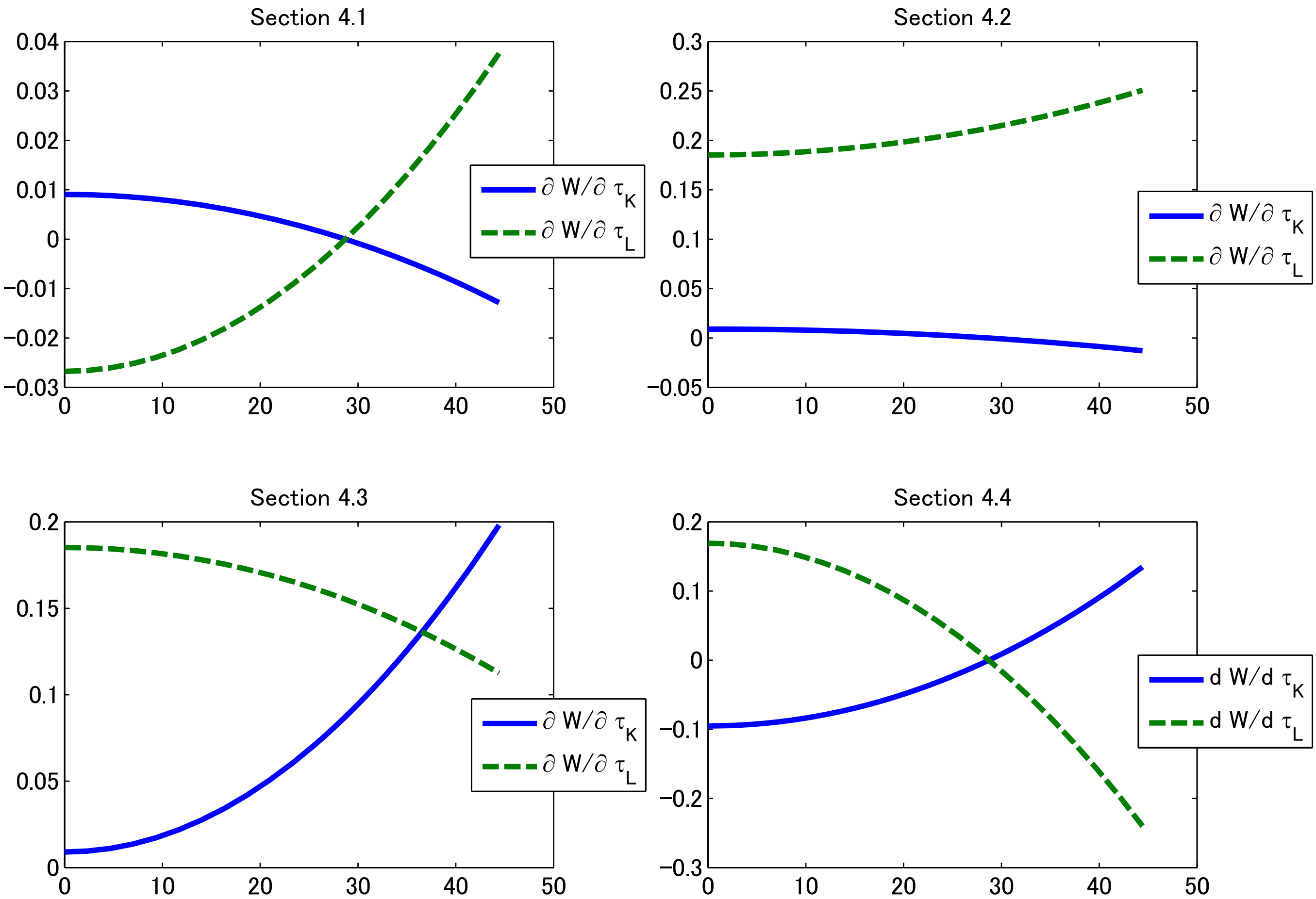

Figure 2: Marginal effects on the ex-ante welfare of the capital and labor taxation under alternative assumptions on lump-sum transfers. The horizontal axis measures the standard deviation of the initial endowments (percent).

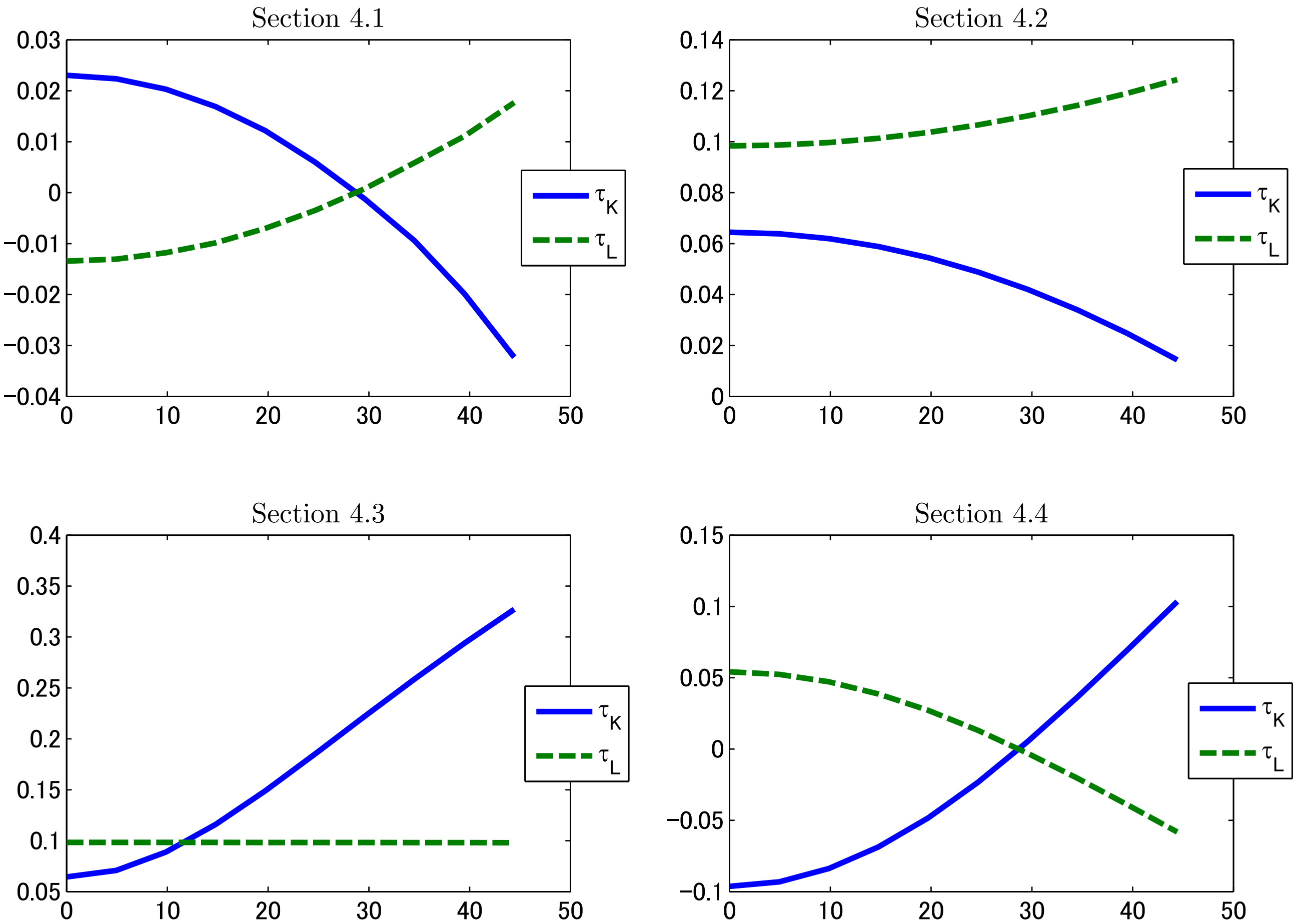

Figure 3: The optimal tax rates on capital and labor under alternative assumptions on lump-sum transfers. The horizontal axis measures the standard deviation of the initial endowments (percent).