Verifying Competitive Equilibria in Dynamic Economies*

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Abstract

In this paper I examine ϵ-equilibria of stationary dynamic economies with heterogeneous agents and possibly incomplete financial markets. I give a simple example to show that even for arbitrarily small $\epsilon > 0$ allocation and prices can be far away from exact equilibrium allocations and prices. That is, errors in market clearing or individuals' optimality conditions do not provide enough information to assess the quality of an approximation. I derive a sufficient condition for an ϵ-equilibrium to be close to an exact equilibrium. If the economic fundamentals are semi-algebraic, one can verify computationally whether this condition holds. The condition can be interpreted economically as a robustness requirement on the set of ϵ-equilibria which form a neighborhood of the computed approximation.

I illustrate the main result and the computational method using an infinite horizon economy with overlapping generations and incomplete financial markets.

Keywords: Dynamic general equilibrium, semi-algebraic economy, computational methods

JEL Classifications: C63 D52 G12

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1 Introduction

Given a numerically computed ϵ-equilibrium of a dynamic stochastic economy with heterogeneous agents, how can one verify whether this is a good approximation to an exact competitive equilibrium? In this paper, I show that if economic fundamentals are semi-algebraic, i.e. per period utility- and production-functions can be described by finitely many polynomials (see e.g. Blume and Zame (1993) or Kubler and Schmedders (2009)), one can validate numerically whether a computed ϵ-equilibrium provides a good approximation to an exact equilibrium. The basic idea is as follows. One can create a set of ϵ-equilibria by constructing a neighborhood around the computed approximation. Under some regularity conditions explained in detail below, one can ensure that this set contains an exact equilibrium by verifying that, if endogenous variables next period lie in the set, the conditions necessary for competitive equilibrium imply that endogenous variables in the current period must also lie in the set. I show that this verification can be done relatively efficiently by using methods from polynomial optimization and that therefore the method can be used for medium-sized dynamic stochastic models. While this is only a sufficient condition for the computed approximation to be close to an exact equilibrium, I argue that under an economically intuitive robustness requirement on the exact equilibrium it is also necessary.

Applied researchers routinely compute ϵ-equilibria of dynamic stochastic economies although almost nothing is known about the nature of exact equilibria in these models. For dynamic models for which the solution can be characterized by a planner's problem, Santos and Vigo (1998) and Santos (2000) have developed sufficient conditions under which they can give explicit error bounds both on policy functions and on allocations. Under these conditions, error bounds on allocations can be derived from Euler equation residuals and ϵ-equilibria are always close to exact equilibria. Unfortunately, however, these results do not generalize to models with heterogeneous agents and incomplete markets or overlapping generations. In these models, it is not known if recursive equilibria exist or if policies are continuous functions of the state (see e.g. Kubler and Polemarchakis (2003)). Even if for given prices each agent makes only a small error in his individual problem, it is possible that the exact market clearing price is far from the computed approximation. The set of ϵ-equilibria in these economies might therefore be very large and the computed approximation might be nowhere close to an exact equilibrium. In Section 2, I construct a simple example of a deterministic economy with overlapping generations where I find two approximate solutions. I can show that one ϵ-equilibrium is close to an exact equilibrium while the other ϵ-equilibrium which exhibits similar errors in market-clearing and optimality conditions is far away from any exact solution.

So far, no sufficient conditions were known which allow the derivation of error bounds on computed equilibrium prices and allocations in the models considered in this paper. Kubler and Schmedders (2005) show that in these models, ϵ-equilibria can be interpreted as exact equilibria of close-by economies. Their paper does not make any statements about the set of all ϵ-equilibria or how ϵ-equilibria are related to exact equilibria of the given stationary economy. However, it suggests that if a competitive equilibrium is well-behaved (in a sense to be made precise) perturbations of exogenous variables should not lead to large perturbations in the equilibrium and if one considers the set of ϵ-equilibria that results from such perturbations it should form a well-behaved neighborhood around the exact equilibrium. In this paper, I build on this idea, I derive a simple condition that ensures that a set of ϵ-equilibria contains an exact equilibrium and I argue that in well-behaved cases a neighborhood of a computed ϵ-equilibrium which is close to an exact equilibrium should satisfy this condition.

I define a set of ϵ-equilibria to be *robust* if it satisfies the following property. Suppose at date T economic fundamentals are perturbed so that from T onwards the new competitive equilibrium realizes in the ϵ-equilibrium set. Then robustness of this set requires that up to date T the endogenous variables of the competitive equilibrium in which all agents anticipate the changes at T also realize in the set. The main reason for introducing this concept is that robustness ensures that an ϵ-equilibrium set contains an exact equilibrium and that one can effectively check if a (semi-algebraic) ϵ-equilibrium set is robust. For semi-algebraic economies, robust ϵ-equilibrium sets that are semi-algebraic exist for all $\epsilon > 0$. It is economically sensible to focus on competitive equilibria whose neighborhoods form robust ϵ-equilibrium sets since otherwise small perturbations of fundamentals will lead to large changes in the equilibrium.

While the theoretical analysis is conducted using abstract equilibrium sets, one has to consider recursive ϵ-equilibria for the practical error analysis. Computational algorithms typically use recursive methods to approximate equilibria numerically and for this ϵ-equilibria are written as functions mapping the state of the economy into current endogenous variables. If this is the case, one of course wants to verify that the computed recursive ϵ-equilibrium is close to an exact equilibrium for all permissable values of the state. In order to do so, I construct candidate robust ϵ-equilibrium sets by creating a strip around a computed recursive ϵ-equilibrium. I show that one can verify that the ϵ-equilibrium set is robust for all relevant initial values of the state by solving a polynomial optimization problem. While this is not a convex programming problem, there now exist algorithms which find lower bounds on global minima of relatively large constrained polynomial problems (see Laurent (2008) for an overview).

Throughout the paper, I take as given that a candidate approximate equilibrium has been computed by some existing method and that this is described by a continuous (policy-) function $\hat{\rho}$ that is semi-algebraic. While the existence of a recursive exact equilibrium is in general not guaranteed (Citanna and Siconolfi (2008) give conditions for generic existence of recursive equilibria in economies with overlapping generations), recursive ϵ-equilibria always exist. However, typically there is no guarantee that a recursive ϵ-equilibrium can be described by continuous functions. This paper is not about the computation of ϵ-

equilibria (see Judd (1998) for detailed descriptions of algorithms for the computation of equilibria in dynamic stochastic models) but asks how it can be verified that a recursive ϵ-equilibrium is close to an exact equilibrium. Even if there is a recursive ϵ-equilibrium with a continuous policy-function, it is not guaranteed that the construction of a candidate robust ϵ-equilibrium set always works. However, I argue below that if this is not the case, the economy is likely to be so 'badly'behaved' that it seems hopeless to derive accurate numerical solutions at all.

As an example, I study a stochastic economy with overlapping generations which is a generalization of Samuelson (1958) and Diamond (1965) to uncertainty. In many applications (e.g. Rios-Rull (1996) among many others) researchers routinely compute approximate equilibria for versions of this model and find that low degree polynomials suffice for very good approximations of policy-functions. This might strike one as surprising since it is well known that in many dynamic economic models with overlapping generations *the set* of competitive equilibria can be almost arbitrarily wild. In particular the issue of indeterminacy of equilibria in deterministic olg models received a lot of attention and it is now well understood that extremely restrictive assumptions are needed to guarantee local uniqueness (see e.g. Kehoe and Levine (1990)). But this obviously says little about the existence (or non-existence) of simple equilibria that can be approximated by low-degree polynomials or by piece-wise polynomials. As Kehoe and Levine (1990) point out, in deterministic olg models the fact that the equilibrium set is complicated does not necessarily imply anything about the practical computation of equilibria.

In the literature on Markovian equilibria in stochastic games with continuous state-space, it has been observed that for games arising from non-exponential discounting, there exist very ill-behaved equilibria (see e.g. Krusell and Smith (2003)). Numerical work, on the other hand, finds very well-behaved epsilon equilibria. However, there are almost no proofs for existence of continuous or smooth consumption rules (i.e. well behaved equilibria) (see Judd (2008) or Morris (2002) for some examples). In this paper, I focus on competitive equilibria and the methods are not directly applicable to games. As Duffie et al. (1994) point out, equilibrium in stochastic games fits into their general characterization of dynamic equilibria (which is similar to the one I use in this paper) only if one focuses on correlated equilibrium with publicly observed messages. It is subject to further research how the methods can used to make statements about Markov-perfect equilibria in stochastic games.

The rest of the paper is organized as follows. In Section 2, I give a simple example to illustrate the main points of the paper. In Section 3 I abstractly describe the economy, define robust ϵ-equilibrium sets and prove the main theoretical result that relates robust ϵ-equilibria to exact competitive equilibrium. Section 4 introduces recursive methods and relates the theoretical result to polynomial optimization. I argue in this section that it can be verified if a recursive ϵ-equilibrium is close to an exact equilibrium by solving a series of constrained optimization problems. Section 5 applies the methods to examine simple dynamic equilibria in stochastic models with overlapping generations.

2 An example

To illustrate the main ideas I examine one of the simplest examples in which serious problems can arise. Consider an exchange economy with a single perishable commodity and overlapping generations. Time extends from zero to infinity, $t = 0, 1, \ldots$. At each t a representative agent is born and lives for 3 periods. Each period individuals receive endowments that depend on their age, e_a being the endowment of an agent of age $a = 1, \ldots, 3$. Utility is time separable with the utility of an agent born at time t given by

$$U_t(c_1, c_2, c_3) = \sum_{a=1}^{3} \beta^a \frac{c_a^{1-\sigma}}{1-\sigma}.$$

At each t agents can trade in a risk-free bond with price $q(t)$. Let $\theta_a(t)$ denote the bond-holding of an agent of age a at time t. At $t = 0$, the initial conditions of the economy are determined by the bond-holding of the initially alive agents of ages $a = 1, 2$.

A competitive equilibrium is defined as usual by market clearing and agent optimality, that is, it is given by a sequence $\left(q(t), (c_a(t))_{a=1}^{3}, (\theta_a(t))_{a=1}^{2}\right)$ such that for each t, $\sum_{a=1}^{2} \theta_a(t) = 0$ and

$$\begin{aligned} (c_1(t), \ldots, c_3(t+2), \theta_1(t), \theta_2(t+1)) \quad \in \quad & \arg\max\nolimits_{c,\theta} U_t(c(t), \ldots, c(t+2)) \text{ s.t.} \\ & c(t) = e_1 - q(t)\theta(t) \\ & c(t+1) = e_2 + \theta(t) - q(t+1)\theta(t+1) \\ & c(t+2) = e_3 + \theta(t+1). \end{aligned}$$

Since utility is concave and satisfies an Inada condition and agents are finitely lived the first order conditions for agents' optimality are necessary and sufficient. A competitive equilibrium can therefore be described by first order conditions and market clearing. It is useful to define $z(t)$ to be the vector of all endogenous variables relevant at time t, i.e. $z(t) = \left(\theta_1(t-1), q(t), (c_a(t))_{a=1}^{3}, (\theta_a(t))_{a=1}^{2}\right)$. Given initial conditions $\theta_1(-1)$, a competitive equilibrium can then be characterized as a sequence $(z(t))_{t=0}^{\infty}$ with $c_a(t) \geq 0$ for all t and all $a = 1, 2, 3$ that satisfies $h(z(t), z(t+1)) = 0$ for all $t = 0, 1, \ldots$, where

$$h(z(t), z(t+1)) = \begin{cases} -q(t)\frac{1}{c_1(t)^\sigma} + \frac{\beta}{c_2(t+1)^\sigma} \\ -q(t)\frac{1}{c_2(t)^\sigma} + \frac{\beta}{c_3(t+1)^\sigma} \\ c_1(t) - e_1 + q(t)\theta_1(t) \\ c_2(t) - e_2 - \theta_1(t-1) + q(t)\theta_2(t) \\ c_3(t) - e_3 + \theta_1(t-1) \\ \theta_1(t) + \theta_2(t) \end{cases} \tag{1}$$

The beginning-of period period wealth of the middle aged at time t is given by $\theta_1(t-1)$, i.e. the savings of the young in the last period. It is convenient to build market-clearing into the definition of z and h and take the beginning of period wealth of the old to be $-\theta_1(t-1)$. Clearly in any competitive equilibrium the endogenous variables, z, can only take values in the following set.

$$K_0 = \mathbb{R}^2 \times \mathbb{R}_+ \times \mathbb{R}^3_+ \times \mathbb{R}^2 \tag{2}$$

A competitive equilibrium is a steady state if there is a $\bar{z}$ such that $z_t = \bar{z}$ for all t, i.e. if there is a $\bar{z}$ with $h(\bar{z}, \bar{z}) = 0$. There always exists at least one steady state in this example (the situation will be quite different once uncertainty is introduced). Generally however, one is interested in competitive equilibria for initial conditions which are not part of a steady state. Ideally one would like to describe competitive equilibria for an entire interval of initial conditions. In this case one needs a convenient method to numerically describe or approximate the equilibrium. In this paper I assume throughout that the approximate solution is in the form of a *recursive* ϵ-equilibrium. In this example the natural state space Θ consists of beginning of period cash at hand of the middle aged and there are functions approximate policy-functions that map the state, $\theta_- \in \Theta$ into current period consumptions, savings and prices, $\hat{\rho} : \Theta \to \mathbb{R}_+ \times \mathbb{R}^3_+ \times \mathbb{R}^2$. That is for each t, $z(t)$ can be written as

$$z(t) = \left(\theta_1(t-1), q(t), (c_a(t))_{a=1}^3, (\theta_a(t))_{a=1}^2\right) = (\theta_1(t-1), \hat{\rho}(\theta_1(t-1))$$

and satisfies for each t, $\|h(z_t, z_{t+1})\| < \epsilon$.

2.1 A numerical specification

For the concrete example suppose there is no discounting ($\beta = 1$), the coefficient of relative risk aversion is given by $\sigma = 3$ and individual endowments are $e_1 = 1, e_2 = 10.575$ and $e_3 = 0.5$. Suppose one is interested in equilibria for initial conditions $\theta_- = -6$.

I turns out that for this specification there exist recursive ϵ -equilibria for which $\hat{\rho}$ can be written as a polynomial of degree 4. It is not clear if polynomials are always the right choice in these economies as policies could be backward bending - however, for this example (and all the ones consider in Section 5 below) it turns out fine. There are the following two candidate solutions.

(i) The admissible state space is $\Theta = [-6, 0]$, approximate savings policy of the middle aged is given by

$$\hat{\rho}_{\theta^2}(\theta_-) = 0.000433\theta_-^4 + 0.00224\theta_-^3 - 0.000585\theta_-^2 + 0.0289\theta_- - 0.154$$

The bond price is given by

$$\hat{\rho}_q(\theta_-) = -0.0012\theta_-^4 + 0.0382\theta_-^3 + 1.35\theta_-^2 + 14.9\theta_- + 52.7$$

(ii) The admissible state space is $\Theta = [-6, -5.4]$, the approximate savings policy (of the middle aged) is given by the following polynomial

$$\hat{\rho}_{\theta^2}(\theta_-) = 0.245\theta_-^3 + 4.54\theta_-^2 + 28.7\theta_- + 55.4.$$

The bond price is given by

$$\hat{\rho}_q(\theta_-) = 0.0385\theta_-^3 + 0.717\theta_-^2 + 4.56\theta_- + 10.$$

Figure 1 shows the approximate portfolio-policies for the two candidate solutions. Clearly the two equilibria are quite different. Solution i) is inefficient, with very low consumption of the old, Solution ii) is efficient with high consumption of the old.

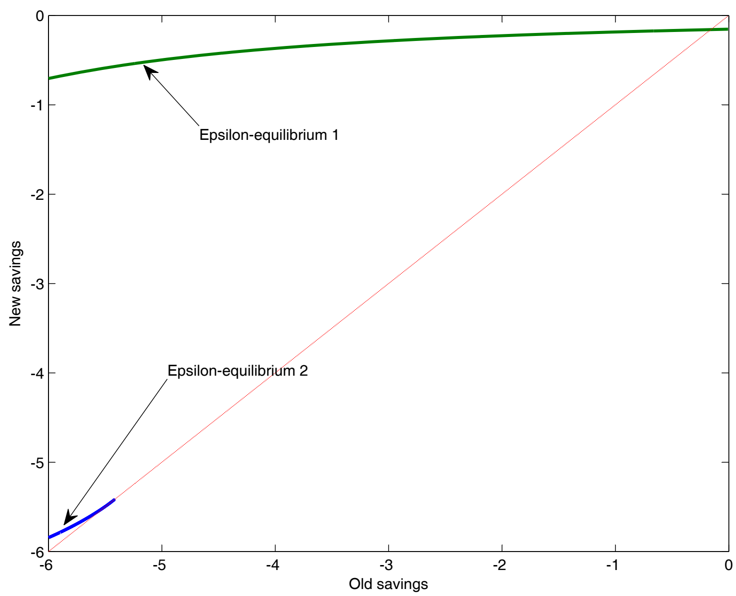

Figure 1

In this simple example, one can verify that there is something 'wrong' with the second approximate solution as the model has a unique steady state at around $\theta_- = -0.15839$. The second solution seems to converge to an approximate steady state at around $\theta_- = -5.44571$ but all exact steady states in this model are characterized by a finite number of polynomial equations and the methods in Kubler and Schmedders (2009) can be used to show that there is a unique steady state in this economy. In fact, building on this, I show in Appendix A of the paper, that there cannot be a competitive equilibrium anywhere close to candidate solution ii), i.e. this ϵ-equilibrium cannot be part of an cyclical equilibrium either.

The question is how to determine which one of these candidate solutions provides a good approximation to an exact equilibrium without knowing the steady states. As mentioned above, in models with stochastic shocks, there are no steady-state equilibria and the quality of an approximation has to be judged differently. The standard measure is relative errors in Euler equations. That is, one can impose market clearing and the budget-constraints on the approximate equilibrium by computing consumptions from the budget constraints. The only error is then in the Euler equations and it is useful (see Judd (1998)) to report the maximum relative error

$$\epsilon^r = \max_t \left\{ \max \left(\beta \frac{c_2(t+1)^{-\sigma}}{q(t)c_1(t)^{-\sigma}} - 1, \frac{\beta c_3(t+1)^{-\sigma}}{q(t)c_2(t)^{-\sigma}} - 1 \right) \right\}.$$

In both cases these errors turn out to be very small (in solution 2 the maximum error is 10^{-5}, in solution 1 around 2×10^{-5}) and therefore this criterion cannot discriminate between the good approximation and the candidate solution that is far from any exact equilibrium. In fact, it is clear from Santos (2000) that one can only infer from these errors how far away agents choices are from optimal choices, given fixed prices. In this example both choices and prices are approximate.

Another possibility would be to increase the precision and try to compute equilibria with smaller errors, hoping that eventually the error for candidate solution 2 would not decrease further. For this it is useful to use a back-ward time iteration scheme as the algorithm to compute equilibrium. Define an approximate equilibrium set, $\mathcal{Z}^\epsilon$ as the graph of the policy function, i.e. $\mathcal{Z}^\epsilon = \text{graph}(\hat{\rho})$, and define the backward operator as follows.

$$\mathbf{B}(K_0, \mathcal{Z}^\epsilon) = \{\bar{z} \in K_0 \text{ with } \bar{\theta}_- \in \Theta : \exists z \in \mathcal{Z}^\epsilon, \text{ such that } h(\bar{z}, z) = 0\}, \tag{3}$$

where the set K_0 is as in Equation (2). For a given set $\mathcal{Z}^\epsilon$ this operator describes the equilibrium values in the current period, given that the equilibrium in the next period is described by $\hat{\rho}$. Clearly if $\mathbf{B}(K_0, \mathcal{Z}^\epsilon) = \mathcal{Z}^\epsilon$, the function $\hat{\rho}$ describes an exact recursive equilibrium. The reason why this operator is defined on sets rather than functions will become clear below.

One would hope that if one applies this operator repeatedly one converges to an exact equilibrium, i.e. given an approximate equilibrium $\mathcal{Z}_0^\epsilon$, if one takes $\mathcal{Z}_i^\epsilon = \mathbf{B}(K_0, \mathcal{Z}_{i-1}^\epsilon)$, $i = 1, 2, \ldots$ then eventually $\mathcal{Z}_i^\epsilon$ will be close to some exact equilibrium set. However, it might be far away from $\mathcal{Z}_0^\epsilon$ and since determining the limit requires infinitely many exact computations this is not a possible solution. However, what can be done is to verify that if one starts with any approximate equilibrium in some neighborhood of the computed approximation, the new (hopefully better) approximation stays in this neighborhood. If, in addition, one can prove that $\mathcal{Z}_i^\epsilon$ is non-empty for all i, this suffices to prove that there is an exact equilibrium in this neighborhood of the computed approximation. More formally, define for a given $\delta > 0$ the set

$$\widetilde{\mathcal{Z}}^\epsilon = \{z = (\theta_-, q, c_1, c_2, c_3, \theta_1, \theta_2) : \theta_- \in \Theta, \|(q, c_1, c_2, c_3, \theta_1, \theta_2) - \hat{\rho}(\theta_-)\| < \delta\}.$$

If it is the case that

$$\mathbf{B}(K_0, \widetilde{\mathcal{Z}}^\epsilon) \subset \widetilde{\mathcal{Z}}^\epsilon, \tag{4}$$

and if $\mathcal{Z}_i^\epsilon$ is non-empty for all i, then obviously $\mathcal{Z}_i^\epsilon$ has to remain in $\widetilde{\mathcal{Z}}^\epsilon$ and this set should contain an exact equilibrium. This is proven formally in Theorem 1 below (the only technical complication is to ensure existence in the limit).

Condition (4) is interesting because it can be verified by solving a constraint maximization problem. It is equivalent to the statement that the optimal value of the following problem is below δ.

$$\max_{\epsilon,\theta_-\in\Theta,\eta\in K_0} (\eta - \hat{\rho}(\theta_-))^2 \ s.t.$$
$$h\left((\theta_-,\eta),(\theta_1,\hat{\rho}(\theta_1)+\epsilon)\right) = 0$$
$$\|\epsilon\|^2 \leq \delta$$

I explain in detail in Section 4 below how good upper bounds can be obtained for this problem using semi-definite programming and sum-of-squares relaxation. The key is to write $h(.)$ as a polynomial system. This requires the utility function to be semi-algebraic. For the numerical example above, the function h as defined in Equation (1) is not polynomial since because of the first two equations. However, for rational σ they can be easily rewritten as polynomials.

To summarize, to check whether a computed solution is close to an exact equilibrium one has to solve a global polynomial optimization problem and one has to prove that $\mathcal{Z}_i^\epsilon$ are non-empty for all i. This procedure can be applied easily to candidate solution (i) to prove that there exists an exact equilibrium with a neighborhood of 10^{-4} of the computed approximation. In Section 5, I give a general proof that $\mathcal{Z}_i^\epsilon$ is non-empty which can be applied to this case. I also revisit this example explain exactly how the computations are done and explain why this method of verification must fail for candidate Solution (ii). In Appendix A, I prove that there is no exact equilibrium close to this ϵ-equilibrium.

3 An abstract model

In order to formally present the main theoretical result, I first introduce a general stochastic dynamic framework that fits both models with overlapping generations and models with infinitely lived agents and incomplete markets. At this stage I do not assume that the approximate equilibrium is recursive, since the main result is easiest to prove in a general framework.

3.1 The Dynamic Economy

I consider a general abstract formulation of dynamic general equilibrium. Duffie et al. (1994) use a similar framework (not assuming differentiability and semi-algebraic fundamentals, as I do) and show in their paper that it encompasses general equilibrium models with overlapping generations as well as models with infinitely lived agents. It will turn out that most dynamic general equilibrium models used in applications fit the framework.

Time and uncertainty are represented by a countably infinite tree Σ. Each node of the tree, $\sigma \in \Sigma$, is a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ for a given initial shock s_0. The process of shocks (s_t) is assumed to be a Markov chain with finite support $\mathcal{S}$. To indicate that $s^{t'}$ is a successor of s^t (or s^t itself), I write $s^{t'} \succeq s^t$. The number of elements in $\mathcal{S}$ is S. The $S \times S$ transition matrix is denoted by π. By a slight abuse of notation, for $\sigma' \succeq \sigma$, I write $\pi(\sigma'|\sigma)$ to denote the conditional probability of σ' given σ.

I consider dynamic economic models where an equilibrium can be characterized by a system of semi-algebraic equalities and weak inequalities relating current-period exogenous and endogenous variables to endogenous and exogenous variables one period ahead. Examples of such conditions are individuals' Euler equations, firms' first-order conditions, and market-clearing equations for goods or financial assets.

A subset $A \subset \mathbb{R}^n$ is a semi-algebraic subset of $\mathbb{R}^n$ if it can be written as the finite union and intersection of sets of the form $\{x \in \mathbb{R}^n : g(x) > 0\}$ or $\{x \in \mathbb{R}^n : f(x) = 0\}$ where f and g are polynomials in x with coefficients in $\mathbb{R}$, that is, $f, g \in \mathbb{R}[x]$. Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. A function $\theta : A \to \mathbb{R}^m$ is semi-algebraic if its graph $\{(x, y) \in A \times \mathbb{R}^m : y = \theta(x)\}$ is a semi-algebraic subset of $\mathbb{R}^{n+m}$. Blume and Zame (1993) and Kubler and Schmedders (2009) discuss in detail the assumption of semi-algebraic fundamentals in finite exchange economies.

Current period endogenous variables are denoted by $z \in \mathbb{R}^M$. I assume that the system of inequalities characterizing equilibrium can be written as follows.

$$h(\bar{s}, \bar{z}, z_1, \ldots, z_S) = 0, \quad g(\bar{s}, \bar{z}) \geq 0, \tag{5}$$

where for each fixed $s \in \mathcal{S}$, h and g are continuous semi-algebraic functions. The arguments $(\bar{s}, \bar{z})$ denote the exogenous state and endogenous variables for the current period. The vector $z_s \in \mathbb{R}^M$ denotes endogenous variables in the subsequent period in state s. This is identical to the characterization in the example in Section 2, except that now one also has to consider S possible exogenous shocks in the subsequent period and that I want to allow for additional inequality constraints.

A competitive equilibrium is then a process $(z(s^t))$ such that for each s^t

$$h(s_t, z(s^t), z(s^t, 1), \ldots, z(s^t, S)) = 0, \quad g(s_t, z(s^t)) \geq 0 \tag{6}$$

As explained for the example in Section 2, it is useful to describe a competitive equilibrium not by infinite sequences but by a set that consists at least all elements of the sequence but might also contain several equilibria at the same time.

Definition 1 *An equilibrium set is a set $\mathcal{Z} = \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_S \subset \mathbb{R}^{MS}$, such that for all $\bar{s} \in \mathcal{S}$ and all $\bar{z} \in \mathcal{Z}_{\bar{s}}$, $g(\bar{s}, \bar{z}) \geq 0$ and there exist $(z_1, \ldots, z_S) \in \mathcal{Z}$ such that*

$$h(\bar{s}, \bar{z}, z_1, \ldots, z_S) = 0.$$

Given the equilibrium equations (5), I define a *backward operator* to map variables next period into variables in the current period that are consistent with the equilibrium conditions. That is, given sets $K_0, K_1, ..., K_S \subset \mathbb{R}^M$, I define for each $\bar{s}$,

$$\mathbf{B}_{\bar{s}}(K_0, (K_1, \ldots, K_S)) = \{\bar{z} \in K_0 : \exists z_s \in K_s, s = 1, ..., S \text{ such that } h(\bar{s}, \bar{z}, z_1, \ldots, z_S) = 0, \\ g(\bar{s}, \bar{z}) \geq 0\}.$$

Again, this is the same object as in the example in Section 2, except that it has to be defined for all possible shocks. Very roughly speaking in the subsequent analysis this operator will play a role similar to the role of the Bellman operator in dynamic programming. The main difference is that it is defined on sets and not on functions.

3.2 Robust ϵ-equilibria

Given any $\epsilon \geq 0$, define an ϵ-equilibrium set to be a set $\mathcal{Z}^\epsilon = \mathcal{Z}_1^\epsilon \times \ldots \times \mathcal{Z}_S^\epsilon \subset \mathbb{R}^{MS}$, such that for all $\bar{s} \in \mathcal{S}$ and all $\bar{z} \in \mathcal{Z}_{\bar{s}}^\epsilon$, $g(\bar{s}, \bar{z}) \geq 0$ and there exist $(z_1, \ldots, z_S) \in \mathcal{Z}^\epsilon$ such that

$$\|h(\bar{s}, \bar{z}, z_1, \ldots, z_S)\| \leq \epsilon.$$

The following is an abstract definition of a robust ϵ-equilibrium set.

Definition 2 *An ϵ-equilibrium set $\mathcal{Z}^\epsilon \subset \mathbb{R}^{MS}$ is robust if it is closed and bounded and if for all $\bar{s} \in \mathcal{S}$,*

$$\mathbf{B}_{\bar{s}}(\mathbb{R}^M, (\mathcal{Z}_1^\epsilon, \ldots, \mathcal{Z}_S^\epsilon)) \subset \mathcal{Z}_{\bar{s}}^\epsilon.$$

The definition requires that for all endogenous variables in the ϵ-equilibrium set which could realize in the subsequent period, all variables in the current period that are consistent with equilibrium must also lie in the set. I present a more intuitive economic interpretation of the concept below after characterizing robust ϵ-equilibrium in terms of exact equilibrium. The following lemma provides the theoretical foundation for this.

Lemma 1 *Suppose that there are (non-empty) closed and bounded set $(K_1^0, ..., K_S^0)$ such that if one defines recursively, for each $\bar{s}$,*

$$K_{\bar{s}}^i = \mathbf{B}_{\bar{s}}(K_{\bar{s}}^0, (K_1^{i-1}, \ldots, K_S^{i-1}))$$

each K_s^i is non-empty and closed. Then there exists an equilibrium set with $\mathcal{Z}_s \subset K_s^0$ for all $s = 1, ..., S$

Although the lemma follows directly from Duffie et al (1994) I give the proof for completeness - the proof also helps to understand the subsequent analysis.

Proof of the Lemma. The main step of the proof consists in showing, by induction, that $K_s^i \subset K_s^{i-1}$ for all i and all s. By definition $K_s^1 \subset K_s^0$ for all s. To show that if $K_s^i \subset K_s^{i-1}$, it must be that also $K_s^{i+1} \subset K_s^i$, observe that if for a given $\bar{s}, \bar{z}$, there exist $z_s \in K_s^i, s = 1, ..., S$ such that $h(\bar{s}, \bar{z}, z_1, ..., z_S) = 0, g(\bar{s}, \bar{z}) \geq 0$ then, since $K_s^i \subset K_s^{i-1}$, there must also exist $z_s \in K_s^{i-1}$, $s = 1, \ldots, S$ satisfying this property, and hence $\bar{z}$ must lie in $K_{\bar{s}}^i$. Since the intersection of nested closed non-empty sets is non-empty, one can now define for each $s \in \mathcal{S}$, $\mathcal{Z}_s = \cap_{i=0}^\infty K_s^i$. Clearly the collection of sets $(\mathcal{Z}_1, \ldots, \mathcal{Z}_S)$ satisfy the conditions of an equilibrium set. $\square$

The lemma states that if one has candidate equilibrium sets K^0 and one can somehow prove that the recursively defined K^i are non-empty for all i, then one can infer that each K_s^0 in fact contains an equilibrium set $\mathcal{Z}_s$. If one takes K^0 to be a robust ϵ equilibrium set, in order to apply the lemma one still somehow needs to verify that each K^i is non-empty.

It is useful to do this via showing existence of truncated equilibria.

DEFINITION 3 *Given arbitrary sets $(Z_1, ..., Z_S)$, $Z_s \subset \mathbb{R}^M$, define a T-truncated equilibrium with terminal condition $(Z_1, ..., Z_S)$ as a finite horizon process $(z(s^t))_{t \leq T}$ such that for each s^t, $t \leq T-1$, the equilibrium conditions (6) hold and such that $z(s^T) \in Z_{s_T}$ for all terminal s^T.*

The concept is closely related to the standard definition of equilibrium in truncated economies. The only difference is that at the final period, T, agents face prices, consumptions and investments prescribed by Z and not as in the standard concept, zero asset prices and no new trade. Showing existence of a T-truncated equilibrium with terminal condition turns out to be not much harder than showing existence of equilibria for truncated economies (which is part of standard existence proofs in these models). A sufficient condition for existence is typically that each Z_s contains a continuous function and that there are constraints on trades which guarantee that it is never feasible to leave the specified state space. I illustrate this with an example below.

If for an arbitrary ϵ-equilibrium set $\mathcal{Z}^\epsilon$, there exists a truncated equilibrium with terminal condition $\mathcal{Z}^\epsilon$ for all T, it is not guaranteed that the set contains an exact equilibrium since it is not guaranteed that the truncated equilibria take values in $\mathcal{Z}^\epsilon$. However, if the ϵ-equilibrium set is robust, it is clear by the definition that the truncated equilibria (if they exist) must take values in the set. One can apply Lemma 1 and ensure that there must be an exact equilibrium set contained in the ϵ-equilibrium set. The following theorem states this formally.

THEOREM 1 *Suppose $\mathcal{Z}^\epsilon$ constitutes a robust ϵ -equilibrium and that for each T there exists a T-truncated equilibrium with terminal condition $\mathcal{Z}^\epsilon$. Then there exists an exact equilibrium set $\mathcal{Z}$ with $\mathcal{Z}_s \subset \mathcal{Z}_s^\epsilon$ for each $s \in \mathcal{S}$.*

While the definition of robust ϵ-equilibrium makes no mention of an exact equilibrium this theorem allows for the following interpretation of robust ϵ-equilibrium in terms of exact equilibrium.

One typically hopes that competitive equilibria in infinite horizon models are good approximations to equilibria in models with large finite horizons and that these equilibria converge to the infinite equilibrium. In fact one hopes that changes in exogenous variables in the far future have negligible effect on endogenous variables today. As Kubler and Schmedders (2005) show, ϵ-equilibria can be interpreted as equilibria of a *perturbed* economy, i.e. equilibria of an economy with slightly different endowments or preferences. Robustness of an ϵ-equilibrium sets requires that no matter how exogenous variables in the future are (locally) perturbed, as long as the new equilibrium realizes in the ϵ-equilibrium set, the effect on endogenous variables today must be no larger than the effect on endogenous variables at the date of the perturbation.

Note that a reverse interpretation is not possible. If up to some T all endogenous variables realize in any ϵ-equilibrium set, the only way that from T onwards all equilibrium conditions hold exactly is that at the value of all endogenous variables at T, $z(s^T)$ already lies in an exact equilibrium set.

3.3 Verification and existence of robust ϵ-equilibria

The main advantage of the concept is that semi-algebraic robust ϵ-equilibria exist under reasonable assumptions on the fundamentals and that (at least in principle) one can always check numerically whether a given semi-algebraic set constitutes a robust ϵ-equilibrium. This follows directly from the so-called Tarski-Seidenberg principle and the quantifier elimination algorithm (see e.g. Bochnak et al. (1998, Chapter 5) for the Tarski-Seidenberg principle and Basu et al. (2003) for computational issues). More precisely, there is an algorithm that decides for a given semi-algebraic set of ϵ-equilibria if it is a robust ϵ-equilibrium set[1]. If it is, it must contain an exact equilibrium. This raises the question if one should expect robust ϵ-equilibrium sets to exist and to be semi-algebraic.

3.3.1 Existence of robust ϵ-equilibrium

Suppose from the economic model one can find a priori bounds on all endogenous equilibrium variables (the example below shows this is usually not very difficult, these bound typically come from non-negativity constraints in consumption, market-clearing etc.) Let K^0 satisfy these bounds and without loss of generality, impose the bounds in the equilibrium inequality $g(.) \geq 0$.

It is clear that if K^0 is semi-algebraic the constructed K_s^i are semi-algebraic for all $i = 1, ...$ and each s. Fixing an $\epsilon > 0$ there must now exist a sufficiently large i such that K^i actually constitutes an ϵ-equilibrium set. By boundedness of K^0, for each $\delta > 0$ there must be an i such that for all s, if $z \in K_s^i$ there must be a $\tilde{z} \in K_s^{i+1}$ with $\|z - \tilde{z}\| < \delta$. For each ϵ there must be a δ so that if $h(\bar{s}, \tilde{z}, z_1, \ldots, z_S) = 0$ then $\|h(\bar{s}, z, z_1, \ldots, z_S)\| \leq \epsilon$ whenever $\|z - \tilde{z}\| < \delta$.

The construction also implies that K^i is robust - if there existed some $\bar{s}, \bar{z}$ with $\bar{z} \in \mathbf{B}_{\bar{s}}(\mathbb{R}^M, (K_1^i, \ldots, K_S^i))$ but $\bar{z} \notin K_{\bar{s}}^i$, clearly by Lemma 1, we must have $\bar{z} \notin K_{\bar{s}}^0$. But this is impossible because the inequality $g(.) \geq 0$ imposes $\bar{z} \in K_{\bar{s}}^0$ by constructions.

Unfortunately, so far the analysis did not put enough structure on the equilibrium to ensure that the ϵ-equilibrium set is actually *small* and Theorem 1 has some content. More structure is also needed to develop efficient algorithms for verifying robustness of a candidate robust ϵ-equilibrium set. Feng et al. (2009) develop a method to approximate an robust

[1] It is well known that quantifier elimination is hopelessly inefficient. I introduce more a more tractable method for this below.

ϵ-equilibrium set by discretization. It is subject to further research to determine if for a given level of discretization it can actually be proven that their method produces a robust ϵ-equilibrium set.

4 A recursive formulation

So far, the analysis was done for abstract (ϵ) equilibrium sets. However, while it is easy to compute one 'recursive' approximate equilibrium, researchers typically do not explicitly compute entire sets of equilibria. In this Section, I use the theoretical results above to show that it is possible to verify that a recursive ϵ-equilibrium is close to an exact (not necessarily recursive) competitive equilibrium.

For this, I need to impose a bit more structure on the abstract economy and define a recursive ϵ-equilibrium. As in Section 2, I write the vector of endogenous variables as $z = (\theta_-, \eta)$, with θ_- being the 'endogenous state'. The relevant endogenous state space $\Theta = (\Theta_1, \ldots, \Theta_S)$ with each $\Theta_s \subset \mathbb{R}^D$ depends on the underlying model and is determined by the payoff-relevant pre-determined endogenous variables; that is, by variables sufficient for the optimization of individuals at every date-event, given the prices. If Θ is the 'endogenous state space' there must exist set-valued functions $\rho_s : \Theta_s \rightrightarrows \mathbb{R}^{M-D}$ such each $\mathcal{Z}_s = \text{graph}(\rho_s)$ for all $s \in \mathcal{S}$.

The function $h(.)$ typically uniquely determines θ_{-s} for each shock s, as a function of $\bar{z}$. In the simplest example, the beginning-of-period portfolio holding is the endogenous state and this is equal to last period's choices across agents. I illustrate this point in the next section.

The value of the state variables $s_0 \in \mathcal{S}$,$\theta_-(0) \in \Theta_{s_0}$ in period 0 is called 'initial condition' and is part of the description of the economy. It will often be useful to make this explicit. In particular I often want to require that an equilibrium set describes a family of equilibria arising from different initial conditions in a set of $\theta_-(0)$ that contains an open set. Through this requirement, the state-space Θ is partly specified exogenously, but it is of course endogenous in the sense it must contain all realizations of θ_- that occur in equilibrium. In some models with exogenous constraints on trades, Θ can be taken as exogenous since the realizations of θ_- are predetermined through these restrictions. This applies for example in models with asset markets and short-sale constraints on these assets. For the purpose of this section it will be useful to assume that there are sufficient constraints on trades that ensure that in fact Θ is specified exogenously. I will give an example below where this is not the case and show that it is without loss of generality to assume that in fact agents face trading constraints that are never binding in equilibrium.

A recursive ϵ-equilibrium consists of sets $\hat{\Theta}_s$, and functions $\hat{\rho}_s : \hat{\Theta}_s \to \mathbb{R}^{M-D}$, $s \in \mathcal{S}$ such that if $\mathcal{Z}_s^\epsilon = \text{graph}(\hat{\rho}_s)$ for all $s \in \mathcal{S}$, then $\mathcal{Z}^\epsilon$ constitutes an ϵ-equilibrium set. Note that the comment made about the exogeneity of Θ also applies to $\hat{\Theta}$. Again I will assume that $\hat{\Theta}$ is given through the description of the economy and that therefore $\Theta_s = \hat{\Theta}_s$ for all

$s \in \mathcal{S}$. From now I will also assume that $\hat{\rho}_s$ is a continuous semi-algebraic function (which implies that $\hat{\Theta}_s$ is a semi-algebraic set) for all $s \in \mathcal{S}$.

It is easy to see that $\hat{\rho}_s$ itself, since it is a function, will never describe a robust ϵ-equilibrium set (unless it is an exact equilibrium). The first step is therefore to create a function strip around $\hat{\rho}$ that describes an entire set of ϵ-equilibria.

4.1 Constructing robust ϵ-equilibrium sets

For a given approximate recursive equilibrium $(\hat{\Theta}_s, \hat{\rho}_s)_{s \in \mathcal{S}}$ and fixed $\delta > 0$, I take as a candidate ϵ -equilibrium set

$$\mathcal{Z}_s^\epsilon = \{(\theta_-, \eta) : \theta_- \in \hat{\Theta}_s, \quad \|\eta - \hat{\rho}_s(\theta_-)\| \le \delta\}, \quad s \in \mathcal{S}. \tag{7}$$

Note that the exact relation between δ and ϵ in this definition is not important for what follows. What is important is that one would *hope* that if for sufficiently small δ, $\mathcal{Z}^\epsilon$ contains an exact competitive equilibrium, $\mathcal{Z}^\epsilon$ should also be a robust ϵ-equilibrium set. At this abstract level this is not entirely clear. This should certainly be trued if close to the exact equilibrium the backward operator is monotone. In Section 5 I illustrate this point in detail.

Given the analysis above, in order to verify robustness, one now needs to verify that for all $s \in \mathcal{S}$,

$$\mathbf{B}_s(\mathbb{R}^M, (\mathcal{Z}_1^\epsilon, \ldots, \mathcal{Z}_S^\epsilon)) \subset \mathcal{Z}_s^\epsilon.$$

However, this neglects the fact that one would like the recursive ϵ-equilibrium to be close to an exact equilibrium for all $\theta_- \in \hat{\Theta}_s$, $s \in \mathcal{S}$. But since I assumed that Θ is given exogenously through constraints on trades, it is without loss of generality to impose that the inequalities $h(.) \geq 0$ ensure that θ_{-s} always realize in Θ_s. Therefore the definition of robust equilibrium is now equivalent to the following, perhaps more intuitive concept. Define for each $s \in \mathcal{S}$, $\mathcal{Y}_s = \{z = (\theta_-, \eta) \in \mathbb{R}^M : \theta_- \in \hat{\Theta}_s\}$ and require for $\mathcal{Z}^\epsilon$ that for each $s \in \mathcal{S}$,

$$\mathbf{B}_s(\mathcal{Y}_s, (\mathcal{Z}_1^\epsilon, \ldots, \mathcal{Z}_S^\epsilon)) \subset \mathcal{Z}_s^\epsilon. \tag{8}$$

In other words, if endogenous variables next period lie within some δ of $\hat{\rho}$ all endogenous variables this period must also lie within δ of $\hat{\rho}$. Of course, Theorem 1 now needs to be slightly modified and one needs to verify that truncated equilibria exist for all initial conditions in $\hat{\Theta}$. The rest of the argument then remains the same.

If $\mathcal{Z}^\epsilon$ is robust (and the conditions of Theorem 1 are satisfied) it must contain an exact equilibrium, i.e. the computed approximation must be within δ of an exact equilibrium for all values of the state. Note that these are absolute errors. Alternatively, we could have defined

$$\mathcal{Z}_s^\epsilon = \{(\theta_-, \eta) : \theta_- \in \hat{\Theta}_s, \quad \left| \max_i \frac{\eta_i}{\hat{\rho}_{si}(\theta_-)} - 1 \right| \; le\delta\}$$

to obtain relative errors. The exposition in this section is using absolute errors, while I will use relative errors in some of the examples below.

The advantage of working in a recursive framework is now that one can formulate Condition (8) as a constrained optimization problem. Given a fixed $\delta > 0$, I consider the following constrained optimization problem for each $s \in \mathcal{S}$,

$$
\begin{aligned}
\max_{\theta_- \in \hat{\Theta}_s, \epsilon, \eta} \|\eta - \hat{\rho}(\theta_-)\| \text{ s.t. } \quad & \|\epsilon\| \leq \delta \\
& h\left(s, \theta_-, \eta, (\theta_{-1}, \hat{\rho}_1(\theta_{-1}) + \epsilon_1), \ldots, (\theta_{-S}, \hat{\rho}_S(\theta_{-S}) + \epsilon_S)\right) = 0 \\
& g(s, \theta_-, \eta) \geq 0
\end{aligned}
\tag{9}
$$

It is easy to see that if the optimal value of this problem lies below δ the set $\mathcal{Z}^\epsilon$ as defined in (7) is a robust ϵ-equilibrium set.

Since the optimization problem (9) is not a convex programming problem one can generally not find the global maximum. However, in the semi-algebraic case, matters are different. I assume that h,g and $\hat{\rho}$ are continuous semi-algebraic functions and that $\hat{\Theta}_s$ are closed semi-algebraic sets. It follows from Proposition 2.1.8. of Bochnak et al. (1998) that there exist polynomials $p^i : A \to \mathbb{R}^k$ and $q^i : A \to \mathbb{R}^l$, $i = 1, ..., m$, such that

$$h\left(s, \theta_-, \eta, (\theta_{-1}, \hat{\rho}_1(\theta_{-1}) + \epsilon_1), \ldots, (\theta_{-S}, \hat{\rho}_S(\theta_{-S}) + \epsilon_S)\right) = 0, g(s, \theta_-, \eta) \geq 0$$

if and only if for some $i = 1, ..., m$,

$$p^i\left(s, \theta_-, \eta, (\theta_{-1}, \hat{\rho}_1(\theta_{-1}) + \epsilon_1), \ldots, (\theta_{-S}, \hat{\rho}_S(\theta_{-S}) + \epsilon_S)\right) = 0, q^i(s, \theta_-, \eta) \geq 0.$$

Furthermore there exist polynomials ξ^i and ζ^i such that $\hat{\rho}(\theta_-) = \eta \Leftrightarrow \xi^i(\theta, \eta) = 0, \zeta^i(\theta, \eta) \geq 0$ for some i. Therefore one can solve the maximization problem (9) by solving a series of polynomial optimization problems. See also Kubler and Schmedders (2009) for a detailed description of this point in finite economies.

To simplify notation I will assume from now on that h, g and $\hat{\rho}$ are already polynomial functions. It will be the case in the applications below, that the equilibrium conditions can directly be rewritten as polynomial functions and that $\hat{\rho}$ is polynomial to start with.

4.2 Solving the maximization problem (9)

Under the assumption that the Kuhn-Tucker conditions are necessary and have finitely many isolated solutions, algorithms designed to find all solutions to polynomial equations (see Sturmfels (2002) for an overview) can be used to find all critical points and by comparing them, one can find the globally optimal solution to the system (9). However, this ‘brute force’ approach is extremely inefficient. For relatively small models where h consists of around 4-5 equations of low degree, one can use computer-algebra systems to find the so-called Gröbner basis for $h(.)$. This is an equivalent system of polynomials which has a simple triangular structure and solving the system of Kuhn-Tucker conditions using the Gröbner basis often turns out to be much simpler. This approach has the advantage that one does not need to worry about rounding errors and actually formally proves that an approximate equilibrium is close to an exact equilibrium. Unfortunately, it is only feasible for small models and if $\hat{\rho}$ itself is of very low degree.

For larger problems, it turns out to be advantageous to use semi-definite programming and so called sum-of-squares relaxation to solve the polynomial optimization problem. In the following I briefly explain the basic idea of the method.

4.2.1 Sum-of-squares relaxations

In the last decade big advances have been made in polynomial optimization - see Laurent (2008) for an overview. The basic idea, (which is nicely explained in detail in e.g. Parrilo (2003)), is as follows.

A polynomial $p \in \mathbb{R}[x]$ is said to be a sum of squares (of polynomials) if it can be written as

$$p = \sum_{j=1}^{m} u_j^2 \text{ for some } u_j \in \mathbb{R}[x].$$

Clearly, if for a polynomial p, there exists a number γ, such that $p - \gamma$ is a sum of squares, then γ is a lower bound for $p(x)$ for any x.

If the degree of p is d, in order for it to be a sum of squares, there have to exist u_j which are of degree $d/2$. The main insight is now that one can use semi-definite programming to search over all polynomials of degree $d/2$ to establish that p is the sum of squares. The polynomial p can be written as a quadratic form of all the monomials of degree less than or equal to d, i.e. let $z = \left[1, x_1, x_2, ..., x_n, x_1^2, x_1 x_2, ..., x_n^d\right]$ be the vector of all such monomials. Then there must exist a positive-definite matrix Q with

$$p(x) = z^T Q z.$$

This matrix can be found using semi-definite programming (see Parrilo (2003) for details).

Following the same idea, but slightly more complicated, now consider the constrained optimization problem

$$\min f(x) \text{ s.t}$$

$$g_1(x) = \ldots = g_m(x) = 0, \quad h_1(x) \geq 0, \ldots, h_l(x) \geq 0.$$

If there exist a number γ, arbitrary polynomials $q_1, ..., q_m \in \mathbb{R}[x]$ and sum-of-squares polynomials $r_1, ..., r_l \in \mathbb{R}[x]$ such that

$$p := f - \gamma - \sum_{i=1}^{m} g_i q_i - \sum_{i=1}^{l} h_i r_i \tag{10}$$

is a sum-of-squares, then clearly $f(x) \geq \gamma$ for all x satisfying $g(x) = 0, h(x) \geq 0$. So again, γ is a lower bound for the minimization problem. As before, given a candidate p, semi-definite programming can be used to efficiently check if p is a sum-of-squares.

It is quite complicated to derive conditions on g and h that ensure that the converse holds, i.e. if γ solves the minimization problem, one can achieve the sum-of-square representation. The *Positivstellensätze* by Schmüdgen and by Putinar (see Laurent (2008, Theorem 3.16)) provide an answer. Basically, one needs to impose a condition a bit stronger than compactness of the constrained set – it is beyond the scope of this paper to discuss this further.

More important for this paper is the fact that it is not possible to find good bounds on the degree of q and r and therefore on the degree of p in equation (10). While one can still use semi-definite programming to determine for which γ the term in the equation can be written as a sum of squares, one has no a priori-bound on the degree and therefore has to experiment with different values. Fixing the maximal degree of the polynomial p, semi-definite programming can be used to determine efficiently if polynomials q and r of appropriate degree exist. Waki et al. (2006) provide a way to exploit sparseness in the polynomial problem so that the method is applicable to interesting problems. They report solving (sparse) problems with several hundred variables.

In Section 5 below, I use a matlab implementation by the authors called *SparsePOP*, described in Waki et al. (2008) to solve the relatively small problems that arise from the applications. The package produces a lower bound for the problem, γ, from the solution of the semi-definite program, as well as an approximate solution (the minimizer) to the polynomial problem. If these values coincide a true minimum has been found. If γ is smaller than the value of f at the approximate solution, γ is still a lower bound for the problem but neither γ nor the value of f might be the true minimum.

It is important to understand that one cannot always guarantee that the algorithm finds the global minimum. There are essentially two reasons for this. First, it turns out that solving the resulting semi-definite program is a difficult numerical problem and the solver might fail to find a solution (see Waki et al. (2006) for a extensive discussion of this problem and some possible remedies). More importantly, to use SparsePOP one needs to specify the degree of relaxation (the parameter '`param.relaxOrder'`), which is a bound on $d/2$ where d is the degree of the polynomial p in Equation (10). As explained above, one cannot say a priori how large this should be. In the examples below I try the values 3, 4, and for the deterministic example the values 5 and 6. Large values generally lead to severe numerical problems. Obviously, even if there exist γ and polynomials r and u such that p can be written as the sum of squares, it is not guaranteed that it will be of degree 8 or less. One should therefore expect the algorithm to fail in some cases.

However, the output of the algorithm is always a lower bound on the true value of the minimization problem. For my purposes, it is irrelevant what the true value of the problem is. In order to determine if the error set $\mathcal{E}$ is empty for a given δ one actually does not have to solve the maximization problem (9). It suffices, that that δ provides an upper bound for the problem. So if the software package finds a $\gamma < \delta$ for which the problem can be written as a sum-of-squares, existence of a robust equilibrium within δ of the candidate equilibrium is proven. If the software package does not find such a γ, nothing can be said about robustness of the candidate solution.

Note that this is a numerical algorithm, i.e. an approximate numerical solution is computed via floating point arithmetic and rounding errors could potentially lead to problems. Peyrl and Parrilo (2008) develop an algorithm that computes an exact algebraic solution if coefficients are rational.

4.2.2 Practical Considerations

As I will explain in the next section, it is important to formulate the optimization problem so that the degree of the polynomials is relatively low and it is important to find good bounds on the variables. For many interesting economic applications this might not always be possible and it might not be feasible to use existing implementations of these methods for solving the polynomial optimization problem. From a practical perspective the insights from this section are useful nevertheless. There are various efficient methods to find local minima to the optimization problem (9) and while this obvious cannot lead to a guarantee that the computed approximation is close to an exact equilibrium it can be a useful necessary check. In particular, independently of the economy being semi-algebraic, the maximization problem can be viewed as a programming problem with equilibrium constraints. There is a large literature on these problems (see e.g. Luo et al. (1996)) and reliable software to solve large-scale problems (see e.g. Su and Judd (2008) for an application of these methods to economics).

Unfortunately, the optimal value of the optimization problem (9) lying below δ is only a sufficient condition for an exact equilibrium lying in a δ-neighborhood of the recursive ϵ-equilibrium. Nevertheless, it will become clear in the examples below that the maximization problem can be modified in a way that if the optimal value is larger than δ, one can strongly suspect that there is something wrong with the candidate solution.

5 Example: Stochastic overlapping generations

In this section, I illustrate the method using a simple stochastic olg economy. This is the natural extension of the model considered in Section 2 to an environment with uncertainty, Agents live for three periods, there is a single good, a single agent per generation and a Markov chain determines endowments over the life cycle. For now, I assume that there is no production – in Section 5.3 I discuss briefly additional complications that arise in a model with capital accumulation. I assume that financial markets are incomplete[2]. The main purpose of this section is to illustrate the theoretical results above. Clearly there a trade-off between the model and notation being extremely simple and the model being interesting,

[2] In an earlier version of the paper I considered the case of complete markets. This makes the analysis slightly easier. Results are similar and available upon request.

i.e. somewhat realistic or similar to models used in other applications. I will therefore start with a simple model and extend it later.

At each date-event a single individual commences his economic life; he lives for 3 periods. An individual is identified by the date event of his birth, $\sigma = (s^t)$. The age of an individual is $a = 1, 2, 3$; he consumes and has endowments at all nodes $s^{t-1+a} \succeq s^t$, $a = 1, 2, 3$. An agent's individual endowments are a function of the shock and his age alone, i.e. for all $a = 1, 2, 3$, $e^{s^t}(s^{t-1+a}) = \mathbf{e}_a(s_{t-1+a})$ for some function $\mathbf{e}_a : \mathcal{S} \to \mathbb{R}_+$.

The agent has an intertemporal time-separable expected utility function.

$$U^\sigma(c) = \sum_{a=1}^{3} \sum_{s^{t-1+a} \succeq \sigma} \pi(s^{t-1+a}|\sigma) u_a\left(c(s^{t-1+a}), s_{t-1+a}\right)$$

The Bernoulli utility u depends on the age and the current shock alone.

At each s^t, there is a single risk-free bond in zero net supply available for trade. Its price is denoted by $q(s^t) \in \mathbb{R}_+$ and agent σ's bond-holding is $\theta^\sigma(s^t) \in \mathbb{R}$. Agents might face a borrowing constraint of the form $\theta^\sigma(s^t) \geq b$, for some $b < 0$.

At the root node, s_0, there are individuals of all ages s^{-1}, s^{-2} with initial wealth $\theta^{s^{-a}}(s^{-1})$. These determine the 'initial condition' of the economy.

It will turn out to be useful to write $c_a(s^t)$ and $\theta_a(s^t)$ to denote consumption and portfolios of the agent born at s^{t-1+a}.

A competitive equilibrium is a collection of prices and choices of individuals $(q(s^t), (\theta_a(s^t), c_a(s^t))_{a=1,2,3})_{s^t \in \Sigma}$ such that markets clear and agents optimize, i.e. for all nodes $s^t \in \Sigma$ the following holds:

- Market clearing:
$$\sum_{a=1}^{3} \theta_a(s^t) = 0$$

- For each s^t,individual $\sigma = s^t$ maximizes utility:
$$(c^\sigma, \theta^\sigma) \in \arg\max_{c \geq 0, \theta} U^\sigma(c) \text{ s.t.}$$

$$\begin{aligned}
c(s^t) - \mathbf{e}^1(s_t) + q(s^t)\theta(s^t) \leq 0, \quad \theta(s^t) \geq b \\
c(s^{t+1}) - \mathbf{e}^2(s_{t+1}) + q(s^{t+1})\theta(s^{t+1}) - \theta(s^t) \leq 0, \quad \theta(s^{t+1}) \geq b \\
c(s^{t+2}) - \mathbf{e}^3(s_{t+2}) - \theta(s^{t+1}) \leq 0 \\
\text{for all } s^{t+1} \succeq \sigma \text{ and all } s^{t+2} \succeq s^{t+1}
\end{aligned}$$

Optimality conditions for initially alive agents, s^{-1} and s^{-2} are analogous.

As in the deterministic example in Section 2, the natural endogenous state space of this economy consists of beginning of period bond-holdings of the middle aged. Define $c = (c_1, c_2, c_3)$ to be consumption across agents alive in the current period and $\theta = (\theta_1, \theta_2)$ to be new portfolio choices and $\kappa = (\kappa_1, \kappa_2) \in \mathbb{R}^2_+$ be the multipliers associated with the borrowing constraint, as well as θ_- to be beginning of period wealth of the middle aged and the old. Let $z = (\theta_-, q, c, \theta, \kappa)$ denote the vector of endogenous variables in a given period.

It is well known that under the assumption that U^h is differentiable, strictly increasing, strictly quasi-concave and satisfies an Inada-condition, the first order conditions are necessary and sufficient for agent optimality. As in Section 2, the equilibrium equations consist of these first order conditions, budget equations, market-clearing and the equations that determine cash-at-hand in the next period, given choices today.

In this example, the state space does not depend on the shock. Note that if $b = -\infty$ and there are no constraints on trades, and the endogenous state space Θ is completely endogenous. Therefore, the previous analysis has to be slightly modified. One can view a recursive ϵ-equilibrium of an economy without constraints that is given by $\hat{\rho}_s : \hat{\Theta} \to \mathbb{R}^{M-D}$, with $\hat{\Theta}$ as an ϵ-equilibrium of an economy where each agent faces the constraint $\theta_s \in \hat{\Theta}$, but the constraint is simply never binding. Through the error set, one then needs to verify that this ϵ-equilibrium is close to an exact equilibrium with the same constraints in which this constraints are also never binding. In practice I therefore check if a δ-strip around the computed ϵ equilibrium is a robust ϵ-equilibrium set for the economy with constraints and then show that in the exact equilibrium these constraints never bind.

As explained in the introduction, it is not difficult to show that competitive equilibrium and policy-correspondences exist. In order to apply the main result of the paper, Theorem 2, one first needs to establish existence of truncated equilibria with terminal conditions. To prove existence of competitive equilibria in these models, on typically first proves that an equilibrium exists for all finitely truncated economies and then take the limit. The result needed here is very similar.

5.1 Existence of truncated equilibrium

In this section, I prove that in the olg model truncated equilibria with initial conditions Z always exist if the set Z contains the graph of a continuous policy function. Let $\hat{\rho}_q$ denote the approximate pricing function and $\hat{\rho}_\theta$ the approximate policy function of the middle aged. Let $\hat{\Theta} \subset \mathbb{R}$ be a closed interval for each s. Let $\underline{c} \geq 0$ denote a lower bound of an agents' consumption in any equilibrium. In the case where $u(.)$ is continuous on $\mathbb{R}_+$ this is zero, for cases where $u(c) \to -\infty$ as $c \to 0$, this is some positive number determined by the fact that endowments are strictly positive and agents are finitely lived.

As mentioned above, existence of a truncated equilibrium is shown for a slightly modified economy where all agents face additional trading constraints of the form $\theta \geq b$. By market clearing this implies $b \leq \theta_1 \leq -b$ and one obtains a compact state space.

In this setup, it is relatively straightforward to prove the following lemma.

LEMMA 2 *Suppose $\hat{\rho}$ is continuous and that for any $s \in \mathcal{S}$ and any $\theta_- \in \hat{\Theta}$, $\mathbf{e}_2(s) + \theta_- - \hat{\rho}_q(\theta_-, s) \cdot \hat{\rho}_\theta(\theta_-, s) > \underline{c}$. Then for all initial conditions $s_0, \theta_-(s_0) \in \hat{\Theta}$ and for any T, there exists a T-truncated competitive equilibrium with terminal condition $\hat{\rho}$.*

Although the proof of the lemma is a standard application of Kakutani's theorem (Kubler and Polemarchakis (2004) give a similar proof for the existence of truncated equilibria in olg models), I present an outline. The only difficulty lies in making assumptions that ensure that agents' budget sets are non-empty and to modify the problems of agents alive at $T-1$ and T.

Fix initial conditions $s_0, \theta_-(s_0) \in \hat{\Theta}$. Each agent in the T horizon economy who is not active at T takes prices a given and a standard argument shows that his best response is continuous for strictly positive prices. At each node s^t, $t < T$, there is a price player that takes choices at the node as given and solves

$$\max_{(p,q)\in\Delta^2_\eta} p(c_1(s^t) + c_2(s^t) + c_3(s^t)) + q(\theta_1(s^t) + \theta_2(s^t)),$$

where Δ^2_η denotes the two dimensional truncated simplex. His choices are upper-hemi continuous and convex valued in choices of all agents.

Given choices of all agents, let Let $\bar{q}(s^T) = \hat{\rho}_q(\theta^1(s^{T-1}))$ and let $\bar{\theta}(s^T) = \hat{\rho}_\theta(\theta^1(s^{T-1}))$. Clearly this is continuous and non-empty for all admissible choices of agents.

Finally, all agents born at period $T-1$ at node s^{T-1}, take as given $\bar{q}(s^T)$ and $\bar{\theta}(s^T)$, as well as prices at s^{T-2} and as s^{T-1}. Their consumption at T is then $e_2(s_T)+\theta-\bar{q}(s^T)\cdot\bar{\theta}(s^T)$. It is standard to show that their best responses are continuous and non-empty since it was assumed that for some choices $e_2(s_T) + \theta - \bar{q}(s^T) \cdot \bar{\theta}(s^T) > 0$. Kakutani's theorem ensures the existence of a fixed point of the Cartesian product of the best-responses – it is standard to show that for sufficiently small $\eta > 0$, this is a T-truncated equilibrium according to the above definition.

The main insight of the proof is that if for a given T-horizon economy one can show that agents choices are continuous in prices, and if one imposes a continuous policy function at T which allow for positive consumption at some values of the state and if one imposes constraints on trades that ensure that agents' actions will never result in a state outside of the specified space $\hat{\Theta}$.

5.2 Examples

As explained in the introduction, this is paper is not about how to compute approximate equilibrium. For the simple model in this section, there are several reliable methods to do so. I use the time-iteration algorithm which is explained in detail in Kubler and Krueger (2004) for a model with overlapping generations and production. I first revisit the deterministic example form Section 2, then consider the case without borrowing constraints, i.e. $b = -\infty$, and finally provide an example with a binding constraint. Throughout, I assume that for each a and s, $u_a(c, s) = -\beta^a c^{1-\sigma}$, $\beta > 0$, i.e. preferences exhibit constant relative risk aversion with a coefficient of relative risk aversion of $\sigma > 1$. Throughout, I take for each $s \in \mathcal{S}$, Θ_s to be the smallest interval that contains the initial condition and that is feasible in the sense that in the self-validated equilibrium, choices at all nodes ensure that next period's endogenous state is feasible, i.e. lies in the state-space. The initial condition is part of the description of the economy.

5.2.1 Example 1: A deterministic economy

To fix ideas, it is useful to first reconsider the example from Section 2. Inspecting the equilibrium equations (1) one notices that in order to solve the constrained maximization problem (9) one does not need to know the entire policy function but in fact only the consumption policies. It is therefore useful to approximate these separately, i.e. instead of plugging the portfolio- and pricing functions into the budget constraints and so obtaining an approximating polynomial for consumption, it makes more sense to approximate consumption directly, solve the constrained maximization problem using only this function and then ask what it implies for portfolios and prices. Concretely, for the two specifications it turns out that consumption policies are actually better behaved then portfolios and can be well approximated by polynomials of degree 3.

In order to verify that a candidate solution is close to an exact equilibrium one now has to go through several steps: (1) Given a policy function of the middle aged, $\hat{\rho}_c(.)$, the constrained maximization problem to determine if this can be part of a robust ϵ-equilibrium is as follows.

$$\begin{aligned}
&\max_{(\epsilon),\theta_-\in\hat{\Theta},q,\theta} \pm(e^2+\theta_- + q\cdot\theta - \hat{\rho}_c(\theta_-))\ s.t.\\
&-\tilde{q}^{1/3}(\epsilon+\hat{\rho}_c(\theta)) + \beta^{1/3}(e^1-\tilde{q}^3\theta) = 0,\\
&-\tilde{q}^{1/3}(e^3-\theta) + \beta^{1/3}(e^2+\tilde{q}^3\cdot\theta+\theta_-) = 0,\\
&\theta\in\hat{\Theta},\quad q\geq 0,\quad -\delta\leq\epsilon\leq\delta,
\end{aligned} \tag{11}$$

where $\pm$ indicates that first the positive objective function is maximized and secondly the negative objective function is maximized.

Note that it is useful to substitute in budget constraints and to write the system using $\tilde{q} = q^{1/\sigma}$. In its original formulation, the system of constraint would contain the term $q(\epsilon+\hat{\rho}_c(\theta))^3$ – with $\hat{\rho}_c$ being a polynomial of degree 3 that would result in a polynomial of degree 7 and potentially cause numerical problems. Note also that although in its original formulation, the example did not include constraints on trades, I add the constraint $\theta\in\hat{\Theta}$. Since the polynomial function $\hat{\rho}_c$ is defined on all of $\mathbb{R}$, but only makes sense on $\hat{\Theta}$ this cannot be avoided. For candidate solution (i) and $\delta = 10^{-4}$, SparsePOP returns an upper bound of 4.3×10^{-5} for the problem. As a next step, (2), one has to check what a deviation of 10^{-4} in the consumption policy implies for prices and portfolios. For this, one can use SparsePOP to solve $\max_{(\epsilon),\theta_-\in\hat{\Theta},q,\theta} \pm(\theta-\hat{\rho}_\theta(\theta_-))$ and $\max_{(\epsilon),\theta_-\in\hat{\Theta},q,\theta} \pm(q-\hat{\rho}_q(\theta_-))$ subject to the same constraints as above. For Solution (i) SparsePOP returns an upper bound of 3.2×10^{-4}. Finally, in step (3), one now has to verify that if portfolios stay in this region, the additional superficial constraint $\theta\in\hat{\Theta}$ never binds, i.e. in this case, for Solution (i), one needs to check that for all $\theta_-\in[-6,0]$, $\hat{\rho}(\theta_-) - 3.2\times 10^{-4} > -6$ and $\hat{\rho}(\theta_-) + 3.2\times 10^{-4} > 0$

– which is the case.

In comparison, for Solution (ii), Steps (1) and (2) above also yield good results. However, in this case, one obtains that the portfolio policy is within 10^{-2} of $\hat{\rho}(\theta_-)$ but that $\hat{\rho}(\theta_-)+10^{-2}$ lies outside of $\hat{\Theta}$. The method fails to verify that this approximate solution is close to an exact equilibrium - as I show in Appendix A it is not.

5.2.2 Example 2: Endowment uncertainty

The introduction of uncertainty now causes additional numerical problems. If one continues to use the consumption policy-function, for CRRA utility and an economy with S exogenous shocks one has to solve $s = 1, ..., S$ problems of the form

$$\begin{array}{l} \max\limits_{(\epsilon_1,...,\epsilon_S),\theta_-\in\Theta,q,\theta} \quad \pm(e^1(s) + \theta_- + q\theta - \hat{\rho}_c(\theta_-)) \quad s.t. \\ -q\frac{1}{(e^1(s)-q\theta)^\sigma} + \sum_{s'=1}^S \pi(s'|s)\beta\frac{1}{(\epsilon_{s'}+\hat{\rho}_{c^1}(\theta,s'))^\sigma} = 0, \\ -q\frac{1}{(e^2(s)+q\theta+\theta_-)^\sigma} + \sum_{s'=1}^S \pi(s'|s)\beta\frac{1}{(e^3(s)-\theta)^\sigma} = 0 \\ q \geq 0, \;\; \theta \in \hat{\Theta} \text{ and } -\delta \leq \epsilon_{s'} \leq \delta, \quad \text{for all } s' \in \mathcal{S}. \end{array}$$

In this formulation, the equilibrium constraints are not polynomial but of course, if σ is rational, one can obtain polynomial expressions by multiplying out. Unfortunately, even for very simple examples with $\sigma = 3$ and 2 shocks, if the approximate consumption function is cubic this leads to polynomials of degree 13. This is not feasible for SparsePOP. Instead, it is useful to consider the functions

$$\hat{\rho}_{m^1}(\theta_-) = \left(\sum_{s'=1}^S \pi(s'|s)\beta\frac{1}{\hat{\rho}_{c^1}(\theta_-,s')^\sigma}\right)^{-\frac{1}{\sigma}}$$

and

$$\hat{\rho}_{m^2}(\theta_-) = \left(\sum_{s'=1}^S \pi(s'|s)\beta\frac{1}{(e^3_{s'}-\theta_-)^\sigma}\right)^{-\frac{1}{\sigma}}.$$

In the examples below it turns out that these functions can be extremely well approximated by low degree polynomials. With this, the equilibrium constraints of the maximization problem can be trivially written as

$$\begin{array}{c} -\tilde{q}\hat{\rho}_{m^1}(\theta) + (e^1(s) - \tilde{q}^\sigma\theta) = 0, \\ -\tilde{q}\hat{\rho}_{m^1}(\theta) + (e^2(s) + \tilde{q}^\sigma\theta + \theta_-) = 0. \end{array}$$

With this, one has to work with relative errors, since maximal relative consumption errors translate one-to-one to maximal relative errors in $\hat{\rho}_{m^2}$ The problem can be solved easily for interesting values of S and σ. Of course, with this one has to verify in addition that the polynomial approximation $\hat{\rho}_{m^1}$ is sufficiently good.

To illustrate this, I consider a very simple numerical example. Suppose $S = 2$, shocks are iid with $\pi = 1/2$, and endowments are given by

$$e^0 = (e^0(1), e^0(2)) = (1, 2), \quad e^1 = (4, 2), \quad e^2 = (0.5, 0.5),$$

and suppose $\beta = 1$ and $\sigma = 3$.

It turns out in the computed approximate equilibrium, that the endogenous state can be chosen to be $\hat{\Theta} = [-0.2, -0.05]$ and the approximate consumption functions can be chosen to be the quadratic functions $\hat{\rho}_{c^1}(\theta_-, 1) = 1.6369 + 0.2350\theta_- - 0.0055\theta_-^2$ and $\hat{\rho}_{c^1}(\theta_-, 2) = 1.2187 + 0.3227\theta_- - 0.0362\theta_-^2$. With these specifications, a cubic polynomial leads to an excellent approximation and:

$$\max_{\theta_- \in [-0.2, -0.05]} \left| \hat{\rho}_{m^1}(\theta_-) - \left(\sum_{s'=1}^{S} \pi(s'|s)\beta \frac{1}{\hat{\rho}_{c^1}(\theta_-, s')^\sigma} \right)^{-\frac{1}{\sigma}} \right| < 10^{-8}.$$

With this in place, one can now repeat the steps from above and verify that there is an exact equilibrium for which the approximation exhibits a relative error of less than 10^{-4}.

The simple example illustrates the method - the technique can easily handle examples with 6 or more states and risk aversion of 4 or 5.

5.2.3 Interpretation of robustness

Obviously, it is possible that a recursive ϵ-equilibrium is within some δ of an exact equilibrium but that a δ-neighborhood around the consumption functions does not form a robust ϵ-equilibrium set. However, in the previous example it is easy to understand what this implies for the economy. Here, the relative perturbations in next period's consumption can be interpreted as perturbations in the discount factor β. ....

5.3 Extensions to production economies

In many applications in macro-economics researchers consider models with capital accumulation. For example, stochastic versions of Diamond's (1965) model are used for analysing tax-policies and social security. Assume that there is a single firm that takes aggregate capital K and labor L as inputs and produces the consumption good according to the following Cobb-Douglas production function

$$f(K, L; s) = \xi(s)K^\alpha L^{1-\alpha} + K(1 - \delta(s)) \tag{12}$$

where $\xi(.)$ is the stochastic shock to productivity and where $\delta(.)$ can be interpreted as the (possibly) stochastic depreciation rate. Instead of having endowments in the consumption good, households are now endowed with deterministic labor which they sell inelastically to the firm at a stochastic wage. Instead of having access to a risk-free bond they can save and borrow in risky capital. With competitive factor markets, substituting for wages and return to capital, budget constraints now read as

$$c(s^t) - (1-\alpha)\xi(s_t)K(s^t)^\alpha e^1 + \theta(s^t) \leq 0$$
$$c(s^{t+1}) - (1-\alpha)\xi(s_{t+1})K(s^{t+1})^\alpha e^2 + \theta(s^{t+1}) - \theta(s^t)(\alpha\xi(s_{t+1})K(s^{t+1})^{\alpha-1} + (1-\delta(s_{t+1}))) \leq 0$$
$$c(s^{t+2}) - (1-\alpha)\xi(s_{t+2})K(s^{t+2})^\alpha e^3 - \theta(s^{t+1})(\alpha\xi(s_{t+2})K(s^{t+2})^{\alpha-1} + (1-\delta(s_{t+2}))) \leq 0$$
$$\text{for all } s^{t+1} \succeq \sigma \text{ and all } s^{t+2} \succeq s^{t+1}$$

with $K(s^t) = \sum_{a=1}^2 \theta_a(s^{t-1})$ and $e^1 + e^2 + e^3 = 1$.

With this modified model, two additional complications arise. The natural endogenous state space now consists of aggregate capital and the distribution of capital among the middle aged and the old, i.e. it is two dimensional. Furthermore, the expressions for returns to capital and wages have to be rewritten as polynomials. To be precise, suppose for example that $\alpha = 1/3$, $\xi(1) = 0.85$, $\xi(2) = 1.15$, $\delta(1) = \delta(2) = 0.5$, $\sigma = 3$, $e^1 = e^2 = 0.4$ and $e^3 = 0.2$. The endogenous state now consists of the beginning of period capital holding of the middle aged as well as aggregate capital. As in the exchange economy, it turns out that a (complete) polynomial of degree 2 can be used to approximate consumption policies. Similarly to above, define

$$\hat{\rho}_{m^1}(\theta_-, K, s) = \left(\sum_{s'=1}^{S} \pi(s'|s)\beta(\alpha\xi(s')K^{\alpha-1} + (1-\delta(s'))) \frac{1}{\hat{\rho}_{c^1}(\theta_-, s')^\sigma} \right)^{-\frac{1}{\sigma}}.$$

Using an polynomial of degree 2 suffices to approximate $\hat{\rho}_{m^1}$ up to an error of 10^{-9}. With this, the constrained optimization problem is not much more difficult as the one above, except that one has to define $\tilde{K} = K^\alpha$ and use the additional equality constraint $\tilde{K}^3 = K$.

Applying the above steps, one can prove that there is an exact equilibrium with consumption policies within 10^{-4} of $\hat{\rho}_c$.

6 Conclusion

This paper develops a computationally feasible test to verify that a computed candidate equilibrium is close to a competitive equilibrium of a dynamic stochastic economy. The result has both practical and theoretical relevance.

In practice researchers often need to argue that their computation of dynamic equilibria are accurate. The most consistent method is verifying that relative errors in Euler-equations are small, but several other more ad-hoc methods are used in practice. The method in this paper develops a method to compute the exact deviation between computed function and actual equilibrium. The computation is relatively efficient and can be carried out for interesting small problems. I also argue that the method suggests an error analysis that is feasible for large-scale models.

Theoretically, it has been known that the assumption of semi-algebraic preferences and technology allows for an arbitrarily good approximation of all equilibria of a finite economy (see e.g. Kubler and Schmedders (2008)). The papers in the recent book edited by Kubler and Brown (2008) explore other implications of semi-algebraic fundamentals in finite economies. In this paper, I show how to extend the ideas from real algebraic geometry to infinite economies.

Appendix

I show that there is no equilibrium close to the candidate Solution (ii). The method from Kubler and Schmedders (2009) proves that there is a unique steady states which is associated with candidate solution (i). To understand better the dynamics of the model it is useful to identify the pairs of beginning of period bond-holding and consumption of the middle aged, (θ_-, c), which lead to constant portfolios and those that lead to constant consumption. If portfolios are constant at some $\bar{\theta} = \theta = \theta_-$ the budget constraint of the middle aged and the first order condition imply as a necessary condition that there is a price q such that $q\bar{\theta} = c - e_2 - \bar{\theta}, \quad qc^{-3} = (e_3 - \theta)$. Rewriting these equations as polynomials and using Gröbner bases to eliminate q (as in Kubler and Schmedders (2009)) implies that constant portfolios arise if (θ_-, c) satisfy the following polynomial equation

$$320\theta_-^4 + (-320c + 2904)\theta_-^3 + (480c - 4836)\theta_-^2 + (-320c^3 - 240c + 2498)\theta_- + (40c - 423) = 0.$$

If consumption remains constant there must be a θ and a q such that the budget constraint of the middle aged, and both first order conditions hold, i.e.

$$q\theta = c - e_2 - \theta_-, \quad qc^{-3} = (e_3 - \theta), \quad q(e_0 + e_1 + e_2 - c - (e_2 - \theta_-)^{-3} = c^{-3}.$$

Rewriting as polynomials and using Gröbner bases to eliminate (q, θ) on obtains

$$256000c^4 + (-384000\theta_- - 4252800)c^3 + (128000\theta_-^2 + 2771200\theta_- + 14927120)c^2 +$$
$$(192000\theta_-^2 + 44448000\theta_- + 25724280)c + (-64000\theta_-^3 - 2222400\theta_-^2 - 257242800\theta_- - 99252847) = 0$$

These two polynomial equations define curves in (θ_-, c) space and one determine the dynamics above and below these curves. Figure 2 shows the two curves, with θ_- on the x-axis and c on the y-axis in the region close to consumption of the second Candidate solution.

The arrows indicate the dynamics of c and θ. While it is not apparent from the figure one can easily verify that the two curves do not intersect in this region. Therefore there can be no equilibrium with consumption close to the one in candidate solution 2. Any such (θ_-, c) pair leads to a dynamical system that explodes in consumption - either consumption of the middle aged or consumption of the old goes to $-\infty$.

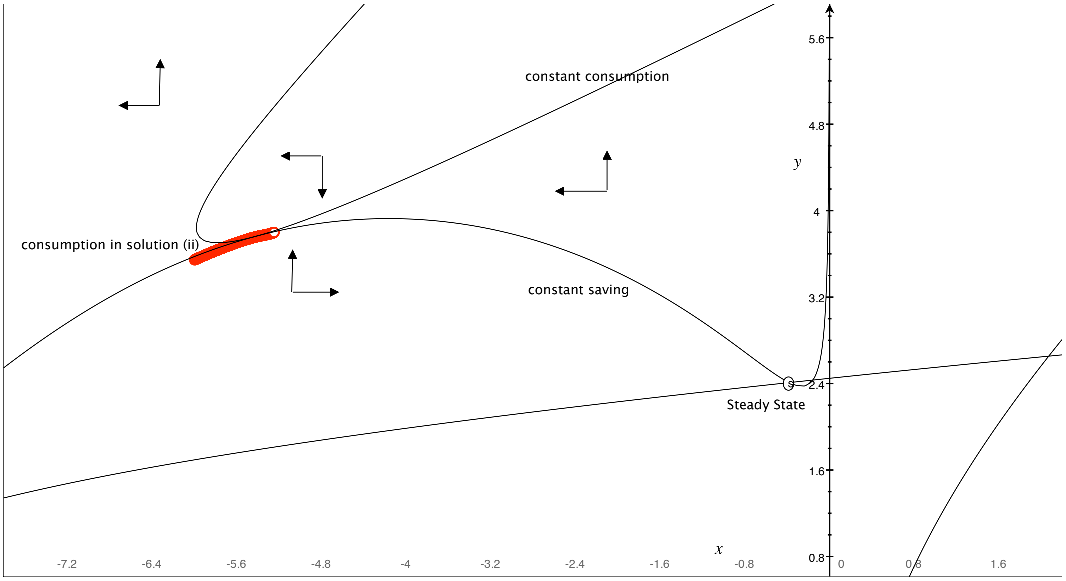

Figure 2

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