Transaction Costs and Planner Intervention

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Abstract

In this paper, I examine a dynamic general equilibrium model with transactions costs imposed on the trading of assets in the financial markets. These transaction costs are general functions in the model and can represent any costs associated with asset trade that result in a real loss of resources. The presence of these transaction costs results in a Pareto inefficient equilibrium allocation. Attempting to fix this problem, the planner will intervene by adjusting the transaction costs and returning the funds to the households through a tax/subsidy scheme. The planner's intervention must satisfy fiscal balance. I prove that over a generic subset of parameters and subject to an upper bound on the number of household types, there exists an open set of planner interventions that lead to a Pareto superior allocation.

1 Introduction

Transaction costs are pervasive in financial markets, both those in the real world and those studied in economic models. Some transaction costs are measurable and apparent such as a tax imposed by the government on the trade of an asset. Other transaction costs are unmeasurable, but are the accepted explanation for why beneficial trade does not occur. The models of derivative asset pricing rely on transaction costs to justify the pricing of an otherwise redundant asset. In other financial models, the holding of an asset is divided into purchases and sales. Without transaction costs, these two variables would be indeterminate.

Recently, interest in explaining the emergence of transaction costs has arisen. One explanation for transaction costs is that they emerge because a financial intermediary is required to facilitate asset trade. This intermediary must be compensated a market

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wage for the labor required to produce such a service. The most recent work that corrects many of the shortfalls of the previous literature is Martins-da-Rocha and Vailakis (2009). My work does not attempt to explain why transaction costs emerge, rather it studies conditions under which an adjustment of these transaction costs can improve market welfare. The welfare criterion that will be used in this paper will be the Pareto criterion.

To illustrate the normative ramifications of transaction costs, suppose that enough assets exist to span all states of uncertainty. If transaction costs were removed from the model, then all households would perfectly insure against future risk by trading financial assets. As a result, the equilibrium allocation would be Pareto optimal, meaning that a planner cannot intervene and make some households better off without making others worse off. However, with transaction costs, the equilibrium allocation is inefficient and there is justification for planner intervention.

The planner will take some fraction of the value of the transaction costs and distribute it to the households with a tax/subsidy scheme. The adjustments made by the planner must satisfy fiscal balance, that is, the sum of all adjustments must have value 0. If the transaction costs are taxes, this statement says that the planner's tax reform must be revenue neutral, that is, the tax revenue collected does not change. The main result states that over a generic subset of parameters and subject to an upper bound on the number of households, there exists an open set of planner interventions that lead to a Pareto superior allocation.

Recent papers by Citanna et al. (2006) and del Mercato and Villanacci (2006) analyze the normative impact of a government tax policy. Both papers, although each has a different setup, arrive at the same conclusion. That conclusion is that with an incomplete markets setup, for a generic subset of endowments and utilities, the introduction of a tax can be Pareto improving. My result differs from both works in two key aspects. First, I focus entirely on one friction (transaction costs) and do not require an incomplete markets setting. Second, these two papers prove that for an economy without tax frictions, introducing taxes to redistribute wealth will lead to a Pareto improvement. However, tax frictions must certainly be present in any economy before a government can redistribute wealth. What I do is prove the regularity of a transaction costs equilibrium (an equilibrium in which tax frictions are already present) and then prove my generic planner intervention result given that equilibrium.

My paper is a descendant of the works by Cass and Citanna (1998) and Elul (1995) which questioned whether financial innovation is always welfare improving. In a setting of incomplete financial markets, both of the above papers prove that there is an open set of payoffs for the new assets (under additional dimensional restrictions) such that the introduction of this new asset actually makes all households worse off. As governments are not in the business of creating new assets, I claim that it is more interesting to study the planner adjustments of transaction costs (taxes), a frequent action performed by governments. The framework used to prove the Cass and Citanna (1998) result, the Citanna et al. (2006) result, the del Mercato and Villanacci (2006) result, and the result presented in this paper was developed by Citanna, Kajii, and Villanacci (1998).

This paper is organized into three remaining sections. In section 2, I introduce the general equilibrium model with transaction costs in the financial markets and define an equilibrium. In section 3, I state and prove the main result of this paper. In section 4, I provide the proofs of two lemmas stated in section 2.

2 The Model

Consider a 2 period general equilibrium model with S states of uncertainty in the second time period. Denoting the first period as the $s = 0$ state, I will number the states as $s \in \mathcal{S} = \{0, ..., S\}$. At each state, $H \geq 2$ households trade and consume $L \geq 2$ physical commodities. There are a finite number of both households and physical commodities with $h \in \mathcal{H} = \{1, .., H\}$. The commodities are denoted by the variable x. Define the total number of goods as $G = L(S+1)$ and then the consumption set is the entire nonnegative orthant: $x^h \in \mathbb{R}^G_+ \ \ \forall h \in \mathcal{H}$. Concerning notation, the vector $x \in \mathbb{R}^{HG}_+$ contains the consumptions for all households, the vector $x^h(s) \in \mathbb{R}^L_+$ contains the consumption by household h in state s (of all commodities), and the scalar $x^h_l(s) \in \mathbb{R}_+$ is the consumption by household h of the good (s, l) or the l^{th} physical commodity in state s.

Households are endowed with commodities in all states. These endowments are denoted by e. I assume that all households have strictly positive endowments:

Assumption 1 $\quad e^h >> 0 \ \ \forall h \in \mathcal{H}$.[1]

In addition to endowments, the household primitives include the utility functions $u^h : \mathbb{R}^G_+ \to \mathbb{R}$ subject to the following assumptions:

Assumption 2 $\quad u^h$ is C^3, differentiably strictly increasing (i.e., $Du^h(x^h) >> 0$ $\forall x^h \in \mathbb{R}^G_+$), differentiably strictly concave (i.e., $D^2u^h(x^h)$ is negative definite $\forall x^h \in \mathbb{R}^G_{++}$), and satisfies the boundary condition ($clU^h(x^h) \subset \mathbb{R}^G_{++}$ where $U^h(x^h) = \{x' \in \mathbb{R}^G_{++} : u^h(x') \geq u^h(x^h)\}$) $\forall h \in \mathcal{H}$.

Define the commodity prices as $p \in \mathbb{R}^G \backslash \{0\}$. Under assumption 2, the prices satisfy $p >> 0$. Of all the physical commodities in each state, the final one ($l = L$) is called the numeraire commodity, meaning that all other commodities are priced relative to this one. For simplicity, I normalize the price of the numeraire commodity $p_L(s) = 1$ in every state $s \in \mathcal{S}$.

The commodities are perishable, so the households require financial markets to transfer wealth between states. I assume that there are J assets ($J \leq S$). These assets are numeraire assets meaning that the payout of each asset is in terms of the numeraire commodity $l = L$. The payouts are assumed to be nonnegative and are collected in the $S \times J$ yields matrix Y :

$$Y = \begin{bmatrix} r_1(1) & ... & r_J(1) \\ \vdots & ... & \vdots \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

[1] The notation $e^h >> 0$ means that $e^h_l(s) > 0 \ \ \forall (l, s)$.

To get the payoff in terms of the unit of account, I make the preserving transformation

$$Y = \begin{bmatrix} p_L(1) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & p_L(S) \end{bmatrix} \cdot \begin{bmatrix} r_1(1) & ... & r_J(1) \\ : & ... & : \\ r_1(S) & ... & r_J(S) \end{bmatrix}.$$

Summarizing what I said so far concerning the parameter Y :

Assumption 3 Y is a nonnegative and full rank $S \times J$ yields matrix.

The assets are in zero net supply and are denoted by the variable θ. As with consumption, $\theta^h \in \mathbb{R}^J$ is the portfolio held by an individual household h, $\theta \in \mathbb{R}^{HJ}$ are the portfolios of all households, and $\theta_j^h \in \mathbb{R}$ is the amount of asset j held by household h.

For each asset j, there exists an asset price $q_j \in \mathbb{R}$, which can be viewed as the payoff of the asset (in terms of the unit of account) in state $s = 0$. Combining the endogenous asset prices with the exogenous payouts, I will represent the overall returns of the financial markets in the $(S+1) \times J$ returns matrix R :

$$R = \begin{bmatrix} -q \\ Y \end{bmatrix}.$$

I will model the transaction costs as costs on the trade of financial assets. Initially, all households have zero asset holdings. Upon trading assets, households must pay the real transaction costs. The transaction costs for the entire portfolio (paid in units of account) will be determined by the mapping

$$\begin{aligned} F^h &: \mathbb{R}^J \to \mathbb{R}_+ \\ F^h\left(\theta^h\right) &= \sum_j q_j \cdot f_j^h\left(\theta^h\right). \end{aligned}$$

For any portfolio θ^h, the value $F^h\left(\theta^h\right)$ is the value that must be paid as transaction costs. The transaction costs are nonnegative. The transaction costs depend (linearly) upon the asset price level (this is natural since the transaction costs will represent a loss of some physical amount of assets). The transaction costs are heterogeneous across households.

The mapping f_j^h is the actual amount of asset j that must be paid as transaction costs. These costs (in terms of asset j) will depend not only on the position of asset j, θ_j^h, but also on the position of the other assets θ_{-j}^h. The transaction costs mappings (parameters of the model) are given by the vector-valued function $f^h = \left(f_1^h, .., f_J^h\right)$ where

$$f_j^h : \mathbb{R}^J \to \mathbb{R}_+ \ \ \forall j.$$

I impose the following assumptions on f_j^h $\forall h$ and $\forall j$.

Assumption 4 f_j^h is C^3, differentiably strictly convex in θ_j^h, and satisfies $f_j^h(\theta) = 0$ for any $\theta \in \mathbb{R}^J$ with $\theta_j = 0$. By strict convexity in θ_j^h, I mean that $a^T D^2 f_j^h\left(\theta^h\right) a \geq 0$ $\forall a$ with strict inequality if $a_j \neq 0$ and $\theta_j^h \neq 0$ $\forall j$.

Claim 1 *Given $q >> 0$, F^h is C^3, differentiably strictly convex, and satisfies $F^h(0) = 0$.*

Proof. The first and the last are obvious. For the second, note that the $J \times J$ Hessian for F^h can be equivalently written as:

$$D^2 F^h\left(\theta^h\right) = \sum\nolimits_j q_j \cdot D^2 f_j^h\left(\theta^h\right)$$

where q_j is a scalar multiplier of the Hessians $D^2 f_j^h\left(\theta^h\right)$. With $q >> 0$, since $a^T D^2 f_j^h\left(\theta^h\right) a \geq 0\ \forall a$, then $a^T D^2 F^h\left(\theta^h\right) a \geq 0\ \forall a$. For strict inequality, if $\theta_j^h \neq 0\ \forall j$ and since $a^T D^2 f_j^h\left(\theta^h\right) a > 0\ \ \forall a : a_j \neq 0$ and this holds $\forall j$, then $a^T D^2 F^h\left(\theta^h\right) a > 0$ $\forall a \neq 0$. ■

Though this paper does not offer an explanation for why the transaction costs are strictly convex, the recent work by Martins-da-Rocha and Vailakis (2009) may shed some light on the question. Their work models transaction costs as an endogenous result of the labor that must be input to produce financial intermediation. The labor to intermediate a financial transaction can be supplied by any of the households in the economy (pure competition). The production set for intermediation needs to be convex. Further, households receive a convex disutility from labor. As a result, the equilibrium transaction costs for a portfolio $\theta^h \in \mathbb{R}^J$ are a function of the utility loss from providing the labor necessary to intermediate θ^h. Martins-da-Rocha and Vailakis implement a linear transaction costs structure (a constant commission paid by households on all asset trades). With a linear transaction costs structure, there is only one variable to endogenize: the slope. However, with a convex production set and convex disutility, the intuitive idea (though harder to implement) is that the per-unit transaction costs will strictly increase with the size of the trade. This would endogenously generate the strict convexity of transaction costs that I assume in my model.

Define the **canonical representation** for the transaction costs mappings as that specification whereby the transaction costs are independent across assets. In this case, f_j^h is only a function of θ_j^h and $D^2 F^h\left(\theta^h\right)$ is a positive definite, diagonal matrix.

The transaction costs are paid in terms of the numeraire assets and can be likened to a sieve which collects a certain percentage of the total asset trade. Since the assets are numeraire and pay out in the real physical commodity $l = L$, the transaction costs have a real effect in that the sieve is removing the commodities $l = L$ from the total resources of the economy. A *transaction costs equilibrium* is thus defined as follows.

Definition 1 $\left(\left(x^h, \theta^h\right)_{h \in \mathcal{H}}, p, q\right)$ *is a transaction costs equilibrium if*

1. *$\forall h \in \mathcal{H}$, given (p, q),*
   $\left(x^h, \theta^h\right)$ is an optimal solution to the household's maximization problem

$$\begin{array}{ll} \max\limits_{x \geq 0, \theta} & u^h(x) \\ subj\ to & p(0)(e^h(0) - x(0)) - q\theta - \sum_j q_j \cdot f_j^h(\theta) \geq 0\ \cdot \\ \forall s > 0 & p(s)(e^h(s) - x(s)) + \sum_j r_j(s)\theta_j \geq 0 \end{array} \tag{HP}$$

2. Markets Clear

$$\begin{array}{l}\sum_h x_l^h(s) = \sum_h e_l^h(s) \quad \forall (l,s) \notin \{(L,1),..,(L,S)\}. \\ \sum_h x_L^h(s) = \sum_h e_L^h(s) + \sum_h \sum_j r_j(s) \cdot \theta_j^h \quad \forall s > 0. \\ \sum_h \theta_j^h + \sum_h f_j^h\left(\theta^h\right) = 0 \quad \forall j.\end{array}$$

The existence of such an equilibrium is well-known and hence its proof is omitted.

The total financial payout in $s = 0$ for some asset j including both the asset price and the transactions costs is given by $-q_j \cdot \tilde{f}_j^h\left(\theta^h\right)$ where

$$\begin{aligned}\tilde{f}_j^h &: \Theta_E \to \tilde{f}_j^h\left(\Theta_E\right) \\ \tilde{f}_j^h\left(\theta^h\right) &= \theta_j^h + f_j^h\left(\theta^h\right).\end{aligned}$$

Θ_E is the set containing all potential equilibrium portfolios (E for equilibrium), that is, assets that satisfy household optimization and market clearing. So far nothing I have said indicates that $\Theta_E \neq \mathbb{R}^J$, but claims 2 and 3 will do just that. By construction $\tilde{f}_j^h$ satisfies the conditions of assumption 4. Let $\tilde{f}^h = \left(\tilde{f}_1^h, .., \tilde{f}_J^h\right)$ be the Cartesian product of $\left(\tilde{f}_j^h\right)_{\forall j}$ with $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right).$

Claim 2 *In equilibrium,* $q \cdot D\tilde{f}^h\left(\theta^h\right) >> 0 \ \ \forall \theta^h \in \Theta_E.$

Proof. The following are the first order conditions of the household's problem (HP) with respect to θ^h where λ^h are the Lagrange multipliers:

$$\lambda^h \begin{pmatrix} -\left(q_1, .., q_J\right) \cdot D\tilde{f}^h\left(\theta^h\right) \\ r(1) \\ \vdots \\ r(S) \end{pmatrix} = 0_{1\times J}. \tag{1}$$

This is best seen as the (j,k) element of $D\tilde{f}^h\left(\theta^h\right)$ is $\frac{\partial \tilde{f}_j^h\left(\theta^h\right)}{\partial \theta_k}$ and the first order condition for any one asset θ_k^h is given as:

$$\lambda^h \begin{pmatrix} -\sum_j q_j \frac{\partial \tilde{f}_j^h\left(\theta^h\right)}{\partial \theta_k} \\ r_k(1) \\ \vdots \\ r_k(S) \end{pmatrix} = 0. \tag{2}$$

From (2) with $\sum_{s>0} \lambda^h(s) r_k(s) > 0$ and $\lambda^h(0) > 0$, then $-\sum_j q_j \frac{\partial \tilde{f}_j^h\left(\theta^h\right)}{\partial \theta_k} < 0 \ \ \forall k.$ This finishes the proof. ■

Take the canonical representation for the transaction costs in which f_j^h is only a function of θ_j^h. Then $\tilde{f}_j^h$ is only a function of θ_j^h and $D\tilde{f}^h\left(\theta^h\right)$ is a diagonal matrix.

Claim 3 *Under the canonical representation, equilibrium conditions imply $q >> 0$ and $\tilde{f}^h : \Theta_E \to \tilde{f}^h(\Theta_E)$ is an invertible function.*

Proof. From the previous claim, $q \cdot D\tilde{f}^h(\theta^h) >> 0 \ \ \forall \theta^h \in \Theta_E$. Under the canonical representation, $D\tilde{f}^h(\theta^h)$ is a diagonal matrix. Thus, if I can show that $q >> 0$, then $q \cdot D\tilde{f}^h(\theta^h) >> 0$ implies that $D\tilde{f}^h(\theta^h)$ has strictly positive diagonal elements for all $\theta^h \in \Theta_E$. Applying the Inverse Function Theorem would yield that $\tilde{f}^h$ is an invertible function.

Consider any asset j and suppose for contradiction that $q_j \leq 0$. Then $q_j < 0$ and $\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} < 0 \ \ \forall h$ from (2). By the definition of

$$\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} = 1 + Df_j^h(\theta_j^h) < 0,$$

then $Df_j^h(\theta_j^h) < -1$. Since $f_j^h : \mathbb{R} \to \mathbb{R}_+$ has the global minimum at $\theta_j^h = 0$, then $\theta_j^h < 0$. From the market clearing condition:

$$\textstyle\sum_h \tilde{f}_j^h(\theta_j^h) = 0,$$

there exists some households such that $\tilde{f}_j^h(\theta_j^h) \leq 0$. For these households, the financial payout in state $s = 0$ is given by

$$-q_j \cdot \tilde{f}_j^h(\theta_j^h) \leq 0$$

and the payout in states $s > 0$ is given by

$$\begin{pmatrix} \vdots \\ r_j(s)\theta_j^h \\ \vdots \end{pmatrix} < 0.$$

As a result, these households are not optimizing as $\theta_j^h = 0$ is affordable and strictly preferred. Thus $q_j < 0$ and $\frac{\partial \tilde{f}_j^h(\theta^h)}{\partial \theta_j} < 0$ cannot be an equilibrium outcome for any household h. ■

To proceed, I will need to use the inverse function of $\tilde{f}^h : \Theta_E \to \tilde{f}^h(\Theta_E)$. Under the canonical representation, $\tilde{f}^h$ is invertible. Without the canonical representation, $\tilde{f}^h$ may not be invertible. I will return to this point in lemma 2. For now, I state the results conditional on $\tilde{f}^h$ being an invertible function.

Claim 4 *If $\tilde{f}^h : \Theta_E \to \tilde{f}^h(\Theta_E)$ is an invertible mapping and $Y \cdot \left[D\tilde{f}^h(\theta^h)\right]^{-1}$ is a nonnegative matrix for all equilibrium θ^h, then $q >> 0$.*[2]

[2] Since Y is nonnegative by assumption 3 and under the canonical representation $D\tilde{f}^h(\theta^h)$ is a strictly positive, diagonal matrix, then for an open set of matrices around the canonical representation, $Y \cdot \left[D\tilde{f}^h(\theta^h)\right]^{-1}$ is nonnegative. For this open set, the nonnegativity assumption in claim 4 need not be stated.

Proof. Since $\tilde{f}^h$ is invertible, the matrix $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ has full rank. Thus $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is a full rank matrix. From the first order conditions given in (1) :

$$qD\tilde{f}^h\left(\theta^h\right) = \frac{\left(\lambda^h(1), .., \lambda^h(S)\right)}{\lambda^h(0)} \cdot Y.$$

Thus, the asset prices q are given by:

$$q = \frac{\left(\lambda^h(1), .., \lambda^h(S)\right)}{\lambda^h(0)} \cdot Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}.$$

Since $Y \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-1}$ is a nonnegative, full rank matrix, there exists at least one strictly positive element in each column. As $\lambda^h >> 0$, then $q >> 0$. ■

I will define the new asset variable $\eta^h \in \mathbb{R}^J$ such that

$$\eta^h = \tilde{f}^h\left(\theta^h\right) \quad \text{or } \eta_j^h = \tilde{f}_j^h\left(\theta^h\right) \quad \forall j.$$

If $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right)$ is invertible, then $\exists g^h : \tilde{f}^h\left(\Theta_E\right) \to \Theta_E$ such that

$$\begin{aligned} g^h &= \left(\tilde{f}^h\right)^{-1} \\ g^h\left(\eta^h\right) &= \theta^h. \end{aligned}$$

The vector $g^h = \left(g_1^h, .., g_J^h\right)$ is such that $g_j^h : \tilde{f}^h\left(\Theta_E\right) \to \mathbb{R}$ is C^3 $\forall j$. Further, if $\eta_j^h \neq 0$ $\forall j$, then $\theta_j^h \neq 0$ $\forall j$ since $\eta_j^h = \tilde{f}_j^h\left(\theta^h\right) = \theta_j^h + f_j^h\left(\theta^h\right)$.

With this alternative asset, I will redefine a transaction costs equilibrium.

Definition 2 $\left(\left(x^h, \eta^h\right)_{h\in\mathcal{H}}, p, q\right)$ *is a* $\beta-$*transaction costs equilibrium if*

1. *$\forall h \in \mathcal{H}$, given (p, q),*
   $\left(x^h, \eta^h\right)$ is an optimal solution to the household's maximization problem

$$\begin{array}{cc} \max\limits_{x\geq 0,\eta} & u^h(x) \\ \text{subj to} & p(0)(e^h(0) - x(0)) - q\eta \geq 0 \\ \forall s > 0 & p(s)(e^h(s) - x(s)) + \sum_j r_j(s) g_j^h(\eta) \geq 0 \end{array} . \tag{HP2}$$

2. *Markets Clear*

$$\begin{array}{l} \sum_h x_l^h(s) = \sum_h e_l^h(s) \quad \forall (l, s) \notin \{(L, 1), .., (L, S)\}. \\ \sum_h x_L^h(s) = \sum_h e_L^h(s) + \sum_h \sum_j r_j(s) g_j^h\left(\eta^h\right) \quad \forall s > 0. \\ \sum_h \eta_j^h = 0 \quad \forall j. \end{array}$$

Define the total financial payout in each state $s > 0$ as the function

$$\begin{aligned} G_s^h &: \mathbb{R}^J \to \mathbb{R} \\ G_s^h\left(\eta^h\right) &= \sum\nolimits_j r_j(s) \cdot g_j^h\left(\eta^h\right). \end{aligned}$$

Then $G^h : \mathbb{R}^J \to \mathbb{R}^S$ defined as the Cartesian product $G^h = \left(G_1^h, ..., G_S^h\right)$ is given equivalently by:

$$G^h\left(\eta^h\right) = Y \cdot \begin{pmatrix} g_1^h(\eta^h) \\ \vdots \\ g_J^h(\eta^h) \end{pmatrix}$$

where $\begin{pmatrix} g_1^h(\eta^h) \\ \vdots \\ g_J^h(\eta^h) \end{pmatrix} = g^h\left(\eta^h\right).$ Thus, the derivative of $G^h\left(\eta^h\right)$ (an $S \times J$ matrix) is given by:

$$DG^h\left(\eta^h\right) = Y \cdot Dg^h\left(\eta^h\right).$$

$Dg^h\left(\eta^h\right)$ has full rank and so $DG^h\left(\eta^h\right)$ has full column rank.

Define the $(S+1) \times G$ price matrix

$$P = \begin{bmatrix} p(0) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & p(S) \end{bmatrix}.$$

I can characterize the *$\beta-$transaction costs equilibria* with a system of equations Φ. Define $n = H(G+J+S+1)+J+G-(S+1)$ as the number of variables. Given parameters $\sigma = \left(e^h, u^h, f^h\right)_{h\in\ \mathcal{H}}$, the variables $\xi = \left(\left(x^h, \lambda^h, \eta^h\right)_{h\in\mathcal{H}}, p, q\right)$ constitute a *$\beta-$transaction costs equilibrium* iff $\Phi(\xi, \sigma) = 0 \in \mathbb{R}^n$ where

$$\begin{array}{ll} \Phi(\xi,\sigma) \quad = & \\ & \vdots \\ (FOCx) & Du^h(x^h) - \lambda^h P \\ & \vdots \\ & \vdots \\ (BC) & \begin{array}{l} p(0)(e^h(0)-x^h(0)) - q\eta^h \\ p(s)(e^h(s)-x^h(s)) + \sum_j r_j(s) g_j^h\left(\eta^h\right) \ \ \forall s>0 \end{array} \\ & \vdots \\ & \vdots \\ (FOC\eta) & \lambda^h \begin{pmatrix} -q \\ Y \cdot Dg^h(\eta^h) \end{pmatrix} \\ & \vdots \\ (MCx) & \sum_{h\in\mathcal{H}} (e_l^h(s) - x_l^h(s)) \ \ \forall l \neq L, \forall s \geq 0 \\ (MC\eta) & \sum_{h\in\mathcal{H}} \eta^h \end{array}$$

Claim 5 *If $\tilde{f}^h : \Theta_E \to \tilde{f}^h\left(\Theta_E\right)$ is an invertible mapping and $D^2F^h\left(\theta^h\right) \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is a positive semidefinite matrix for all equilibrium θ^h, then $\sum_{s>0} \lambda^h(s) \cdot D^2G_s^h\left(\eta^h\right)$*

is a negative semidefinite matrix.[3]

Proof. I will employ the Einstein summation convention in this proof for notational simplicity. A good reference is Lee (2006).

As $g^h = \left(\tilde{f}^h\right)^{-1}$, then for any θ^h :

$$\begin{aligned} g^h \circ \tilde{f}^h\left(\theta^h\right) &= \theta^h \\ Dg^h\left(\tilde{f}^h\left(\theta^h\right)\right) \cdot D\tilde{f}^h\left(\theta^h\right) &= I_J. \end{aligned}$$

Define $\eta^h = \tilde{f}^h\left(\theta^h\right)$ and $\phi^j = \sum_{s>0} \lambda^h(s) r^j(s) > 0$. Then premultiply the above equation by $\left(\phi^1, .., \phi^J\right)$ to obtain:

$$\left(\phi^1, .., \phi^J\right) Dg^h\left(\eta^h\right) \cdot D\tilde{f}^h\left(\theta^h\right) = \left(\phi^1, .., \phi^J\right). \tag{3}$$

Equation (3) is equivalent to (using the Einstein summation convention):

$$\phi^j Dg^h_j\left(\eta^h\right) \cdot D\tilde{f}^h\left(\theta^h\right) = \left(\phi^1, .., \phi^J\right).$$

Taking a second derivative yields:

$$\phi^j D^2 g^h_j\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \phi^j D_k g^h_j\left(\eta^h\right) D^2 \tilde{f}^h_k\left(\theta^h\right) = 0. \tag{4}$$

Define $\left(\psi^1, .., \psi^J\right)$ such that $\psi^k = \phi^j D_k g^h_j\left(\eta^h\right)$. Then (4) can be written as:

$$\phi^j D^2 g^h_j\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \psi^k D^2 \tilde{f}^h_k\left(\theta^h\right) = 0. \tag{5}$$

From the first order conditions with respect to η^h_k of the problem $(HP2)$:

$$\lambda^h \begin{pmatrix} -q_k \\ r^j(1) D_k g^h_j\left(\eta^h\right) \\ \vdots \\ r^j(S) D_k g^h_j\left(\eta^h\right) \end{pmatrix} = 0.$$

By the definition of ϕ^j and ψ^k, the terms $\psi^k = \lambda^h(0) q_k \ \ \forall k$. Thus, (5) reduces to

$$\phi^j D^2 g^h_j\left(\eta^h\right) \cdot \left(D\tilde{f}^h\left(\theta^h\right)\right)^2 + \lambda^h(0) q^k D^2 \tilde{f}^h_k\left(\theta^h\right) = 0. \tag{6}$$

By definition, $F^h\left(\theta^h\right) = q^k f^h_k\left(\theta^h\right)$. Since $\tilde{f}^h\left(\theta^h\right) = \theta^h + f^h\left(\theta^h\right)$, then

$$D^2 F^h\left(\theta^h\right) = q^k D^2 \tilde{f}^h_k\left(\theta^h\right).$$

[3] Since $D^2 F^h\left(\theta^h\right)$ is positive semidefinite from claim 1 and under the canonical representation $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is a strictly positive, diagonal matrix, then for an open set of matrices around the canonical representation, $D^2 F^h\left(\theta^h\right) \cdot \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is positive semidefinite. For this open set, the semidefinite assumption in claim 5 need not be stated.

By definition, $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) = \phi^j D^2 g_j^h \left(\eta^h\right).$

Thus, inserting these definitions into (6) and rearranging terms yields the final equation:

$$\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right) = -\lambda^h(0) D^2 F^h \left(\theta^h\right) \left[D\tilde{f}^h \left(\theta^h\right)\right]^{-2}. \tag{7}$$

If $D^2 F^h \left(\theta^h\right) \left[D\tilde{f}^h \left(\theta^h\right)\right]^{-2}$ is a positive semidefinite matrix, then $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ is a negative semidefinite matrix. This completes the proof. ∎

As a well-known regularity result extended to this model, I state without proof the following lemma (with associated well-known corollary).

Lemma 1 *The matrix $D_\xi \Phi_{|\Phi(\xi,\sigma)=0}$ has full row rank on a generic subset of endowments $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$.*

Corollary 1 *Over a generic subset of endowments $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$, then (i) $\exists l < L$ (without loss of generality, $l = 1$) such that $(e_1^h(s) - x_1^h(s)) \neq 0 \quad \forall s > 0$, $\forall h$ and (ii) $\eta_j^h \neq 0 \quad \forall j, \forall h$.*

Critical in defining the $\beta-$transaction costs equilibrium is that the mapping $\tilde{f}^h : \Theta_E \to \tilde{f}^h \left(\Theta_E\right)$ is invertible. Up until now, the results only hold conditional on the mapping $\tilde{f}^h$ being invertible. Under the canonical representation, the mapping $\tilde{f}^h$ is invertible. No known conditions exist to guarantee that $\tilde{f}^h$ is always invertible for the general representation.[4] Lemma 2 will find an open set in which all mappings $\tilde{f}^h$ will be invertible and this is possible as the set of invertible matrices is an open set. The proof of lemma 2 is contained in section 4.

Lemma 2 *There exists an open set of transaction costs mappings $\left(f^h\right)_{h\in\ \mathcal{H}}$ such that for mappings in this set and endowments $\left(e^h\right)_{h\in\ \mathcal{H}}$ in a generic subset of $\mathcal{E}$, the mapping $\tilde{f}^h : \Theta_E \to \tilde{f}^h \left(\Theta_E\right)$ is invertible $\forall h \in \mathcal{H}$.*

The next lemma will be useful in the proof of the main theorem. The result from the lemma is sufficient to prove that, over a generic subset of endowments, all equilibrium allocations are Pareto inefficient. The proof of lemma 3 is contained in the appendix.

Lemma 3 *With $H \leq S$, the matrix*

$$\begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & \ldots. & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & \ldots. & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix}$$

has full column rank on a generic subset of endowments $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$.

[4] It is possible to use investment constraints to restrict the asset trade to a subset such that $\left(\tilde{f}^h\right)_{h\in\mathcal{H}}$ are invertible over the restricted domains. This, however, adds an unwanted additional friction to this transaction costs model.

The fact that g_j^h is strictly concave leads to the inefficiency of the equilibrium allocation. If g_j^h was a linear function of η_j^h only ($\forall h$ and $\forall j$), then the equilibrium would exactly equal the GEI equilibrium. With complete markets $J = S$, the equilibrium allocation would be Pareto optimal.

This inefficiency in the equilibrium allocation (a generic result given lemma 3) justifies planner intervention. The planner will make adjustments to the transaction costs while satisfying fiscal balance. For asset j, $\sum_{h\in\mathcal{H}} f_j^h\left(\theta^h\right)$ is the total amount of asset lost due to the transaction costs. The planner will intervene by taking the value $\gamma_j \sum_{h\in\mathcal{H}} f_j^h\left(\theta^h\right)$ and returning it to the households using an anonymous subsidy/tax scheme. The planner choice γ_j can be either positive (a reduction in transaction costs) or negative (an increase in transaction costs).[5] Fiscal balance requires that the net adjustment has value zero:

$$\sum_j \gamma_j \sum_{h\in\mathcal{H}} f_j^h\left(\theta^h\right) = 0. \tag{BB}$$

I will call this the budget balance (BB) equation.

The planner has tools given by the vector $\gamma = \left(..., \gamma_j, ..\right) \in \mathbb{R}^J$. I will call any equilibrium that results following planner intervention the planner updated equilibrium. This is in contrast to the original $\beta-$transaction costs equilibrium. If $\gamma = \overrightarrow{0}$, the planner is taking no action and the planner updated equilibrium is identical to the original $\beta-$transaction costs equilibrium.[6]

Under the tax/subsidy scheme, the households are likely to make different optimizing decisions. Define the asset choices of the planner updated equilibrium as $\left(\hat{\theta}^h\right)_{h\in\mathcal{H}}$. Notice that the original asset choice θ^h under the $\beta-$transaction costs equilibrium will still be feasible given the newly introduced tax/subsidy scheme.

The tax/subsidy scheme will be such that after planner intervention, the households will have asset payouts given by:

$$\begin{aligned} r_j^h(s) &= \rho_j r_j(s) \text{ if } \hat{\theta}_j^h < 0 \\ r_j^h(s) &= (2-\rho_j) r_j(s) \text{ if } \hat{\theta}_j^h \geq 0. \end{aligned}$$

The variable $\left(\rho_j\right)_{\forall j}$ is determined uniquely by γ from the materials balance condition:

$$\rho_j \sum_{h\in\mathcal{H}} \left(\hat{\theta}_j^h\right)^- + (2-\rho_j) \sum_{h\in\mathcal{H}} \left(\hat{\theta}_j^h\right)^+ = \left(1-\gamma_j\right) \sum_{h\in\mathcal{H}} \hat{\theta}_j^h.$$

The materials balance condition can be equivalently rewritten (as $\hat{\eta}_j^h \neq 0 \ \ \forall j$ and $\forall h \in \mathcal{H}$ from corollary 1 implies $\sum_{h\in\mathcal{H}} \hat{\theta}_j^h < 0 \ \forall j$):

$$\rho_j = \frac{\left(1-\gamma_j\right) \sum_{h\in\mathcal{H}} \hat{\theta}_j^h - 2 \sum_{h\in\mathcal{H}} \left(\hat{\theta}_j^h\right)^+}{\sum_{h\in\mathcal{H}} \hat{\theta}_j^h - 2 \sum_{h\in\mathcal{H}} \left(\hat{\theta}_j^h\right)^+}.$$

[5] Implicitly, I require that $\gamma_j \leq 1$, since it is not well-defined for a planner to reduce the transaction costs by more than 100%.

[6] Planner inaction trivially satisfies the budget balance equation.

If $\gamma_j > 0$, then $\rho_j < 1$, so debtors $\hat{\theta}_j^h < 0$ pay back less and creditors $\hat{\theta}_j^h \geq 0$ receive a higher payout.

The planner tool γ is of dimension J. However due to the budget balance equation (BB), the planner only has $J-1$ degrees of freedom in choosing γ. To obtain the result that the planner can use the vector γ to generically effect a Pareto improvement, there must be as many free tools as households. Thus, throughout this work, the assumption $H \leq J-1$ is essential.[7]

3 The Result

Theorem 2 *Under assumptions 1-4 with both $2 \leq H \leq J-1$ and $L \geq 2$ and for parameters $\sigma = \left(e^h, u^h, f^h\right)_{h \in \mathcal{H}}$ belonging to a generic subset of $\Sigma = \mathcal{E} \times \mathcal{U} \times \mathcal{F}$ where $\mathcal{E} = \{(e^h)_{h \in \mathcal{H}} : e^h >> 0\}$, $\mathcal{U}$ is the set of utility functions satisfying assumption 2, and $\mathcal{F}$ is the set of transaction costs mappings satisfying assumption 4 and belonging to the open set given in lemma 2, then given the original $\beta-$transaction costs equilibrium allocation, there exists a planner policy satisfying fiscal balance such that the planner updated allocation is Pareto superior and the new equilibrium is regular.*

Proof The implication of the theorem is that an open set of γ exists (call it A) such that if $\gamma \in A$, then all households are made better off, provided that $H \leq J-1$. The proof of this theorem will follow the framework of Citanna, Kajii, and Villanacci (1998), henceforth simply CKV. The principal task will be to show that the vector of household utilities $U(\xi, \gamma) = \left(u^1(x^1), .., u^H(x^H)\right)$ is a submersion.

Picking a vector of parameters $\bar{\sigma} = \left(\bar{e}^h, \bar{u}^h, \bar{f}^h\right)_{h \in \mathcal{H}}$ such that $\left(\bar{e}^h\right)_{h \in \mathcal{H}}$ belongs to a generic subset of $\mathcal{E}$, then all resulting *$\beta-$transaction costs equilibria* are regular values of Φ. In particular, this means that there exists an open set Σ' around $\bar{\sigma}$ such that for any parameters $\sigma \in \Sigma'$, the resulting equilibria satisfy the rank condition of lemma 1. The set of $\left(x^h\right)_{h \in \mathcal{H}}$ such that $\left(u^1(x^1), .., u^H(x^H)\right) >> \left(u^1(\bar{x}^1), .., u^H(\bar{x}^H)\right)$ is an open set where $\left(\bar{x}^h\right)_{h \in \mathcal{H}}$ is the equilibrium allocation resulting from the original parameters $\bar{\sigma}$. As such, if for some planner tool γ^*, the resulting planner updated allocation is Pareto superior, then all planner updated equilibrium allocations given γ in an open neighborhood around γ^* are Pareto superior as well.

Given parameters $\sigma = \left(e^h, u^h, f^h\right)_{h \in \mathcal{H}}$, the variables $\hat{\xi} = \left(\left(\hat{x}^h, \hat{\eta}^h\right)_{h \in \mathcal{H}}, \hat{p}, \hat{q}\right)$ and policy parameters γ constitute a planner updated equilibrium iff $\Gamma(\hat{\xi}, \gamma, \sigma) = 0$. Γ has one more equation than the system Φ used to define a *$\beta-$transaction costs equilibrium* and is defined as:

[7] If the assumption $H \leq J-1$ appears restrictive, using the idea from Cass and Citanna (1998), the parameter H can be viewed as the number of different types of households. All households of the same type will have parameters (endowments, utilities, and transaction costs mappings) that lie in an open set around the specified parameters for $h \in \mathcal{H}$.

$$
\begin{array}{lll}
\Gamma(\hat{\xi},\gamma,\sigma) & = & \\
 & & \vdots \\
FOCx() & & Du^h(\hat{x}^h)-\hat{\lambda}^h\hat{P} \\
 & & \vdots \\
 & & \vdots \\
(BC) & & \begin{array}{l} \hat{p}(0)(e^h(0)-\hat{x}^h(0))-\hat{q}\hat{\eta}^h \\ \hat{p}(s)(e^h(s)-\hat{x}^h(s))+\sum_j r_j^h(s)g_j^h\left(\hat{\eta}^h\right)\ \forall s>0 \end{array} \\
 & & \vdots \\
 & & \vdots \\
(FOC\eta) & & \hat{\lambda}^h\begin{pmatrix} -\hat{q} \\ \hat{Y}^h\cdot Dg^h(\hat{\eta}^h) \end{pmatrix} \\
 & & \vdots \\
(MCx) & & \sum_{h\in\mathcal{H}}(e_l^h(s)-\hat{x}_l^h(s))\ \ \forall l\neq L,\forall s\geq 0 \\
(MC\eta) & & \sum_{h\in\mathcal{H}}\hat{\eta}^h \\
(BB) & & \sum_j\gamma_j\sum_{h\in\mathcal{H}}\left(\hat{\eta}_j^h-g_j^h(\hat{\eta}^h)\right)
\end{array}
$$

The matrix $\hat{Y}^h$ is the payout function with terms $\left(r_j^h(s)\right)$ as defined at the end of section 2.

By definition, if $\Gamma(\hat{\xi},\overrightarrow{0},\sigma)=0$ and $\Phi(\xi,\sigma)=0$, then $\hat{\xi}=\xi$.

Define the matrix $(H+n+1)\times(n+J)$ matrix Ψ_0 :

$$
\Psi_0=\begin{pmatrix} D_\xi U(\hat{\xi},\gamma) & 0 \\ D_\xi\Gamma(\hat{\xi},\gamma,\sigma) & D_\gamma\Gamma(\hat{\xi},\gamma,\sigma) \end{pmatrix}.
$$

From CKV, if Ψ_0 has full row rank, $\exists\hat{\xi}\neq\xi$ s.t. $\hat{\xi}$ satisfies $\Gamma=0$ (for some γ) and $U(\hat{\xi})>U(\xi)$. The matrix Ψ_0 is square if $H+1=J$, but if $H+1<J$, then there are more columns than rows and I must remove some columns (it does not matter which) in order to obtain a square matrix Ψ. This matrix Ψ does not have full rank iff $\exists\nu\in\mathbb{R}^{H+n+1}$ s.t. $\Phi'(\hat{\xi},\gamma,\nu,\sigma)=0$ where

$$
\Phi'(\hat{\xi},\gamma,\nu,\sigma)=\begin{pmatrix} \Psi^T\nu \\ \nu^T\nu/2-1 \end{pmatrix}.
$$

I will have proven the theorem if I can show that for a generic choice of $\sigma\in\Sigma$, there does not exist (ξ,ν) s.t.

$$
\begin{array}{rcl}
\Phi(\xi,\sigma) & = & 0 \\
\Phi'(\xi,\overrightarrow{0},\nu,\sigma) & = & 0.
\end{array}
\qquad (\Phi,\Phi')
$$

Counting equations and unknowns, (Φ,Φ') has n equations in Φ, n variables ξ, $H+n+2$ equations in Φ', and only $H+n+1$ variables ν. I must show that over a generic subset of parameters (exactly which generic subset will be discussed next), the system of equations (Φ,Φ') (more equations than variables) has full rank. To show full rank of (Φ,Φ'), I will reference the (ND) condition of CKV, which is a sufficient condition for the full rank of (Φ,Φ'). The condition states that for $\gamma=\overrightarrow{0}$

and $\hat{\xi} = \xi$, the matrix

$$\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_\sigma \Phi' \right) \text{ has full row rank} \tag{ND}$$

where σ are the parameters on which the genericity statement is made.

For simplicity, I divide the vector ν^T into subvectors that each represent a certain equation in Ψ. Define $\nu^T = \left(\Delta u^T, \Delta x^T, \Delta \lambda^T, \Delta \eta^T, \Delta p^T, \Delta q^T, \Delta b\right) \in \mathbb{R}^{H+n+1}$ where each subvector corresponds sensibly to an equation (row) in Ψ as follows:

$$\begin{array}{rcl} \Delta u^T & \Longleftrightarrow & U(\xi, \hat{\xi}) \\ \Delta x^T & \Longleftrightarrow & FOCx \\ \Delta \lambda^T & \Longleftrightarrow & BC \\ \Delta \eta^T & \Longleftrightarrow & FOC\eta \\ \Delta p^T & \Longleftrightarrow & MCx \\ \Delta q^T & \Longleftrightarrow & MC\eta \\ \Delta b & \Longleftrightarrow & BB. \end{array}$$

With $\gamma = \overrightarrow{0}$, the variables $\left(\left(\hat{x}^h, \hat{\lambda}^h, \hat{\eta}^h\right)_{h \in \mathcal{H}}, \hat{p}, \hat{q}\right) = \left(\left(x^h, \lambda^h, \eta^h\right)_{h \in \mathcal{H}}, p, q\right)$. A subset of the equations $\nu^T \Psi = 0$ are given by (corresponding to derivatives with respect to $\left((x^h, \lambda^h, \eta^h)_{h \in \mathcal{H}}\right)$ in that order):

$$\begin{array}{c} \vdots \\ \Delta u_h D u^h(x^h) + \Delta x_h^T D^2 u^h(x^h) - \Delta \lambda_h^T P - \Delta p^T \Lambda = 0. \quad (8.a) \\ \vdots \end{array} \tag{8}$$

$$\begin{array}{c} \vdots \\ -\Delta x_h^T P^T + \Delta \eta_h^T \begin{pmatrix} -q \\ Y D g^h(\eta^h) \end{pmatrix}^T = 0. \quad (8.b) \\ \vdots \end{array}$$

$$\begin{array}{l} \vdots \\ \Delta \lambda_h^T \begin{pmatrix} -q \\ Y D g^h(\eta^h) \end{pmatrix} + \Delta \eta_h^T \sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) + \\ \Delta q^T + \Delta b \left(\overrightarrow{\gamma} \cdot \left(I_J - D g^h(\eta^h)\right)\right) = 0. \\ \vdots \end{array} \quad (8.c)$$

where Λ is the $(G - S - 1) \times G$ matrix

$$\Lambda = \begin{bmatrix} \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} I_{L-1} & 0 \end{pmatrix} \end{bmatrix}$$

and $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is the $J \times J$ negative definite matrix defined in section 2.[8]

[8] The term $\overrightarrow{\gamma} \cdot \left(I_J - D g^h(\eta^h)\right)$ is the $1 \times J$ derivative matrix of the budget balance equation with respect to η^h. The proof of the result requires me to prove the (ND) condition for $\overrightarrow{\gamma} = 0$. Thus, this term has value 0 and will be ignored in future analysis.

For simplicity, I break up the analysis into two cases. Those are Case I: $\Delta x_h^T \neq 0$ $\forall h \in \mathcal{H}$ and Case II: $\Delta x_h^T = 0$ for some $h \in \mathcal{H}$. In Case I, I show that (ND) holds over a generic subset of parameters. In Case II, I show that the system of equations (Φ, Φ') will generically not have any solution.

Case I: $\Delta x_h^T \neq 0 \;\; \forall h \in \mathcal{H}$

Claim 6 $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) \neq 0 \;\; \forall h \in \mathcal{H}$

Proof. Suppose that $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) = 0$ for some h. Then $(8.a)$ reads

$$\Delta x_h^T D^2 u^h(x^h) - \Delta \lambda_h^T P = 0.$$

Postmultiplying by Δx_h and using $(8.b)$, I obtain

$$\Delta x_h^T D^2 u^h(x^h) \Delta x_h = \Delta \lambda_h^T \begin{pmatrix} -q \\ Y D g^h(\eta^h) \end{pmatrix} \Delta \eta_h$$

and using $(8.c)$ with $\Delta q^T = 0$, I finally reach

$$\Delta x_h^T D^2 u^h(x^h) \Delta x_h = -\Delta \eta_h^T \left(\sum\nolimits_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)\right) \Delta \eta_h. \tag{9}$$

As $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h \left(\eta^h\right)$ is a negative semidefinite matrix, then the right-hand side of (9) is nonnegative. Meanwhile, from assumption 2, the left-hand side is strictly negative. The contradiction finishes the claim. ■

Claim 7 *For* $\gamma = \overrightarrow{0}$, *then* $D_u \Phi' = \begin{pmatrix} A^* \\ \overrightarrow{0} \end{pmatrix}$ *where* A^* *has full row rank and corresponds to the rows for derivatives with respect to* $\left(x^h\right)_{h \in \mathcal{H}}$.

Proof. Consider the space of utility functions $\left(u^h\right)_{h \in \mathcal{H}} \in \mathcal{U}$, where u^h satisfies assumption 2. The space $\mathcal{U}$ is infinite-dimensional and is endowed with the C^3 uniform convergence topology on compact sets. This means that a sequence of functions $\{u^\nu\}$ converges uniformly to u iff $\{Du^\nu\}$, $\{D^2 u^\nu\}$, and $\{D^3 u^\nu\}$ uniformly converge to Du, $D^2 u$, and $D^3 u$, respectively. Additionally, any subspace of $\mathcal{U}$ is endowed with the subspace topology of the topology of $\mathcal{U}$. I will use the regularity result from lemma 1 to define utility functions as locally belonging to the finite-dimensional subset $\mathcal{A} \subseteq \mathcal{U}$.

Using lemma 1, pick a regular value $\bar{\sigma}$. For that $\bar{\sigma}$, there exist finitely many equilibria $\left(\bar{\xi}_i, \bar{\sigma}\right)$ $i = 1, .., I$. Further, there exist open sets Σ' and $A_i'^h$ s.t. $\bar{x}_i^h \in A_i'^h$, the sets $A_i'^h$ are disjoint across i, and $\forall \sigma \in \Sigma'$, $\exists!$ equilibrium $x_i^h \in A_i'^h$. Choose $A_i'^h$ such that the closure $\bar{A}_i'^h$ is compact and there exist disjoint open sets $\tilde{A}_i'^h$ s.t. $A_i'^h \subset \bar{A}_i'^h \subset \tilde{A}_i'^h$.

For each household, define a bump function $\delta^h : \mathbb{R}_+^G \to [0,1]$ with I bumps as $\delta^h = 1$ on $A_i'^h$ and $\delta^h = 0$ on $(\tilde{A}_i'^h)^c$. Now, I define u^h in terms of a $G \times G$ symmetric matrix A^h as:

$$u^h(x^h; A^h) = \bar{u}(x^h) + \frac{1}{2} \delta^h(x^h) \sum\nolimits_i \left[(x^h - \bar{x}_i^h)^T A^h (x^h - \bar{x}_i^h)\right]. \tag{10}$$

Thus, the space of symmetric matrices $A^h \in \mathcal{A}$ is a finite dimensional subspace of $\mathcal{U}$. Since $\mathcal{A}$ has the subspace topology of $\mathcal{U}$, then $u^h(\cdot; A^\nu) \to u^h(\cdot; A)$ iff $A^\nu \to A$. This can be seen by taking derivatives and noting that the function $\bar{u}$ stays fixed at the regular value.

Taking derivatives with respect to $x^h \in A_i'^h$ yields:

$$\begin{aligned} D_x u^h(x^h; A^h) &= D\bar{u}(x^h) + A^h(x^h - \bar{x}_i^h) \\ D_{xx}^2 u^h(x^h; A^h) &= D^2\bar{u}(x^h) + A^h. \end{aligned}$$

$\mathcal{A}$ is a $G(G+1)/2$ dimensional space, so write A^h as the vector

$$\left((A_{i,i}^h)_{i=1,..,G}, (A_{i,j}^h)_{i<j,i=1,..,G-1}\right).$$

Postmultiply D_{xx}^2 by Δx_h :

$$D_{xx}^2 u^h(x^h; A^h)\Delta x_h = D^2\bar{u}(x^h)\Delta x_h + A^h \Delta x_h.$$

Taking derivatives with respect to the parameter u^h is equivalent to taking derivatives with respect to A^h :

$$D_u\left(D_{xx}^2 u^h(x^h; A^h)\Delta x_h\right) = D_A\left(A^h \Delta x_h\right)$$

$$= \begin{pmatrix} \Delta x_h^1 & 0 & 0 & & & \\ 0 & ... & 0 & \Sigma(1) & ... & \Sigma(G-1) \\ 0 & 0 & \Delta x_h^G & & & \end{pmatrix} \in \mathbb{R}^{G,G(G+1)/2}$$

where the submatrix $\Sigma(i)$ is defined as

$$\Sigma(i) = \begin{pmatrix} 0 \in \mathbb{R}^{i-1,G-i} \\ \begin{pmatrix} \Delta x_h^{i+1} & ... & \Delta x_h^G \\ \Delta x_h^i & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \Delta x_h^i \end{pmatrix} \end{pmatrix} \in \mathbb{R}^{G,G-i}.$$

Thus, since $\Delta x_h \neq 0$ (without loss of generality $\Delta x_h^1 \neq 0$), then

$$rank D_A\left(D_{xx}^2 u^h(x^h; A^h)\Delta x_h\right) = G. \tag{11}$$

Out of all the rows Ψ^T, the utility function u^h only appears in the row for derivatives with respect to x^h. This row in Ψ^T for one household h is given by (as in (8.a)):

$$\begin{array}{cccccc} U(\xi,\hat{\xi}) & FOCx & BC & FOC\eta & MCx & MC\eta \\ \left(Du^h(x^h)\right)^T & D^2u^h(x^h) & -P^T & 0 & -\Lambda^T & 0 \end{array}.$$

Thus, taking the derivative $D_{A^h}\Phi' = D_{A^h}\Psi^T\nu$, the only nonzero element is

$$D_{A^h}\left(\left(Du^h(x^h)\right)^T \Delta u_h + D^2u^h(x^h)\Delta x_h - P^T\Delta\lambda_h - \Lambda^T\Delta p\right)$$

$$= D_{A^h}\left(\left(Du^h(x^h;A^h)\right)^T \Delta u_h\right) + D_{A^h}\left(D^2u^h(x^h;A^h)\Delta x_h\right).$$

From the construction of A^h, $D_x u^h(x^h;A^h) = D\bar{u}(x^h) + A^h(x^h - \bar{x}^h_i) = D\bar{u}(x^h)$ for $\gamma = \overrightarrow{0}$ (since $x^h = \bar{x}^h_i$). Thus $D_{A^h}(D_x u^h(x^h;A^h)\Delta u_h) = 0$. Using (11), then $D_{A^h}\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right)$ is a full rank matrix of size $G \times (G(G-1)/2)$. Thus $A^* = \begin{bmatrix} ... & 0 & 0 \\ 0 & D_{A^h}\left(D^2_{xx}u^h(x^h;A^h)\Delta x_h\right) & 0 \\ 0 & 0 & ... \end{bmatrix}$ has full row rank. ■

Consider the space of transaction costs mappings for all households $\left(F^h\right)_{h\in\mathcal{H}}$. By definition, $F^h\left(\theta^h\right) = \sum_j q_j \cdot f^h_j\left(\theta^h\right)$ depends on the endogenous asset prices. In equilibrium, $q >> 0$ and all the results follow by letting the asset prices $(q_j)_{\forall j}$ be fixed at some strictly positive values.

Claim 8 *For $\gamma = \overrightarrow{0}$ and if $\Delta\eta_h \neq 0 \quad \forall h \in \mathcal{H}$, then $D_F\Phi' = \begin{pmatrix} \overrightarrow{0} \\ B^* \\ \overrightarrow{0} \end{pmatrix}$ where B^* has full row rank and corresponds to the rows for derivatives with respect to $\left(\eta^h\right)_{h\in\mathcal{H}}$.*

Proof. Consider the space of functions $\left(F^h\right)_{h\in\mathcal{H}} \in \mathcal{F}$, where $F^h\left(\theta^h\right) = \sum_j q_j \cdot f^h_j\left(\theta^h\right)$ as in section 2 and f^h_j satisfies assumption 4. The space $\mathcal{F}$ is infinite-dimensional and is endowed with the C^3 uniform convergence topology on compact sets (same as $\mathcal{U}$). I will use the regularity result from lemma 1 to define utility functions as locally belonging to the finite-dimensional subset $\mathcal{B} \subseteq \mathcal{F}$.

Exactly as with utility functions, I define F^h in terms of a $J \times J$ symmetric matrix B^h as:

$$F^h(\theta^h;B^h) = \bar{F}(\theta^h) + \frac{1}{2}\delta^h(\theta^h)\sum_i\left[(\theta^h - \bar{\theta}^h_i)^T B^h(\theta^h - \bar{\theta}^h_i)\right]. \tag{12}$$

Thus, the space of symmetric matrices $B^h \in \mathcal{B}$ is a finite dimensional subspace of $\mathcal{B}$.

Taking derivatives with respect to θ^h yields:

$$\begin{aligned} D_\theta F^h(\theta^h;B^h) &= D\bar{F}(\theta^h) + B^h(\theta^h - \bar{\theta}^h_i) \\ D^2_\theta F^h(\theta^h;B^h) &= D^2\bar{F}(\theta^h) + B^h. \end{aligned}$$

Recall the analysis in the proof of claim 5, namely equation (7) :

$$\sum\nolimits_{s>0}\lambda^h(s)\cdot D^2G^h_s\left(\eta^h\right) = -\lambda^h(0)D^2F^h\left(\theta^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}. \tag{7}$$

Thus, I replace $D^2F^h\left(\theta^h\right)$ by $D^2\bar{F}(\theta^h) + B^h$ and post-multiply both sides of (7) by $\Delta\eta_h$ to yield:

$$\sum\nolimits_{s>0}\lambda^h(s)\cdot D^2G^h_s\left(\eta^h\right)\Delta\eta_h = -\lambda^h(0)\left(D^2\bar{F}(\theta^h) + B^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}\Delta\eta_h.$$

Taking derivatives with respect to the parameter F^h is equivalent to taking derivatives with respect to B^h :

$$D_B\left(-\lambda^h(0)\left(D^2\bar{F}(\theta^h)+B^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}\Delta\eta_h\right)=$$

$$=-\lambda^h(0)\left(\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}\left(\begin{matrix}\Delta\eta_h^1 & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & \Delta\eta_h^J\end{matrix}\quad \Sigma(1)\quad ...\quad \Sigma(J-1)\right)\right)\in\mathbb{R}^{J,J(J+1)/2}$$

where the submatrix $\Sigma(i)$ is defined as

$$\Sigma(i)=\left(\begin{matrix}0\in\mathbb{R}^{i-1,J-i}\\ \begin{pmatrix}\Delta\eta_h^{i+1} & ... & \Delta\eta_h^J\\ \Delta\eta_h^i & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & \Delta\eta_h^i\end{pmatrix}\end{matrix}\right)\in\mathbb{R}^{J,J-i}.$$

With $\lambda^h(0)>0$ and $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ a full rank matrix, I only need to verify that $\left(\begin{matrix}\Delta\eta_h^1 & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & \Delta\eta_h^J\end{matrix}\quad \Sigma(1)\quad ...\quad \Sigma(J-1)\right)$ has full rank. If $\Delta\eta_h\neq 0$ (without loss of generality $\Delta\eta_h^1\neq 0$), then

$$rank D_{B^h}\left(\sum\nolimits_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\Delta\eta_h\right)=J. \tag{13}$$

As the above development with utilities u^h reveals, although the function g^h appears in both rows for derivatives with respect to λ^h and η^h (see equations $(8.b)$ and $(8.c)$), the only nonzero derivatives $D_{B^h}\left(\Psi^T\nu\right)$ are those due to the second derivative $\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)$ in $(8.c)$. Using (13) and if $\Delta\eta_h\neq 0$, then the $J\times(J(J-1)/2)$ matrix $D_{B^h}\left(\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\Delta\eta_h\right)$ has full rank. Thus, if $\Delta\eta_h\neq 0\ \forall h\in\mathcal{H}$, then $B^*=\begin{bmatrix}... & 0 & 0\\ 0 & D_{B^h}\left(\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\Delta\eta_h\right) & 0\\ 0 & 0 & ...\end{bmatrix}$ has full row rank. ■

The matrix $\left(\begin{pmatrix}\Psi^T\\ \nu^T\end{pmatrix}\quad D_A\Phi'\quad D_B\Phi'\right)$ is given below (where the rows correspond to the equilibrium variables $\left((x^h,\lambda^h,\eta^h)_{h\in\mathcal{H}},p,q\right)$, policy variables (γ), and vector v^T in that order). To conserve on space, I will employ the following conventions:

$$c\left(A^h\right)=\begin{pmatrix}A^1\\ \vdots\\ A^H\end{pmatrix},\quad r\left(A^h\right)=\begin{pmatrix}A^1 & ... & A^H\end{pmatrix},\quad d\left(A^h\right)=\begin{pmatrix}A^1 & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & A^H\end{pmatrix}$$

where (c,r,d) stand for column, row, and diagonal, respectively. Further, define $\Omega^h=\begin{pmatrix}-q\\ YDg^h(\eta^h)\end{pmatrix}$, $\Xi^h=(\gamma_1,..,\gamma_J)\left(I_J-Dg^h(\eta^h)\right)$, $\bar{D}^2g^h=\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)$,

$\dot{A}^h = D_{A^h}\left(D^2u^h(x^h)\Delta x_h\right)$, and finally $\dot{B}^h = D_{B^h}\left(\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\Delta\eta_h\right)$. The matrix $\left(\begin{pmatrix}\Psi^T\\ \nu^T\end{pmatrix} \quad D_A\Phi' \quad D_B\Phi'\right)$ is given by:

$$\begin{pmatrix}
r\left(Du^h(x^h)^T\right) & d\left(D^2u^h\right) & d(-P^T) & 0 & c(-\Lambda^T) & 0 & 0 & d\left(\dot{A}^h\right) & 0\\
0 & d\left(-P\right) & 0 & d(\Omega^h) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & d\left((\Omega^h)^T\right) & d(\bar{D}^2g^h) & 0 & c(I_J) & c\left((\Xi^h)^T\right) & 0 & d\left(\dot{B}^h\right)\\
0 & r(-\Lambda_2) & ** & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & r(-\eta^h|\overrightarrow{0}) & r(-\lambda^h(0)I_J) & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \Upsilon_1 & \Upsilon_2 & 0 & 0 & \Upsilon_3 & 0 & 0\\
r(\Delta u_h) & r(\Delta x_h^T) & r(\Delta\lambda_h^T) & r(\Delta\eta_h^T) & \Delta p^T & \Delta q^T & \Delta b & 0 & 0
\end{pmatrix}$$

where Λ_2 is the $(G-S-1)\times G$ matrix

$$\Lambda_2 = \begin{bmatrix}\left(\lambda^h(0)I_{L-1} \quad 0\right) & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & \left(\lambda^h(S)I_{L-1} \quad 0\right)\end{bmatrix}$$

and submatrix $(**)$ will not be considered in the analysis. The submatrix Υ_1 is the $J\times H(S+1)$ matrix defined as the transpose of the derivative of the budget constraints with respect to γ. The submatrix Υ_2 is the $J\times HJ$ matrix defined as the transpose of the derivative of $(FOC\eta)$ with respect to γ. The submatrix Υ_3 is the $J\times 1$ matrix defined as the transpose of the derivative of the budget balance equation (BB) with respect to γ. It is defined as:

$$\Upsilon_3 = \begin{pmatrix}\vdots\\ \sum_{h\in\mathcal{H}}\left(\eta_j^h - g_j^h(\eta^h)\right)\\ \vdots\end{pmatrix}.$$

Claim 9 *The submatrices Υ_1 and Υ_2 are given by $\Upsilon_1 = r\left(\overrightarrow{0}|T_1^h\cdot Y^T\right)$ and $\Upsilon_2 = r\left(T_2^h\cdot Dg^h(\eta^h)\right)$, respectively where T_1^h and T_2^h are full-rank diagonal matrices.*

Proof. For simplicity, define $\alpha_j = \frac{-\sum_{h\in\mathcal{H}}\theta_j^h}{\sum_{h\in\mathcal{H}}\theta_j^h - 2\sum_{h\in\mathcal{H}}\left(\theta_j^h\right)^+}$. Since generically $\eta_j^h \neq 0$ $\forall h\in\mathcal{H}$ (corollary 1), then $-1\leq\alpha_j<0$ $\forall j$.

Choose any household $h\in\mathcal{H}$. From the budget constraints and since

$$\begin{aligned}
r_j^h(s) &= \left(\frac{(1-\gamma_j)\sum_{h\in\mathcal{H}}\theta_j^h - 2\sum_{h\in\mathcal{H}}\left(\theta_j^h\right)^+}{\sum_{h\in\mathcal{H}}\theta_j^h - 2\sum_{h\in\mathcal{H}}\left(\theta_j^h\right)^+}\right)\cdot r_j(s) \text{ if } \theta_j^h<0 \text{ and} \qquad (14)\\
r_j^h(s) &= \left(2-\frac{(1-\gamma_j)\sum_{h\in\mathcal{H}}\theta_j^h - 2\sum_{h\in\mathcal{H}}\left(\theta_j^h\right)^+}{\sum_{h\in\mathcal{H}}\theta_j^h - 2\sum_{h\in\mathcal{H}}\left(\theta_j^h\right)^+}\right)\cdot r_j(s) \text{ if } \theta_j^h\geq 0,
\end{aligned}$$

then taking derivatives of $\sum_j r_j^h(s)g_j^h\left(\eta^h\right) = \sum_j r_j^h(s)\theta_j^h$ with respect to $\gamma\in\mathbb{R}^J$ yields:

$$\begin{pmatrix}-\alpha_1\left|\theta_1^h\right| & 0 & 0\\ 0 & ... & 0\\ 0 & 0 & -\alpha_J\left|\theta_J^h\right|\end{pmatrix}\cdot Y^T.$$

Define $T_1^h = \begin{pmatrix} -\alpha_1 \left|\theta_1^h\right| & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & -\alpha_J \left|\theta_J^h\right| \end{pmatrix}$. Both terms are nonzero: $\alpha_j < 0 \ \ \forall j$ and $\left|\theta_j^h\right| > 0 \ \forall j$. The second inequality follows from $\eta_j^h \neq 0 \ \forall j, \forall h$ (corollary 1). Thus, T_1^h is a full-rank diagonal matrix.

From the $(FOC\eta)$ and using the definition of $r^h(s)$ from (14), then taking the derivatives of $\sum_{s>0} \lambda^h(s) r^h(s) Dg^h(\eta^h)$ with respect to $\gamma \in \mathbb{R}^J$ yields:

$$\begin{pmatrix} ... & 0 & 0 \\ 0 & \left(\alpha_j \cdot 1\{\theta_j^h < 0\} - \alpha_j \cdot 1\{\theta_j^h \geq 0\}\right) \cdot \sum_{s>0} \lambda^h(s) r_j(s) & 0 \\ 0 & 0 & ... \end{pmatrix} \cdot Dg^h(\eta^h).$$

Define $T_2^h = \begin{pmatrix} ... & 0 & 0 \\ 0 & \left(\alpha_j \cdot 1\{\theta_j^h < 0\} - \alpha_j \cdot 1\{\theta_j^h \geq 0\}\right) \cdot \sum_{s>0} \lambda^h(s) r_j(s) & 0 \\ 0 & 0 & ... \end{pmatrix}$. Both terms $\left(\alpha_j \cdot 1\{\theta_j^h < 0\} - \alpha_j \cdot 1\{\theta_j^h \geq 0\}\right)$ and $\sum_{s>0} \lambda^h(s) r_j(s)$ are nonzero (since $\alpha_j < 0 \ \ \forall j$, $\lambda^h(s) > 0 \ \forall s$, and $r_j = (.., r_j(s), ..)^T > 0$). Thus, T_2^h is a full-rank diagonal matrix. ■

I will consider two subcases:

Subcase A: $\Delta\eta_h^T \neq 0 \ \forall h \in \mathcal{H}$

I want to show that the matrix $\begin{pmatrix} \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} & D_A\Phi' & D_B\Phi' \end{pmatrix}$ has full rank. From claims 7 and 8 and since $\left(\Delta u_h, \Delta p^T, \Delta q^T\right) \neq 0 \ \ \forall h \in \mathcal{H}$ (claim 6), then the first, second, and last row blocks are linearly independent from the others. Thus, the matrix $\begin{pmatrix} \begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} & D_A\Phi' & D_B\Phi' \end{pmatrix}$ has full row rank iff the submatrix

$$\begin{pmatrix} \begin{matrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \end{matrix} & 0 & \begin{matrix} \Omega^1 & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & \Omega^H \end{matrix} & 0 \\ ... \ \ -\Lambda_2 \ \ ... & ** & 0 & 0 \\ 0 & r(-\eta^h|\overrightarrow{0}) & r\left(-\lambda^h(0) I_J\right) & 0 \\ 0 & \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{pmatrix}$$

has full row rank. By the definition of Λ_2, the $[H(S+1)+G-(S+1)] \times HG$ submatrix $\begin{pmatrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \\ ... & -\Lambda_2 & ... \end{pmatrix}$ is a full rank matrix. I have left to show that the matrix $\begin{pmatrix} r(-\eta^h|\overrightarrow{0}) & r\left(-\lambda^h(0) I_J\right) & 0 \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{pmatrix}$ has full rank. Since $\Upsilon_1 = r\left(\overrightarrow{0}|T_1^h \cdot Y^T\right)$ (claim 9) for a full rank matrix T_1^h and Y^T has full row rank, then the final row is linearly independent. The matrix $r\left(-\lambda^h(0) I_J\right)$ has full rank, so the submatrix

$\begin{pmatrix} r(-\eta^h|\overrightarrow{0}) & r\left(-\lambda^h(0)I_J\right) & 0 \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{pmatrix}$ is a full rank matrix. This concludes the proof under subcase A.

Subcase B: $\Delta\eta_h^T = 0$ for some $h \in \mathcal{H}$

Recall the system of equations (a subset of the equations $\nu^T\Psi = 0$):

$$\begin{array}{c} \vdots \\ \Delta u_h Du^h(x^h) + \Delta x_h^T D^2u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T\Lambda = 0. \quad (8.a) \\ \vdots \end{array} \tag{8}$$

$$\begin{array}{c} \vdots \\ -\Delta x_h^T P^T + \Delta\eta_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T = 0. \quad (8.b) \\ \vdots \end{array}$$

Suppose $\exists h' \in \mathcal{H}$ such that $\Delta\eta_{h'}^T = 0$. Postmultiply $(8.a)$ by $\Delta x_{h'}$ and use $(8.b)$ and Φ to obtain:

$$\Delta x_{h'}^T D^2u^{h'}(x^{h'})\Delta x_{h'} - \Delta p^T\Lambda\Delta x_{h'} = 0.$$

The left term is strictly negative (by assumption 2). Thus $\Delta p^T \neq 0$.

The matrix $\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \quad D_B\Phi' \right)$ is given by:

$$\begin{pmatrix}
r\left(Du^h(x^h)^T\right) & d\left(D^2u^h\right) & d(-P^T) & 0 & c(-\Lambda^T) & 0 & 0 & d\left(\dot{A}^h\right) & 0 \\
0 & d\left(-P\right) & 0 & d(\Omega^h) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & d\left((\Omega^h)^T\right) & d(\bar{D}^2g^h) & 0 & c(I_J) & c\left((\Xi^h)^T\right) & 0 & d\left(\dot{B}^h\right) \\
0 & r(-\Lambda_2) & ** & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & r(-\eta^h|\overrightarrow{0}) & r(-\lambda^h(0)I_J) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \Upsilon_1 & \Upsilon_2 & 0 & 0 & \Upsilon_3 & 0 & 0 \\
r(\Delta u_h) & r(\Delta x_h^T) & r(\Delta\lambda_h^T) & r(\Delta\eta_h^T) & \Delta p^T & \Delta q^T & \Delta b & 0 & 0
\end{pmatrix}$$

From claim 7 and since $\Delta p^T \neq 0$, then the first and last row blocks are linearly independent from the others. As in subcase A, it is known that the submatrix $\begin{pmatrix} -P & 0 & 0 \\ 0 & .. & 0 \\ 0 & 0 & -P \\ ... & -\Lambda_2 & ... \end{pmatrix}$ is a full rank matrix. Thus, the matrix $\left(\begin{pmatrix} \Psi^T \\ \nu^T \end{pmatrix} \quad D_A\Phi' \quad D_B\Phi' \right)$ has full rank iff the submatrix

$$\begin{pmatrix}
(\Omega^1)^T & 0 & 0 & \bar{D}^2g^1 & 0 & 0 & I_J & (\Xi^1)^T \\
0 & ... & 0 & 0 & ... & 0 & \vdots & \vdots \\
0 & 0 & (\Omega^H)^T & 0 & 0 & \bar{D}^2g^H & I_J & (\Xi^H)^T \\
(-\eta^1|\overrightarrow{0}) & ... & (-\eta^H|\overrightarrow{0}) & -\lambda^1(0)I_J & ... & -\lambda^H(0)I_J & 0 & 0 \\
\left(\overrightarrow{0}|T_1^1 \cdot Y^T\right) & ... & \left(\overrightarrow{0}|T_1^H \cdot Y^T\right) & T_2^1 \cdot Dg^1(\eta^1) & ... & T_2^H \cdot Dg^H(\eta^H) & 0 & \Upsilon_3
\end{pmatrix} \tag{15}$$

has full row rank where Υ_1, Υ_2 have been replaced using claim 9. By definition, $(\Omega^h)^T = \begin{pmatrix} -q \\ YDg^h(\eta^h) \end{pmatrix}^T$. If the submatrix

$$M = \begin{pmatrix} (YDg^1(\eta^1))^T & 0 & 0 & I_J \\ 0 & ... & 0 & \vdots \\ 0 & 0 & (YDg^H(\eta^H))^T & I_J \\ T_1^1 \cdot Y^T & ... & T_1^H \cdot Y^T & 0 \end{pmatrix} \tag{16}$$

has full row rank, then since $\begin{pmatrix} -\lambda^1(0)I_J & ... & -\lambda^H(0)I_J \end{pmatrix}$ has full rank, the submatrix (15) would have full row rank.

Claim 10 *The matrix M as defined in* (16) *has full row rank.*

Proof. To verify full row rank of M, I will pre-multiply M by $\omega^T = \left(\left(\omega_{\eta^h}^T\right)_{h\in\mathcal{H}}, \omega_\gamma^T\right)$ and verify that $\omega^T M = 0$ implies $\omega^T = 0$.

Take any household $h \in \mathcal{H}$. The equations of $\omega^T M = 0$ are given by:

$$\begin{aligned} &\vdots \\ \omega_{\eta^h}^T (YDg^h(\eta^h))^T + \omega_\gamma^T T_1^h \cdot Y^T = 0. \quad (17.a) \\ &\vdots \\ \sum\nolimits_{h\in\mathcal{H}} \omega_{\eta^h}^T = 0. \quad (17.b) \end{aligned} \tag{17}$$

Since $(YDg^h(\eta^h))^T = \left(Dg^h(\eta^h)\right)^T Y^T$, then equation $(17.a)$ becomes:

$$\left(\omega_{\eta^h}^T \left(Dg^h(\eta^h)\right)^T + \omega_\gamma^T T_1^h\right) Y^T = 0.$$

With Y^T full row rank, then

$$\omega_{\eta^h}^T \left(Dg^h(\eta^h)\right)^T + \omega_\gamma^T T_1^h = 0$$

$$\omega_{\eta^h} = -\left(Dg^h(\eta^h)\right)^{-1} T_1^h \omega_\gamma \tag{18}$$

where equation (18) follows by taking transposes and noting that $Dg^h(\eta^h)$ is invertible and T_1^h is diagonal. This equation (18) holds $\forall h \in \mathcal{H}$. From $(17.b)$:

$$\sum\nolimits_{h\in\mathcal{H}} \left(Dg^h(\eta^h)\right)^{-1} T_1^h \omega_\gamma = 0. \tag{19}$$

By definition, $\left(Dg^h(\eta^h)\right)^{-1} = D\tilde{f}^h(\theta^h)$ and $T_1^h = \begin{pmatrix} -\alpha_1 \left|\theta_1^h\right| & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & -\alpha_J \left|\theta_J^h\right| \end{pmatrix}$ with strictly positive diagonal terms. If f^h is given by the canonical representation, then $D\tilde{f}^h(\theta^h)$ is a diagonal matrix with strictly positive diagonal terms. Thus, $D\tilde{f}^h(\theta^h) \cdot T_1^h$ is a diagonal matrix with strictly positive terms and this holds $\forall h \in \mathcal{H}$. Adding up over all households, the matrix $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h)\cdot T_1^h$ is diagonal with strictly positive terms.

The matrix $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h)\cdot T_1^h$ has full rank under the canonical representation for $\left(f^h\right)_{h\in\mathcal{H}}$. The transaction costs mappings that are used in the statement of the theorem are those defined in lemma 2 as belonging to an open set around the canonical representation. In this open set, $\sum_{h\in\mathcal{H}} D\tilde{f}^h(\theta^h)\cdot T_1^h$ has full rank.

Therefore, $\sum_{h\in\mathcal{H}} \left(Dg^h(\eta^h)\right)^{-1}\cdot T_1^h$ has full rank and so (19) implies that $\omega_\gamma = 0$. From (18), $\omega_{\eta^h} = 0\ \forall h\in\mathcal{H}$. As $\omega^T = 0$, the matrix M has full row rank. ■

This concludes the proof under case I (both subcases).

Case II: $\Delta x_h^T = 0$ for some $h\in\mathcal{H}$

I will show that over a generic subset of $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$, the system of equations (Φ,Φ') has no solution. Recall the system of equations (a subset of the equations $\nu^T\Psi = 0$):

$$\begin{array}{c}\vdots\\ \Delta u_h Du^h(x^h) + \Delta x_h^T D^2u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T\Lambda = 0. \quad (8.a)\\ \vdots\end{array} \tag{8}$$

$$\begin{array}{c}\vdots\\ -\Delta x_h^T P^T + \Delta\eta_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h)\end{pmatrix}^T = 0. \quad (8.b)\\ \vdots\end{array}$$

$$\begin{array}{c}\vdots\\ \Delta\lambda_h^T \begin{pmatrix} -q \\ YDg^h(\eta^h)\end{pmatrix} + \Delta\eta_h^T \sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right) + \Delta q^T = 0. \quad (8.c)\\ \vdots\end{array}$$

Suppose $\exists h'\in\mathcal{H}$ such that $\Delta x_{h'}^T = 0$. From $(8.a)$ and Φ, I obtain

$$\begin{aligned} \Delta u_{h'} Du^{h'}(x^{h\prime}) - \Delta\lambda_{h'}^T P - \Delta p^T\Lambda &= 0 \\ Du^{h'}(x^{h'}) - \lambda^{h\prime} P &= 0 \end{aligned}$$

which together imply that $\Delta p^T = 0$ and $\Delta\lambda_{h'}^T = \Delta u_{h'}\lambda^{h\prime}$. From $(8.b)$, since $YDg^h(\eta^h)$ has full column rank, then $\Delta\eta_{h'}^T = 0$. From $(8.c)$ and Φ, after plugging in $\Delta\lambda_{h'}^T = \Delta u_{h'}\lambda^{h\prime}$ and $\Delta\eta_{h'}^T = 0$ to $(8.c)$, then $\Delta q^T = 0$. For all other $h\neq h'$, postmultiply $\Delta u_h Du^h(x^h)$ by Δx_h and use both first order conditions in Φ and $(8.b)$ to get $\Delta u_h Du^h(x^h)\Delta x_h = 0$. Next, postmultiply $(8.a)$ by Δx_h and use $(8.b)$ and $(8.c)$ (as in the proof of claim 6) to arrive at

$$\Delta x_h^T D^2u^h(x^h)\Delta x_h = -\Delta\eta_h^T\left(\sum\nolimits_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\right)\Delta\eta_h. \tag{9}$$

In (9), the left hand side is strictly negative if $\Delta x_h \neq 0$ and the right hand side is nonnegative. Thus $\Delta x_h^T = 0\ \forall h\in\mathcal{H}$. From $(8.b)$, since $YDg^h(\eta^h)$ has full column rank, then $\Delta\eta_h^T = 0\ \ \forall h\in\mathcal{H}$.

Thus $\forall h \in \mathcal{H}$, $\Delta\lambda_h^T = \Delta u_h \lambda^h$ and $\left(\left(\Delta u_h, \Delta\lambda_h^T\right)_{h\in\mathcal{H}}, \Delta b\right)$ are the only nonzero elements of ν. As such, the following is the equation from $\nu^T\Psi = 0$ corresponding to derivatives with respect to p :

$$\textstyle\sum_{h\in\mathcal{H}} \Delta\lambda_h^s \left(e_{\backslash L}^h(s) - x_{\backslash L}^h(s)\right)^T = 0 \quad \forall s \geq 0 \tag{20}$$

where $\left(e_{\backslash L}^h(s) - x_{\backslash L}^h(s)\right)$ is the $(L-1)-$dimensional vector of household negative excess demand with the numeraire commodity excluded. For the analysis to hold at this point, I must use the assumption that $L \geq 2$. Plugging in $\Delta\lambda_h^T = \Delta u_h \lambda^h$ into equation (20) and only considering the first physical commodity $l = 1$ and the final $s > 0$ states, I have

$$\textstyle\sum_{h\in\mathcal{H}} \Delta u_h \lambda^h(s) \left(e_1^h(s) - x_1^h(s)\right)^T = 0 \quad \forall s > 0$$

or in matrix notation

$$\begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & .... & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & .... & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix} \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_H \end{pmatrix} = 0.$$

From lemma 3, generically on $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$, $\Delta u_h = 0 \ \ \forall h \in \mathcal{H}$. Thus $\Delta\lambda_h^T = 0 \ \ \forall h \in \mathcal{H}$. The following is the equation from $\nu^T\Psi = 0$ corresponding to derivatives with respect to γ :

$$\sum_{h\in\mathcal{H}} \Delta\lambda_h^T \left(\Upsilon_1\right)^T + \Delta\eta_h^T \left(\Upsilon_2\right)^T + \Delta b \left(\Upsilon_3\right)^T = 0.$$

Since $\Delta\lambda_h^T = \Delta\eta_h^T = 0 \ \ \forall h \in \mathcal{H}$ and $\left(\Upsilon_3\right)^T$ has generic full row rank by corollary 1, then $\Delta b = 0$. The entire vector $\nu^T = 0$, which cannot be since Φ' guarantees that $\nu^T\nu/2 = 1$. I conclude that generically case II is not possible. This completes the proof of the theorem. ■

4 Proofs of lemmas

Proof of Lemma 2

Proof. From lemma 1, take any endowment $\left(e^h\right)_{h\in\mathcal{H}}$ from the generic subset of $\mathcal{E} = \{(e^h)_{h\in\ \mathcal{H}} : e^h >> 0\}$. Then, given the canonical representation for f^h, the resulting equilibrium variables will be regular values of Φ. Define the parameters as $\bar{\sigma} = \left(\bar{e}^h, \bar{u}^h, \bar{f}^h\right)_{h\in\mathcal{H}}$ (where $\bar{f}^h$ has the canonical representation). For that $\bar{\sigma}$, there exist finitely many equilibria $\left(\bar{\xi}_i, \bar{\sigma}\right)$ $i = 1, .., I$ where $\bar{\xi}_i = \left(\left(\bar{x}_i^h, \bar{\eta}_i^h\right)_{h\in\mathcal{H}}, \bar{p}_i, \bar{q}_i\right)$. Implicit in lemma 1 is the result that there exists an open set Σ' for all regular values $\bar{\sigma}$ and open sets Ξ_i' such that $\bar{\xi}_i \in \Xi_i' \ \ \forall i = 1, .., I$. Further, the sets Ξ_i' are disjoint across i and $\forall \sigma \in \Sigma'$, $\exists!$ equilibrium $\xi_i^h \in \Xi_i'$.

The parameters $\sigma = \left(e^h, u^h, f^h\right)_{h\in\mathcal{H}}$ in the open set Σ' will be composed of transaction costs mappings $\left(f^h\right)_{h\in\mathcal{H}}$ in an open set around the canonical representation. The mapping $\tilde{f}^h(\theta^h) = \left(\theta_1^h, ..., \theta_J^h\right) + \left(f_1^h(\theta^h), ..., f_J^h(\theta^h)\right)$ is a function of the parameter f^h and the equilibrium variables θ^h. At $\bar{\sigma}$, then $\tilde{f}^h(\bar{\theta}^h) = \left(\bar{\theta}_1^h, ..., \bar{\theta}_J^h\right) + \left(\bar{f}_1^h(\bar{\theta}^h), ..., \bar{f}_J^h(\bar{\theta}^h)\right)$ and equilibrium conditions imply that the mapping $\tilde{f}^h$ is invertible (claim 3). The set of invertible matrices is open. Thus, for any parameters $\sigma \in \Sigma'$, the mapping $\tilde{f}^h$ defined as $\tilde{f}^h(\theta^h) = \left(\theta_1^h, ..., \theta_J^h\right) + \left(f_1^h(\theta^h), ..., f_J^h(\theta^h)\right)$ will also be invertible. ■

Proof of Lemma 3

Proof To prove this, first define

$$Z = \begin{pmatrix} \lambda^1(1)(e_1^1(1) - x_1^1(1)) & .... & \lambda^H(1)(e_1^H(1) - x_1^H(1)) \\ \vdots & & \vdots \\ \lambda^1(S)(e_1^1(S) - x_1^1(S)) & .... & \lambda^H(S)(e_1^H(S) - x_1^H(S)) \end{pmatrix}.$$

I will show that generically on $\mathcal{E}$, the matrix

$$M' = D_{\xi,\omega}\begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2 - 1 \end{pmatrix} = \begin{pmatrix} D_\xi \Phi_{|\Phi(\xi,\sigma)=0} \\ D_{\xi,\omega} Z\omega \\ \begin{pmatrix} 0 & | & \omega \end{pmatrix} \end{pmatrix}$$

has full row rank. Since M' has more rows than columns, if M' has full row rank, then the equations $\begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2 - 1 \end{pmatrix} = 0$ will generically not hold. Thus, Z will generically have full column rank. To show that generically on $\mathcal{E}$, the matrix M' has full row rank, I have to show that the extended matrix

$$M = \begin{pmatrix} M' & | & \begin{pmatrix} D_e \Phi_{|\Phi(\xi,\sigma)=0} \\ D_e Z\omega \\ 0 \end{pmatrix} \end{pmatrix}$$

has full row rank.

Since this proof is independent from the proof in the body, notation will be repeated. To show that M has full row rank, premultiply by the row vector $u^T = \left(\Delta x^T, \Delta\lambda^T, \Delta\eta^T, \Delta p^T, \Delta q^T, \Delta z^T, \Delta\omega\right)$. The lemma is proved upon showing that $u^T = 0$. For convenience, the vector u^T is divided into the indicated subvectors which correspond sensibly with the following equations of $\begin{pmatrix} \Phi \\ Z\omega \\ \omega^T\omega/2-1 \end{pmatrix}$:

$$\begin{array}{rcl}
\Delta x^T & \Longleftrightarrow & FOCx \\
\Delta \lambda^T & \Longleftrightarrow & BC \\
\Delta \eta^T & \Longleftrightarrow & FOC\eta \\
\Delta p^T & \Longleftrightarrow & MCx \\
\Delta q^T & \Longleftrightarrow & MC\eta \\
\Delta z^T & \Longleftrightarrow & Z\omega \\
\Delta \omega & \Longleftrightarrow & \omega^T\omega/2-1.
\end{array}$$

I shall list the equations of $u^TM = 0$ in the order that is most convenient to obtain $u^T = 0$. At my disposal are $\Phi(\xi,\sigma) = 0$ and $\omega \neq 0$.

First, for the columns corresponding to derivatives with respect to x^h and e^h for any $h \in \mathcal{H}$:

$$\begin{array}{rcl}
\Delta x_h^T D^2 u^h(x^h) - \Delta\lambda_h^T P - \Delta p^T \Lambda - \Delta z^T \Lambda_3^h & = & 0 \\
\Delta\lambda_h^T P + \Delta p^T \Lambda + \Delta z^T \Lambda_3^h & = & 0
\end{array}$$

where the matrices P and Λ are as defined previously and Λ_3^h is the $S \times G$ matrix

$$\Lambda_3^h = \left[0 \quad | \quad \begin{pmatrix} \begin{pmatrix} \lambda^h(1)\omega^h & \overrightarrow{0} \end{pmatrix} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & \begin{pmatrix} \lambda^h(S)\omega^h & \overrightarrow{0} \end{pmatrix} \end{pmatrix} \right].$$

By assumption 2, $\left(\Delta x_h^T, \Delta\lambda_h^T\right) = 0 \ \forall h \in \mathcal{H}$, $\Delta p_l(s) = 0 \ \ \forall (l,s) \notin \{(1,1),..,(1,S)\}$, and

$$\Delta p_1(s) + \Delta z_s \lambda^h(s)\omega^h = 0 \quad \forall s > 0 \text{ and } \forall h \in \mathcal{H}. \tag{21}$$

Second, for the columns corresponding to derivatives with respect to η^h for any $h \in \mathcal{H}$ and q :

$$\begin{array}{rcl}
\Delta\eta_h^T \sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) + \Delta q^T & = & 0 \\
\sum_{h\in\mathcal{H}} \Delta\eta_h^T(-\lambda^h(0)) & = & 0.
\end{array} \tag{22}$$

From corollary 1, for a generic subset of $\mathcal{E}$, $\eta_j^h \neq 0 \ \forall j, \forall h$. By the definition of $\eta_j^h = \tilde{f}_j^h\left(\theta^h\right)$, this implies that $\theta_j^h \neq 0 \ \forall j, \forall h$. For any $h \in \mathcal{H}$, from claim 5, the matrix $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is negative semidefinite. Moreover, from equation (7) (recall the equation is given by

$$\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right) = -\lambda^h(0) D^2 F^h\left(\theta^h\right) \left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}), \tag{7}$$

if $D^2F^h\left(\theta^h\right)\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ is positive definite, then $\sum_{s>0} \lambda^h(s) \cdot D^2 G_s^h\left(\eta^h\right)$ is negative definite. By definition, $D^2F^h\left(\theta^h\right)$ is positive definite so long as $\theta_j^h \neq 0$

$\forall j$. Multiplication by $\left[D\tilde{f}^h\left(\theta^h\right)\right]^{-2}$ preserves the positive definiteness (for open sets of transaction costs mappings $\tilde{f}^h$ around the canonical representation). Thus, $\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)$ is negative definite.

Postmultiply the first equation of (22) by $\Delta\eta_h\lambda^h(0)$. The first term

$$\lambda^h(0)\Delta\eta_h^T\left(\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\right)\Delta\eta_h\leq 0\quad(<0\text{ if }\Delta\eta_h\neq 0)\quad\forall h\in\mathcal{H}.$$

This is because the matrix $\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)$ is negative definite and $\lambda^h(0)>0$. The second term $\Delta q^T\Delta\eta_h\lambda^h(0)$ will be equal to 0 when summed over all households. The only way that

$$\sum_{h\in\mathcal{H}}\lambda^h(0)\Delta\eta_h^T\left(\sum_{s>0}\lambda^h(s)\cdot D^2G_s^h\left(\eta^h\right)\right)\Delta\eta_h+\sum_{h\in\mathcal{H}}\Delta q^T\Delta\eta_h\lambda^h(0)=0$$

is if $\Delta\eta_h^T=0\ \forall h\in\mathcal{H}$.[9] From (22), $\Delta q^T=0$.

Finally, for the columns corresponding to derivatives with respect to λ^h for any $h\in\mathcal{H}$ and ω :

$$\begin{aligned}\Delta z^T\Lambda_4^h &= 0\\ \Delta z^TZ+\Delta\omega\left(\omega\right) &= 0\end{aligned}\tag{23}$$

where Λ_4^h is the $S\times(S+1)$ matrix

$$\Lambda_4^h=\begin{pmatrix} & & (e_1^h(1)-x_1^h(1))\omega^h & 0 & 0\\ 0 & | & 0 & ... & 0\\ & & 0 & 0 & (e_1^h(S)-x_1^h(S))\omega^h\end{pmatrix}.$$

From (23), I obtain that

$$\Delta z_s(e_1^h(s)-x_1^h(s))\omega^h=0\quad\forall h\in\mathcal{H}\text{ and }\forall s>0.$$

Generically (corollary 1), $(e_1^h(s)-x_1^h(s))\neq 0\ \ \forall s,\forall h$ and since $\omega\neq 0$, then for some h, $(e_1^h(s)-x_1^h(s))\omega^h\neq 0\ \ \forall s>0$. Thus $\Delta z_s=0\ \ \forall s>0$. From (21), the remaining terms of Δp^T are equal to 0, namely $\Delta p_1(s)=0$ for $s>0$. With $\omega\neq 0$, the scalar $\Delta\omega=0$ (from (23)). Thus $u^T=0$ and the proof of lemma 3 is complete.■

5 References

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