Gambles with prices, operational measure of riskiness and buying and selling price for risky lotteries

Michal Lewandowski

EUROPEAN UNIVERSITY INSTITUTE

Florence, Italy

michal.lewandowski@eui.eu

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Abstract

In this paper I analyze operational measure of riskiness defined by Foster and Hart (2007). I give simple intuition behind their main result. Then I extend the concept of riskiness measure in two respects - I define a generalized riskiness measure based on decreasing absolute risk aversion utility function. I derive necessary and sufficient conditions for existence of such measure for DARA and CRRA class of utility functions. In addition, I show the way how to compare riskiness of gambles with negative expectation or with nonnegative outcomes only. To this end I use properties of buying and selling price for a lottery and their relations to riskiness measure. In particular, I show how buying and selling price for a lottery concepts may be used complementary to the concept of riskiness measure.

Keywords: operational measure of riskiness, extended measure of riskiness

JEL Classification Numbers: D81, D03, C91

1 Introduction

In economics there is more consensus over how to define risk aversion than how to define risk. Aumann and Serrano (2008) used this startling matter of fact to define a concept of risk derived from risk aversion, i.e. risk is what risk averters hate. This ingenious approach and nice axiomatic treatment led to defining economic index of riskiness. Since it looked like the index was measured in dollars but there was no theoretical support for this claim at the time, Dean Foster and Sergiu Hart started to work on giving the index operational interpretation. This plan did not succeed but instead Foster and Hart (2007) came up with another way to measure riskiness which bears a lot of similarity to Aumann and Serrano (2008) index and has a nice operational interpretation. They define measure of riskiness for a gamble as an amount of initial wealth below which the decision maker should reject the gamble if he wishes to guarantee no-bankruptcy making consistent choices in the long run[1]. In this paper I want to provide simple intuition behind Foster and Hart (2007) measure of riskiness. In particular, I want to show why their result holds and which assumptions are crucial for that. I will discuss assumptions behind the model, in particular the assumption of homogeneous simple strategies. Simple strategies are strategies whether to accept a given gamble or not taking into account only current wealth level and the gamble in question. In a dynamic setting it corresponds to Markov stationary strategy concept. Simple strategies are homogeneous if they are scale-invariant - the decision whether to accept a gamble at a given wealth level should not change when both the gamble outcomes and wealth level are rescaled by some positive factor. I will show that in the expected utility framework this assumption is equivalent to constant relative risk aversion utility function representing the individual's preferences. I will argue that the assumption of scale-invariant simple strategies is crucial for the main result of Foster and Hart (2007) even though the statement of alternative result without this assumption seems similar in mathematical terms. I will try to show that the assumption of scale-invariant simple strategies makes the result of Foster and Hart (2007) particularly strong by imposing strong consistency requirements.

Another restriction of the model by Foster and Hart (2007) is necessary for the existence of the riskiness measure. Gambles are assumed to have positive expectation and negative outcomes with positive probability. To be precise this

[1] If this statement is not clear at this point, it will be clarified later on when the exact formulation and the nature of the consistency requirements are outlined.

assumption is not necessary for existence of the riskiness measure as the authors argue that the riskiness measure can be extended so that all gambles with non-positive expectation have riskiness equal to infinity and all gambles with no losses get zero riskiness. However the fact is that only gambles satisfying the assumption can be compared in terms of their riskiness and gambles which do not satisfy this assumption cannot. This is to say that the assumption restricts the set of gambles to those for which riskiness measure is meaningful. I want to show the way to compare riskiness of the gambles which do not satisfy this assumption. For this purpose I will link the concept of riskiness measure with the concept of buying and selling price for a lottery as defined by Raiffa (1968). I will analyze riskiness measure for gambles with prices. The advantage of this approach is that the riskiness measure in this case is well defined even if it is not well defined for gambles without prices. A couple of theoretical results will make it clear in what sense measuring riskiness for gambles without prices can sometimes be inferred from the riskiness measure for gambles with prices.

The riskiness measure of Foster and Hart (2007) can be linked to expected utility maximization. It is rather straightforward to see from the definition[2] that the riskiness measure of a gamble is equal to the value of initial wealth for which the logarithmic utility maximizer is indifferent between taking the lottery and not taking the lottery. Logarithmic utility function is a member of constant relative risk aversion class of utility functions. I extend the definition of the riskiness measure as follows - given some utility function and a gamble, the riskiness measure is the value of wealth for which the decision maker whose preferences are represented by this utility function is indifferent between taking and not taking the gamble. For the case of decreasing absolute risk aversion and the narrower case of constant relative risk aversion I show what are the necessary and sufficient conditions for existence of such riskiness measure. Obviously for functions other than logarithm it is not necessarily the case that the simple strategy corresponding to such utility function guarantees no-bankruptcy so strictly speaking to allow other utility functions is not really an extension of the riskiness measure by Foster and Hart (2007). However, it turns out that for CRRA utility functions the extended riskiness measure is increasing in relative risk aversion coefficient. It means that for utility functions with relative risk aversion coefficient higher that 1 (the limiting case equivalent to logarithm), the extended riskiness measure in fact guarantees no-bankruptcy. Since these utility functions "reject" more than logarithmic utility function, it is the case

[2] See equation (4).

that the riskiness measure of Foster and Hart (2007) is the lowest and hence least restrictive among all riskiness measures which guarantee no-bankruptcy. This result appears already in Foster and Hart (2007) but in an implicit form and therefore I state it in the paper explicitly.
This paper is organized as follows. In section 2 I introduce the model and its assumptions and in particular I introduce the concepts of buying and selling price for a lottery and the riskiness measure of Foster and Hart (2007). In section 3 I show some intuition behind the riskiness measure. In section 4 I extend the definition of the riskiness measure and give the necessary and sufficient conditions for existence of such measure. In section 5 I show in what relation to each other buying and selling price for a lottery and the extended riskiness measure are. I also discuss how this relation can be helpful in comparing riskiness measures of gambles that do not have positive expectation and negative outcomes. In section 6 I demonstrate equivalence between expected utility decision making, riskiness measure based decision making and buying/selling price based decision making. Finally, I conclude. Most of the results which I refer to in the paper and which have been proved in another paper have been placed in the appendix at the end.

2 The model

Below I define the class of utility functions and the space of lotteries which I will focus on in this paper.

Assumption 1. *Preferences obey expected utility axioms. Bernoulli utility function* $U : \mathbb{R} \to \mathbb{R}$ *is twice continuously differentiable, strictly increasing and strictly concave.*

Definition 1. *A lottery* $\mathbf{x}$ *is a real- and finite-valued random variable with finite support. The space of all lotteries will be denoted* $\mathcal{X}$*. I define the maximal loss of lottery* $\mathbf{x}$ *as:* $\min(\mathbf{x}) = \min \operatorname{supp}(\mathbf{x})$.

The typical lottery will be denoted as $\mathbf{x} \equiv (x_1, p_1; ...; x_n, p_n)$, where $x_i \in \mathbb{R}$ $i \in \{1, 2, ..., n\}$ are outcomes and $p_i \in [0, 1]$ $i \in \{1, 2, ..., n\}$ are the corresponding probabilities. Outcomes should be interpreted here as monetary values. Although most of results that follow are true for more general lotteries, the finite support assumption is sufficient for the purposes of this paper. Now I define buying and selling price for a lottery given wealth level along the lines of Raiffa (1968). To avoid repetitions, I will henceforth skip statements of the form: "Given utility function U satisfying assumption 1, any lottery $\mathbf{x}$ and wealth W...".

2.1 Buying and selling price for a lottery

Here and in the next sections I introduce the key concepts of the paper.

Definition 2. *I define selling price and buying price for a lottery* $\mathbf{x}$ *at wealth* W *as functions denoted, respectively,* $S(W,\mathbf{x})$ *and* $B(W,\mathbf{x})$*. Provided that they exist, values of these functions will be determined by the following equations:*

$$\mathrm{E}U[W+\mathbf{x}] = U[W+S(W,\mathbf{x})] \tag{1}$$

$$\mathrm{E}U[W+\mathbf{x}-B(W,\mathbf{x})] = U(W) \tag{2}$$

If utility function is defined over the whole real line as is the case for constant absolute risk aversion, buying and selling price as functions of wealth exists for any wealth level by assumption 1. If the domain of utility function is restricted to a part of real line as is the case of constant relative risk aversion utility function analyzed here, the domain of buying and selling price for a lottery is also restricted. I will focus mostly on the case of constant relative risk aversion (CRRA) utility function normalized conveniently[3] so that it takes the following form:

$$U_\alpha(x) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & 0<\alpha<1, \quad x\geq 0 \\ \log x, & \alpha=1, \quad x>0 \\ \frac{x^{1-\alpha}-1}{1-\alpha}, & 1<\alpha, \quad x>0 \end{cases} \tag{3}$$

Observe that

$$\lim_{\alpha\to 1}\frac{x^{1-\alpha}-1}{1-\alpha} \overset{\mathrm{H}}{=} \lim_{\alpha\to 1}\frac{-\log x}{-1} = \log x$$

The proposition below establishes the domain and the range of selling and buying price for a lottery if the utility function takes the above form. The statement and the proof is due to Lewandowski (2009a).

Proposition 2.1 (CRRA2)**.** *Given the class of CRRA utility function used in the section the following holds for any non-degenerate lottery* $\mathbf{x}$*: for* $\alpha\geq 1$

- $\lim_{W\to 0} B(W,\mathbf{x}) = \min(\mathbf{x})$

[3] Normalization is done without loss of generality since cardinal utility function is unique only up to affine transformation. That means that I can choose the slope and the shifting constant in a given point without changing the Pratt (1964) risk attitudes characteristics.

- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = \min(\mathbf{x})$

Define $W_L(\mathbf{x}) = U^{-1}[\mathrm{E}U(-\min(\mathbf{x}) + \mathbf{x})]$. *For* $0 < \alpha < 1$

- $\lim_{W \to W_L(\mathbf{x})} B(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$,

- $\lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) = W_L(\mathbf{x}) + \min(\mathbf{x})$

Additionally,

$$\forall \alpha > 0 \lim_{W \to \infty} B(W, \mathbf{x}) = \lim_{W \to \infty} S(W, \mathbf{x}) = \mathrm{E}[\mathbf{x}]$$

As for the intuition behind selling and buying price for the lottery, in economic terms, given an individual with initial wealth W whose preferences are represented by utility function $U(\cdot)$, $S(W, \mathbf{x})$ is the minimal amount of money which he demands for giving up lottery $\mathbf{x}$. Similarly, $B(W, \mathbf{x})$ is the maximal amount of money which he is willing to pay in order to play lottery $\mathbf{x}$.
Buying and selling price exhibit certain properties, many of which are enumerated in the appendix. Proofs of these properties may be found in Lewandowski (2009b). I will make use of these properties when I establish connections between buying and selling price and riskiness measure.

2.2 Riskiness measure

Foster and Hart (2007) define operational measure of riskiness as follows. The initial wealth is $W_1 > 0$. At every period $t = 1, 2, ...$, the decision maker with wealth W_t is offered a gamble $\mathbf{x_t}$. He may accept or reject the gamble. His wealth next period is $W_{t+1} = W_t + \mathbf{x_t}$ if he accepts and $W_{t+1} = W_t$ if he rejects. Simple strategy of the decision maker whether to accept gamble $\mathbf{x_t}$ at time t or not is assumed to be stationary Markov strategy - it depends only on the gamble $\mathbf{x_t}$ and current wealth level W_t. Simple strategy is homogeneous or scale-invariant if "accept $\mathbf{x}$ at W" implies "accept $\lambda\mathbf{x}$ at λW", for any $\lambda > 0$. For characterization results concerning simple strategies and in particular homogeneous simple strategies consult Lewandowski (2009b). If borrowing is not allowed, bankruptcy occurs when wealth converges to zero as time goes to infinity. A given strategy s yields no-bankruptcy for the process $(\mathbf{x_t})_{t=1,2,...}$ and the initial wealth W_1 if probability of bankruptcy is zero, i.e. $\mathrm{P}[\lim_{t \to \infty} W_t = 0] = 0$. Strategy guarantees no-bankruptcy if it yields no-bankruptcy for every process $(\mathbf{x_t})_{t=1,2,...}$ and every initial wealth level W_1. The technical assumptions state that gambles are assumed to be finite-valued, with finite support and such that $\mathrm{E}[\mathbf{x}] > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$, where $\mathrm{P}[E]$ denotes a

probability of an event E (positive expected value and losses are possible). The stochastic process $(\mathbf{x_t})_{t=1,2,...}$ is assumed to be finitely generated.
The main theorem of Foster and Hart (2007) states the following.

Theorem 1 (Foster and Hart (2007))**.** *For every gamble* $\mathbf{x}$ *there exists a unique real number* $R_{FH}(\mathbf{x}) > 0$ *such that: a homogeneous strategy* s *guarantees no-bankruptcy if and only if for every gamble* $\mathbf{x}$ *and wealth* $W > 0$*,*

$$W < R_{FH}(\mathbf{x}) \;\Rightarrow\; s \text{ rejects } \mathbf{x} \text{ at } W$$

Moreover, $R_{FH}(\mathbf{x})$ *satisfies the following equation*

$$\mathrm{E}\left[\log\left(1 + \frac{\mathbf{x}}{R_{FH}(\mathbf{x})}\right)\right] = 0 \tag{4}$$

Foster and Hart (2007) call $R_{FH}(\mathbf{x})$ the measure of riskiness of $\mathbf{x}$.
As I mentioned in the introduction, there is a link between the riskiness measure and expected utility maximizing individuals. Consider an expected-utility maximizer with utility function U:

$$\text{accept } \mathbf{x} \text{ at } W \iff \mathrm{E}U(W + \mathbf{x}) \geq U(W) \tag{5}$$

Notice that for logarithmic utility function I can rewrite condition on the RHS of (5) in relative - instead of absolute - terms, as follows:

$$\mathrm{E}\left[\log\left(1 + \frac{\mathbf{x}}{W}\right)\right] \geq 0$$

It is clear that the index $R_{FH}(\mathbf{x})$ has the property that the logarithmic utility rejects $\mathbf{x}$ if $W < R_{FH}(\mathbf{x})$ and accepts $\mathbf{x}$ if $W \geq R_{FH}(\mathbf{x})$. Hence by the theorem above logarithmic utility represents a strategy that is among those which guarantee bankruptcy. In the next section I will provide further intuition behind the riskiness measure and discuss assumptions underlying it.

3 Riskiness measure - its assumptions and intuition behind

Notice that simple strategies in the theorem are assumed to be homogeneous. It turns out as proved in Lewandowski (2009b) that in expected utility setting homogeneous simple strategy is equivalent to utility function being of constant relative risk aversion (CRRA) type. Within this class logarithmic utility function guarantees no bankruptcy as shown above. Even more is true as will be shown in the next section - in CRRA class all utility functions with relative risk aversion coefficient $\alpha \geq 1$ guarantee no-bankruptcy. Logarithmic utility function is the least restrictive (rejects the least) among all CRRA utility functions that guarantee no-bankruptcy[4]. To understand the intuition behind this result is quite simple. Suppose CRRA utility function is normalized conveniently and it takes the form given in (3).Notice the following fact about this class of functions:

$$\lim_{x\to 0} U_\alpha(x) = \begin{cases} -\infty & \text{for } \alpha \geq 1 \\ A(\alpha) > -\infty & \text{for } 0 < \alpha < 1 \end{cases}$$

Within CRRA class, logarithmic function is a function with the smallest relative risk aversion among those which "assign" $-\infty$ index to bankruptcy $x = 0$. Intuitively, if utility value for bankruptcy is finite as is the case for $0 < \alpha < 1$ then for any initial wealth W it is possible to construct a sequence of gambles such that the decision maker who makes decisions represented by this utility function goes bankrupt with positive probability. Consider a decision maker whose decisions are represented by a CRRA utility function $U(x)$ for which $\lim_{x\to 0} U(x) = A > -\infty$. Suppose his initial wealth level is $W > 0$. I construct a gamble $(-W, p, M, 1-p)$, $1 > p > 0$ ($-W$ with probability p and M with probability $1-p$) such that $pA + (1-p)U\left(1 + \frac{M}{W}\right) \geq 0$. It is possible to construct such a gamble because CRRA utility function is unbounded above. Hence the decision maker will accept this gamble at wealth level W. In one step only the probability that this decision maker goes bankrupt (his wealth is zero) is $p > 0$. As long as $\lim_{x\to 0} U(x) = A > -\infty$, it is possible to construct gambles that make an individual whose preferences are represented by $U(x)$ bankrupt in one step with positive probability.

The situation is different for $\alpha \geq 1$. Here $\lim_{x\to 0} U_\alpha(x) = -\inf$. An individual with initial wealth W whose preferences are represented by such utility function U will never accept a finite-valued gamble that makes him bankrupt with positive probability. It follows that there does not exists a finite sequence of gambles that such individual would accept and which would make him bankrupt with positive probability. What about the infinite sequence of gambles? Here is a useful illustration: Suppose the decision maker has initial wealth $W > 0$ and his preferences can represented by logarithmic utility function. Suppose further that he is offered an infinite sequence of multiplicative gambles of the following form: $\left(\frac{1}{2}, \epsilon; 2^{\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$, where $\epsilon > 0$. Notice first that the decision maker will be indifferent between accepting and rejecting such gamble: $\epsilon \log \frac{1}{2} + (1-\epsilon) \log 2^{\frac{\epsilon}{1-\epsilon}} = \log 1$. Assume that the gambles

[4] It is also independently shown in Foster and Hart (2007).

in this infinite sequence are perfectly positively correlated.[5] That means that after accepting n such gambles the decision maker's wealth may be written as $\left(W\frac{1}{2^n},\epsilon; W2^{n\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$. Let's write the "next" multiplicative gamble in nominal terms: $\left(-\frac{1}{2}W\frac{1}{2^n},\epsilon; +W2^{n\frac{\epsilon}{1-\epsilon}}, 1-\epsilon\right)$. Notice that as n goes to infinity wealth tends to zero with positive probability ϵ. However, in order to achieve this the gambles in the sequence become infinite valued: $\lim_{n\to\infty} W2^{n\frac{\epsilon}{1-\epsilon}} = \infty$ The above illustration shows that in case of logarithmic utility function [6], one needs infinite-valued gambles so that accepting these gambles leads to bankruptcy with positive probability and yet they are accepted.

It is worth noting that Foster and Hart (2007) have also another theorem in which they relax the assumption of homogeneous simple strategies. It seems however that this theorem is much weaker than the one with homogeneous simple strategies. It still says that to guarantee no-bankruptcy it is necessary to reject gamble $\mathbf{x}$ at wealth level W if $W < R_{FH}(\mathbf{x})$. This time however it is required only if wealth is close to zero already. If wealth is higher other strategies are sufficient such as: reject $\mathbf{x}$ at wealth W if $W < \min(\mathbf{x}) + \epsilon$, for $\epsilon > 0$ and small.

Apart from homogeneity assumption, the riskiness measure is defined only for gambles with positive expectation and possible losses. In the subsequent sections I will show a way to infer something about the riskiness of a gamble even if the gamble does not allow losses or/and has non-positive expectation.

4 Extended riskiness measure

In this section I define an extended riskiness measure and analyze conditions which are necessary and sufficient for existence of such measure. I discuss first the more general case of decreasing absolute risk aversion and then I focus on a subset of this, namely constant relative risk aversion.

4.1 Existence, uniqueness and no-bankruptcy for DARA

I focus on decreasing absolute risk aversion class of utility functions. Following Yaari (1969) I define the acceptance set $A_{\mathbf{x}} \equiv \{W : \mathrm{E}U(W+\mathbf{x}) > U(W)\}$ of wealth levels for which an individual with preferences represented by utility function U facing the lottery $\mathbf{x}$ strictly prefers to accept this lottery. Dybvig and Lippman (1983) proved the following result:

[5] For all other correlation the argument works even better.

[6] Actually, for all CRRA functions with $\alpha \geq 1$.

Theorem 2 (Dybvig and Lippman (1983)). *Let U be a strictly increasing concave utility function with continuous second derivative. Then absolute risk aversion A is decreasing if and only if for each gamble $\mathbf{x}$, $A_{\mathbf{x}}$ is an interval of the form $(\theta_{\mathbf{x}}, +\infty)$, where $-\infty \leq \theta_{\mathbf{x}} \leq +\infty$.*

The theorem is adjusted for the purposes of this paper. Define function $\phi(W) = \mathrm{E}U(W + \mathbf{x}) - U(W)$. Below I present my proof of this result as it is shorter and more straightforward than the original.

Proof. Notice that since U is continuous, function ϕ is continuous as well. Hence exactly one of the three possibilities can occur:

- $\phi(W) > 0,\ \forall W$, in which case $A_{\mathbf{x}} = (-\infty, +\infty)$
- $\phi(W) < 0,\ \forall W$, in which case $A_{\mathbf{x}} = (+\infty, +\infty)$
- function ϕ crosses zero axis

In the last case I will show that function ϕ crosses zero axis exactly once. Suppose function ϕ crosses zero axis at W^*, i.e. $\phi(W^*) = 0$. From the definition of selling price, it is clear that $S(W^*, \mathbf{x}) = 0$. Using the corollary 2 to Pratt (1964) theorem which can be found in the appendix, since for DARA utility S is increasing in W, it must be that for $W > W^*$, $S(W, \mathbf{x}) > 0$ and for $W < W^*$, $S(W, \mathbf{x}) < 0$. And hence there can be exactly one such W^* for which function ϕ crosses zero axis. □

The theorem above makes it clear that DARA utility means that wealthier people accept more gambles. It also shows that if there exists number W^* for which $\phi(W^*) = 0$, it must necessarily be unique. Therefore, it makes sense to define, whenever it exists, $R(\mathbf{x}) = W^*$ as an extended riskiness measure. There are two conditions which are necessary for existence of an extended riskiness measure for all functions which are concave and strictly increasing.

Proposition 4.1. *For all utility functions which are concave and strictly increasing and given a lottery $\mathbf{x}$, the following are necessary conditions for existence of $R(\mathbf{x})$:*

a. $\mathrm{E}[\mathbf{x}] > 0$

b. $\mathrm{P}[\mathbf{x} < 0] > 0$

Proof. To see that these two are the necessary conditions for existence of an extended riskiness measure, note that if $\mathrm{E}[\mathbf{x}] \leq 0$, then by Jensen's inequality

$U(R(\mathbf{x})) = \mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) < U[R(\mathbf{x}) + \mathrm{E}(\mathbf{x})] \leq U(R(\mathbf{x}))$, which is a contradiction. If on the other hand losses are not possible and $\mathrm{P}[\mathbf{x} < 0] = 0$ then $\phi(W) > 0,\ \forall W$ so that $R(\mathbf{x})$ does not exist. □

Suppose now that the outcome space is restricted to strictly positive real numbers, the intuition being that zero represents bankruptcy or the worst possible outcome. In this case the riskiness measure, if it exists, can take values in the interval $(L(\mathbf{x}), +\infty)$, where $L(\mathbf{x})$ is defined as the maximal loss of $\mathbf{x}$ and is equal to $-\min(\mathbf{x})$. The following are the necessary and sufficient conditions for the existence of an extended riskiness measure:

Proposition 4.2. *Given DARA utility function $U : (0, +\infty) \to \mathbb{R}$ and a lottery $\mathbf{x}$ satisfying conditions a. and b. stated above, the necessary and sufficient conditions for $R(\mathbf{x}) > L(\mathbf{x})$ to exist are:*

- $\lim_{W \to L(\mathbf{x})^+} \phi(W) < 0$
- $\lim_{W \to +\infty} \phi(W) \geq 0$

Proof. Notice first that due to condition b. above, $L(\mathbf{x}) > 0$. Therefore expression $\lim_{W \to L(\mathbf{x})^+} \phi(W)$ from the first condition above is well defined. Now let's write the definitions of selling price for two wealth levels W and V:

$$\mathrm{E}U(V + \mathbf{x}) - U[V + S(V, \mathbf{x})] = 0$$
$$\mathrm{E}U(W + \mathbf{x}) - U[W + S(W, \mathbf{x})] = 0$$

If $V > W$ and utility is DARA then by corollary 2 to Pratt (1964) theorem I have:

$$\mathrm{E}U(V + \mathbf{x}) - U[V + S(W, \mathbf{x})] > 0$$
$$\mathrm{E}U(W + \mathbf{x}) - U[W + S(V, \mathbf{x})] < 0$$

If $R(\mathbf{x}) = W$ then $S(W, \mathbf{x}) = 0$ and $\phi(V) > 0$ and if $R(\mathbf{x}) = V$, then $S(V, \mathbf{x}) = 0$ and $\phi(W) < 0$. That means that $R(\mathbf{x}) > L(\mathbf{x})$ exists, if $\lim_{W \to L(\mathbf{x})^+} \phi(W) < 0$. The second condition has to be satisfied due to the same reasons for which the proof of theorem 2 is true. Since extended riskiness measure is unique and function ϕ has to be increasing when evaluated at the extended riskiness measure, the value of function ϕ at wealth going to infinity has to be not less than zero. □

The above two conditions which are both necessary and sufficient for existence of an extended riskiness measure are not very informative for the general case of decreasing absolute risk aversion. Therefore I will provide below a pair of more informative conditions, the difference being that these conditions are sufficient but not necessary for existence of an extended measure of riskiness:

- $\lim_{x\to 0^+} U(x) = -\infty$
- $\lim_{x\to +\infty} A(x) = 0$

where $A(x)$ is absolute risk aversion function evaluated at x. To see that these conditions are sufficient for existence of an extended measure of riskiness, observe that if $\lim_{x\to 0^+} U(x) = -\infty$ and $\mathrm{P}[\mathbf{x} < 0] > 0$ then $\lim_{W\to L(\mathbf{x})^+} \phi(W) = -\infty$ due to the fact that lottery $\mathbf{x}$ is bounded-valued. Notice further that $\lim_{x\to +\infty} A(x) = 0$ means that the decision maker becomes risk neutral when he gets extremely rich. Since expected value of the lottery is assumed to be positive, $\mathrm{E}[\mathbf{x}] > 0$, therefore an extremely rich individual will accept this lottery meaning that $\lim_{W\to +\infty} \phi(W) \geq 0$.

It is worth noting that condition $\lim_{x\to 0} A(x) = +\infty$ is not sufficient to ensure $\lim_{W\to L(\mathbf{x})^+} \phi(W) < 0$ and hence to ensure that an extended riskiness measure exists. One needs stronger requirement of $\lim_{x\to 0^+} U(x) = -\infty$, which shall be apparent in the next subsection where CRRA class of utility functions is analyzed. There are CRRA utility function, namely the ones for which relative risk aversion α is smaller than certain cutoff α^* such that $\lim_{x\to 0} A(x) = +\infty$ and yet $\lim_{W\to L(\mathbf{x})^+} \phi(W) > 0$ meaning that an extended riskiness measure does not exists. Observe however that in the next subsection it proves beneficial to use variable $\frac{1}{W}$ instead of W so that the comparison of the two cases must be done with caution.

Before I will proceed to the next subsection, I want to demonstrate that for a certain class of DARA utility functions which are not necessarily CRRA, no-bankruptcy is guaranteed. First I will need the following lemma, which is also of interest for its own sake.

Without loss of generality[7] assume that utility function U satisfies the following: $U(1) = 0$ and $U'(1) = 1$. Given such utility function U define relative risk aversion function as $RRA(x) = -\frac{U''(x)x}{U'(x)}$. For utility function which is denoted U_i I will use notation RRA_i for the corresponding relative risk aversion function. Then the following lemma is true.

Lemma 1. *For some $\delta > 0$, suppose that $RRA_i(y) > RRA_j(y)$ for all y such that $|y| < \delta$. Then $U_i(y) < U_j(y)$ whenever $y \neq 1$ and $|y| < \delta$*

[7]Cardinal utility function is unique only up to affine transformation.

Proof. First, let me say that the proof is very similar to that used in lemma 2 of Aumann and Serrano (2008). They prove a similar proposition for absolute risk aversion.
Let $|y| < \delta$. If $y > 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= \int_1^y [\log U_i'(z)]' dz \\
&= \int_1^y \frac{U_i''(z)}{U_i'(z)} dz \\
&= -\int_1^y \frac{RRA_i(z)}{z} dz \\
&< -\int_1^y \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

If $0 < y < 1$, then

$$\begin{aligned}
\log U_i'(y) &= \log U_i'(y) - \log U_i'(1) \\
&= -\int_y^1 [\log U_i'(z)]' dz \\
&= -\int_y^1 \frac{U_i''(z)}{U_i'(z)} dz \\
&= \int_y^1 \frac{RRA_i(z)}{z} dz \\
&> \int_y^1 \frac{RRA_j(z)}{z} dz = \log U_j'(y)
\end{aligned}$$

Hence $\log U_i'(y) \lessgtr \log U_j'(y)$, when $y \gtrless 1$. It follows that $U_i'(y) \lessgtr U_j'(y)$, when $y \gtrless 1$.
If $y > 1$, then

$$U_i(y) = \int_1^y U_i'(z)dz < \int_1^y U_j'(z)dz = U_j(y)$$

If $0 < y < 1$, then

$$U_i(y) = -\int_y^1 U_i'(z)dz < -\int_y^1 U_j'(z)dz = U_j(y)$$

And hence the lemma is proved. □

Equipped with lemma 1 I can now demonstrate for which DARA utility functions in general the condition of no-bankruptcy is guaranteed.

Proposition 4.3. *For all bounded-valued lotteries and for all DARA utility functions for which $RRA(x) \geq 1$, $\forall x \in D$, where $RRA(x)$ is relative risk* *aversion function evaluated at x and D is the utility function's domain, no-bankruptcy is guaranteed.*

Proof. No-bankruptcy is guaranteed for logarithmic utility function for which relative risk aversion coefficient is equal to one. Take a DARA utility function U for which relative risk aversion is not less than one for all arguments in the domain of U. For any wealth level W I can normalize U without loss of generality so that $U(W) = \log(W)$. By lemma 1, since $RRA(y) \geq 1$ for all finite y, it is true that $U(y) \leq \log(y)$ and by normalization $U(W) = \log(W)$. It follows that if logarithmic utility function "rejects" a lottery $\mathbf{x}$, utility U also "rejects" this lottery. And hence it also guarantees no-bankruptcy. $\square$

4.2 Existence, uniqueness and no-bankruptcy for CRRA

I focus now on the CRRA function which is conveniently normalized[8]:

$$U(x,\alpha) = \begin{cases} \frac{x^{1-\alpha}-1}{1-\alpha}, & \text{for } 1 \neq \alpha > 0 \\ \log x, & \text{for } \alpha = 1 \end{cases}$$

where $x \in [0, \infty]$. I want to define a measure R for lottery $\mathbf{x}$ for CRRA utility function. This measure should satisfy the following condition:

$$\frac{1}{1-\alpha}\mathrm{E}\left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha} - \frac{1}{1-\alpha} = 0 \tag{6}$$

for a given lottery $\mathbf{x}$ and coefficient α. I want to ensure that such measure is well defined and unique. As already proved in the previous subsection measure of riskiness is unique if it exists. The necessary conditions are already provided in the former subsection and in particular, I will focus only on non-degenerate n-dimensional lotteries $\mathbf{x}$ with bounded values[9] such that $\mathrm{P}[\mathbf{x} < 0] > 0$ and $\mathrm{E}(\mathbf{x}) > 0$. Furthermore, I will restrict attention only to wealth levels W, such that $W \geq L(\mathbf{x}) > 0$. The fact that $L(\mathbf{x}) > 0$ follows from the fact that $\mathbf{x}$ may take negative values. Define lottery $\mathbf{y} = 1 + \frac{\mathbf{x}}{W}$. Notice that this lottery takes only non-negative values. It takes the lowest value of zero for $x_i = -L(\mathbf{x})$ for some $i \in \{1, ..., n\}$, since $W \geq L(\mathbf{x})$.

Notice that for the function form above, the following is true: $U(1) = 0$, $U'(y) = y^{-\alpha}$ and $U'(1) = 1$. Suppose there are two different CRRA utility functions with relative risk aversion coefficients equal to α_i and α_j, respectively. Suppose

[8] See (3).

[9] The following condition holds: there exists $\delta > 0$ such that $|x_i| < \delta \quad \forall i \in \{1, ..., n\}$.

further that $\alpha_i > \alpha_j$. Then from lemma 1 I know that $U(y, \alpha_i) < U(y, \alpha_j)$, for $y \in [0, \delta)$, some $\delta > 0$ and $y \neq 1$. Hence,

$$\begin{aligned} & \frac{1}{1-\alpha_i} \mathrm{E} \left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha_i} - \frac{1}{1-\alpha_i} \\ < \quad & \frac{1}{1-\alpha_j} \mathrm{E} \left(1 + \frac{\mathbf{x}}{R(\mathbf{x})}\right)^{1-\alpha_j} - \frac{1}{1-\alpha_j} \end{aligned}$$

Let's define the following function:

$$\begin{aligned} \phi(\lambda, \alpha) &= \frac{1}{1-\alpha} \sum_{i=1}^{n} p_i [1 + \lambda x_i]^{1-\alpha} - \frac{1}{1-\alpha} \qquad (7) \\ 0 \leq \lambda \leq \frac{1}{L(\mathbf{x})}, &\quad x_i \in [-L(\mathbf{x}), +M(\mathbf{x})] \end{aligned}$$

where $M(\mathbf{x})$ is the maximal gain in $\mathbf{x}$ and $L(\mathbf{x})$ is the maximal loss of $\mathbf{x}$, both assumed to be finite.

I want to find out whether this function has a unique $\lambda > 0$, for which this function is equal to zero, given α, and whether it has a unique α for which the function is equal to zero, given that $\lambda = \frac{1}{L(\mathbf{x})}$. It turns out that the answer to both questions is positive, as I will demonstrate below.

Lemma 2. *The following properties characterize function ϕ:*

$$\begin{aligned} & \phi(0, \alpha) = 0 \\ & \frac{\partial \phi(\lambda, \alpha)}{\partial \lambda} = \sum_{i=1}^{n} p_i [x_i (1 + \lambda x_i)^{-\alpha}] \\ & \left.\frac{\partial \phi(\lambda, \alpha)}{\partial \lambda}\right|_{\lambda=0} = \sum_{i=1}^{n} x_i p_i = \mathrm{E}[\mathbf{x}] > 0 \\ & \frac{\partial^2 \phi(\lambda, \alpha)}{\partial^2 \lambda} = \alpha \sum_{i=1}^{n} p_i x_i^2 (1 + \lambda x_i)^{-\alpha-1} < 0 \quad \text{for} \quad \alpha > 0 \\ & \lim_{\lambda \to \frac{1}{L(\mathbf{x})}} \lim_{\alpha \to 1} \phi(\lambda, \alpha) = -\infty \\ & \lim_{\lambda \to \frac{1}{L(\mathbf{x})}} \phi(\lambda, 0) = \lim_{\lambda \to \frac{1}{L(\mathbf{x})}} \sum_{i=1}^{n} p_i (1 + \lambda x_i) - 1 = \frac{1}{L(\mathbf{x})} \mathrm{E}[\mathbf{x}] > 0 \qquad (8) \end{aligned}$$

Furthermore $\lim_{\lambda \to \frac{1}{L(\mathbf{x})}} \phi(\lambda, \alpha)$ is a continuous function of α and it is strictly monotonic in α (see lemma 1). Therefore the following result holds:

Proposition 4.4. *Given function $\phi(\lambda, \alpha)$ and a random variable $\mathbf{x}$ with n values denoted by x_i for $i = 1, ..., n$, where $\mathrm{E}(\mathbf{x}) > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$,* *the following is true. Denote $L = L(\mathbf{x})$ and $M = M(\mathbf{x})$.*

$$\exists! \alpha^* < 1 : \begin{cases} \alpha < \alpha^* & \phi(\frac{1}{L}, \alpha) > 0 \\ \alpha = \alpha^* & \phi(\frac{1}{L}, \alpha) = 0 \\ \alpha > \alpha^* & \phi(\frac{1}{L}, \alpha) < 0 \end{cases}$$

Furthermore, suppose I take $\alpha > \alpha^*$ and fix it. Then:

$$\exists! \lambda^* : \begin{cases} \lambda < \lambda^* & \phi(\lambda, \alpha) > 0 \\ \lambda = \lambda^* & \phi(\lambda, \alpha) = 0 \\ \lambda > \lambda^* & \phi(\lambda, \alpha) < 0 \end{cases}$$

Proof. Follows from the above stated properties of a function ϕ (lemma 2). □

The above proposition states that riskiness measure for CRRA is defined for $\alpha \geq \alpha^*$, where α^* depends on a lottery. In this case the riskiness measure is unique. For different α's from the set of α's satisfying $\alpha > \alpha^*$ I get different λ^*, which is the inverse of the riskiness measure. Let's define a function $\lambda^*(\alpha)$, where $\alpha > \alpha^*$ and $\phi(\lambda^*(\alpha), \alpha) = 0$. I have the following proposition:

Proposition 4.5. *The function $\lambda^*(\alpha)$ is decreasing in α.*

Proof. Suppose $\alpha_1 > \alpha_2$ and that $\alpha_1 > \alpha^*$. Then

$$\begin{aligned} 0 &= \phi(\lambda^*(\alpha_1), \alpha_1) \\ &= \frac{1}{1-\alpha_1} \sum_{i=1}^{n} p_i (1 + \lambda^*(\alpha_1) x_i)^{1-\alpha_1} - \frac{1}{1-\alpha_1} \\ &< \frac{1}{1-\alpha_2} \sum_{i=1}^{n} p_i (1 + \lambda^*(\alpha_1) x_i)^{1-\alpha_2} - \frac{1}{1-\alpha_2} = \phi(\lambda^*(\alpha_1), \alpha_2) \end{aligned}$$

Hence:

$$\phi(\lambda^*(\alpha_1), \alpha_2) > 0$$
$$\phi(\lambda^*(\alpha_2), \alpha_2) = 0$$

Since $\phi(\lambda, \alpha)$ is concave in λ and $\phi(\frac{1}{L}, \alpha) < 0$, I conclude that $\lambda^*(\alpha_2) > \lambda^*(\alpha_1)$. □

The above proposition states that the higher is α, the relative risk aversion coefficient, the higher is riskiness measure, which is the inverse of $\lambda^*(\alpha)$. It confirms a conjecture that since rejecting for wealth being below riskiness measure based on $\alpha = 1$ (Foster and Hart (2007) riskiness measure) guarantees no bankruptcy, also rejecting for wealth below riskiness measure based on $\alpha > 1$

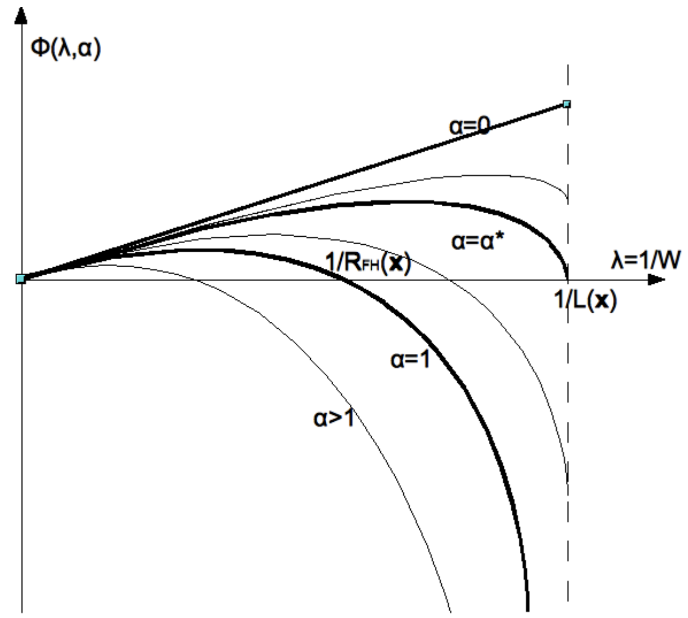

Figure 1: An extended riskiness measure for CRRA utility

guarantees no bankruptcy, as it means more rejection. To illustrate the above propositions and clarify the meaning of the different concepts and variables, look at the graph below:

This graph depicts the shape of $\phi(\lambda, \alpha)$ function for different values of relative risk aversion α within the CRRA class of utility functions. For α between 0 and α^* an extended riskiness measure is not defined since in this case function ϕ does not cross the zero axis. An extended riskiness measure is defined if $\alpha \geq \alpha^*$. Furthermore, it is also clear from the picture that an extended riskiness measure for values of α greater than 1 is necessarily greater than $R_{FH}(\mathbf{x})$ and hence rejecting $\mathbf{x}$ at wealth smaller than the extended riskiness measure in this case also guarantees no-bankruptcy.

5 Relation between riskiness measure and buying and selling price for a lottery

This section provides a link between the concepts of buying and selling price for a lottery and riskiness measure. The first lemma below demonstrates that although the riskiness measure for gambles with negative expectation or gambles without losses is not meaningful, it becomes meaningful and well defined if buying price or selling price for such gamble is subtracted from this gamble.

Lemma 3. *Given a non-degenerate lottery* $\mathbf{x}$ *and wealth level* W*, such that* $B(W, \mathbf{x})$ *and* $S(W, \mathbf{x})$ *exist, both* $R(\mathbf{x} - B(W, \mathbf{x}))$ *and* $R(\mathbf{x} - S(W, \mathbf{x}))$ *are well defined.*

Proof. I have to prove that $R(\mathbf{x} - S(W, \mathbf{x}))$ and $R(\mathbf{x} - B(W, \mathbf{x}))$ exist for any lottery $\mathbf{x}$. Notice that by proposition 4.4, the riskiness measure $R(\mathbf{x})$ exists for a lottery $\mathbf{x}$ and $\alpha \in [\alpha^*, +\infty)$ if and only if $\mathrm{E}(\mathbf{x}) > 0$ and $\mathrm{P}[\mathbf{x} < 0] > 0$. I have to check these two conditions for lotteries $\mathbf{x} - S(W, \mathbf{x})$ and $\mathbf{x} - B(W, \mathbf{x})$. Notice that for an arbitrary lottery $\mathbf{x}$, the following holds by proposition 7.1:

$$\begin{aligned}\mathrm{E}[\mathbf{x} - B(W, \mathbf{x})] &> 0 \\ \mathrm{E}[\mathbf{x} - S(W, \mathbf{x})] &> 0\end{aligned}$$

Notice further that by proposition 7.1, $-L(\mathbf{x}) < B(W, \mathbf{x})$ and $-L(\mathbf{x}) < S(W, \mathbf{x})$. Since all the values in the support of $\mathbf{x}$ get positive probability

$$\begin{aligned}\mathrm{P}[\mathbf{x} - B(W, \mathbf{x}) < 0] &> 0 \\ \mathrm{P}[\mathbf{x} - S(W, \mathbf{x}) < 0] &> 0\end{aligned}$$

Hence the two necessary conditions for $R(\mathbf{x} - S(W, \mathbf{x}))$ and $R(\mathbf{x} - B(W, \mathbf{x}))$ to exists are satisfied. For $\alpha \geq 1$ these conditions are also sufficient for existence of such measures by proposition 4.4. For $0 < \alpha < 1$, the following is true by proposition 2.1[10]:

$$\begin{aligned}\lim_{W \to W_L(\mathbf{x})} B(W, \mathbf{x}) &= W_L(\mathbf{x}) + \min(\mathbf{x}) \\ \lim_{W \to -\min(\mathbf{x})} S(W, \mathbf{x}) &= W_L(\mathbf{x}) + \min(\mathbf{x})\end{aligned}$$

where $W_L(\mathbf{x}) = U^{-1}[\mathrm{E}U(-\min(\mathbf{x}) + \mathbf{x})]$.
Hence $W_L(\mathbf{x}) + \min(\mathbf{x})$ is the minimal value for B and S when $0 < \alpha < 1$. By definition of $W_L(\mathbf{x})$, the following holds:

$$W_L(\mathbf{x}) = R(\mathbf{x} - (W_L\mathbf{x} + \min(\mathbf{x})))$$

So the riskiness measure for $\mathbf{x} - S$ and $\mathbf{x} - B$ for the lowest possible value of S and B (which is equal) is well defined and its value is $W_L(\mathbf{x})$. For higher values of S and B approaching (but not reaching) $\mathrm{E}[\mathbf{x}]$ the riskiness measure is well defined by proposition 4.4 and its value increases. □

The opposite direction of the above lemma is as follows:

[10] Notice that $L(\mathbf{x})$ used in proposition 4.4 is the same as $-\min(\mathbf{x})$ used in proposition 2.1

Lemma 4. *If $R(\mathbf{x}-\delta)$ is well defined, where $\delta \in \mathbb{R}$, then*

$$\delta = S(R(\mathbf{x}-\delta)-\delta, \mathbf{x}) = B(R(\mathbf{x}-\delta), \mathbf{x}) \tag{9}$$

Proof. The relationship in 9 follows from the definitions of R, S and B. Again for $\alpha \geq 1$, values of S and B in the equation above are well defined if $R(\mathbf{x}-\delta)$ is well defined. For $0 < \alpha < 1$ on the other hand the following is true. The riskiness measure is defined for $\alpha > \alpha^*$[11]. By proposition 4.4, $\mathrm{E}U_{\alpha^*}(L(\mathbf{x})+\delta+\mathbf{x}) = U_{\alpha^*}(L(\mathbf{x})+\delta)$. Hence $W_L(\mathbf{x})$ defined in proposition 2.1 for utility function U_{α^*} is equal to $L(\mathbf{x})+\delta = -\min(\mathbf{x})+\delta$. And thus the lowest values of buying and selling price when utility function is U_{α^*} are:

$$S(L(\mathbf{x}), \mathbf{x}) = B(L(\mathbf{x})+\delta, \mathbf{x}) = \delta$$

If buying and selling price are well defined for the lowest wealth levels, they are also well defined for higher wealth levels by proposition 2.1, and hence the lemma is proved. □

The following proposition establishes a simple link between buying and selling price for a lottery and the riskiness measure for this lottery.

Proposition 5.1. *Given wealth level $W \geq 0$, CRRA utility function U with RRA coefficient α in the interval $(\alpha^*, +\infty)$, where α^* satisfies $\phi(\frac{1}{L(\mathbf{x})}, \alpha^*) = 0$, and any non-degenerate lottery $\mathbf{x}$, the following relations hold:*

$$W = R(\mathbf{x}-S(W,\mathbf{x})) - S(W,\mathbf{x}) \tag{10}$$

$$W = R(\mathbf{x}-B(W,\mathbf{x})) \tag{11}$$

Proof. By lemma 3, measures $R(\mathbf{x}-S(W,\mathbf{x}))$ and $R(\mathbf{x}-B(W,\mathbf{x}))$ are well defined for an arbitrary non-degenerate lottery $\mathbf{x}$. Now, it follows from definitions of $R(\mathbf{x})$ and $S(W,\mathbf{x})$, $B(W,\mathbf{x})$ that:

$$\begin{aligned}\mathrm{E}\left[U\left(R(\mathbf{x}-B(W,\mathbf{x}))+\mathbf{x}-B(W,\mathbf{x})\right)\right] &= U(R(\mathbf{x}-B(W,\mathbf{x})))\\ \mathrm{E}\left[U\left(R(\mathbf{x}-S(W,\mathbf{x}))+\mathbf{x}-S(W,\mathbf{x})\right)\right] &= U(R(\mathbf{x}-S(W,\mathbf{x})))\end{aligned}$$

Therefore, it has to be that $W = R(\mathbf{x}-S(W,\mathbf{x})) - S(W,\mathbf{x})$ and $W = R(\mathbf{x}-B(W,\mathbf{x}))$. □

For the next proposition I will need two lemmas. They establish certain delta properties of the riskiness measure.

[11] It is not indicated but α^* is lottery-dependent.

Lemma 5. *Given lottery* $\mathbf{x}$ *and* $\Delta \in \mathbb{R}$ *such that riskiness for* $\mathbf{x}$ *and* $\mathbf{x} + \Delta$ *exists, the following holds:*

$$R(\mathbf{x} + \Delta) \leq R(\mathbf{x}) - \Delta \iff \Delta \geq 0$$

Proof. From the definition of R

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) - U(R(\mathbf{x})) \\ 0 &= \mathrm{E}U(R(\mathbf{x} + \Delta) + \mathbf{x} + \Delta) - U(R(\mathbf{x} + \Delta)) \end{aligned}$$

Since U is increasing, $\Delta \geq 0$ if and only if

$$\mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) - U(R(\mathbf{x})) \leq \mathrm{E}U(R(\mathbf{x}) - \Delta + \mathbf{x} + \Delta) - U(R(\mathbf{x}) - \Delta)$$

And hence

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x} + \Delta) + \mathbf{x} + \Delta) - U(R(\mathbf{x} + \Delta)) \\ &\leq \mathrm{E}U(R(\mathbf{x}) - \Delta + \mathbf{x} + \Delta) - U(R(\mathbf{x}) - \Delta) \end{aligned}$$

Thus by proposition 4.4

$$R(\mathbf{x} + \Delta) \leq R(\mathbf{x}) - \Delta \quad \square$$

Lemma 6. *Given lottery* $\mathbf{x}$ *and* $\Delta \in \mathbb{R}$ *such that riskiness for* $\mathbf{x}$ *and* $\mathbf{x} + \Delta$ *exists, the following holds:*

$$R(\mathbf{x} + \Delta) \leq R(\mathbf{x}) \iff \Delta \geq 0$$

Proof. "Only if" part follows from lemma 5. "If" part can be proved as follows: By definition of R.

$$\begin{aligned} 0 &= \mathrm{E}U(R(\mathbf{x}) + \mathbf{x}) - U(R(\mathbf{x})) \\ 0 &= \mathrm{E}U(R(\mathbf{x} + \Delta) + \mathbf{x} + \Delta) - U(R(\mathbf{x} + \Delta)) \end{aligned}$$

By proposition 4.4, and since $R(\mathbf{x}+\Delta) \leq R(\mathbf{x})$, $\mathrm{E}U(R(\mathbf{x})+\mathbf{x}+\Delta)-U(R(\mathbf{x})) \geq 0$. And since utility is increasing it must be that $\Delta \geq 0$. □

Note that lemma 5 and lemma 6 both imply that it is impossible for $R(\mathbf{x}+\Delta)$ to be between $R(\mathbf{x}) - \Delta$ and $R(\mathbf{x})$.
Now I can state a series of main results of this section which establish a well defined connection between riskiness measure and buying and selling price for a lottery.

Proposition 5.2. *Given wealth W and two lotteries $\mathbf{x}$ and $\mathbf{y}$, if there exist wealth levels W_1, W_2 such that $S(W_1, \mathbf{x}) = S(W, \mathbf{y})$ and $S(W_2, \mathbf{y}) = S(W, \mathbf{x})$. Then:*

$$S(W, \mathbf{x}) \geq S(W, \mathbf{y})$$
$$\iff$$
$$R(\mathbf{y} - S(W, \mathbf{x})) \geq R(\mathbf{x} - S(W, \mathbf{x})) \geq R(\mathbf{y} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{y})) \quad (12)$$

Proof. The requirement that there exist wealth levels W_1, W_2 such that $S(W_1, \mathbf{x}) = S(W, \mathbf{y})$ and $S(W_2, \mathbf{y}) = S(W, \mathbf{x})$ guarantees that $R(\mathbf{x} - S(W, \mathbf{y}))$ and $R(\mathbf{y} - S(W, \mathbf{x}))$ are well defined. This is so due to lemma 3. Also, $R(\mathbf{x} - S(W, \mathbf{x}))$ and $R(\mathbf{y} - S(W, \mathbf{y}))$ are well defined due to this lemma. I will now prove the equivalency stated in the proposition sequentially for the three inequalities in (12).

$\underline{S(W, \mathbf{x}) \geq S(W, \mathbf{y}) \iff R(\mathbf{y} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{y}))}$

Let $\Delta = S(W, \mathbf{x}) - S(W, \mathbf{y})$. By lemma 5, $\Delta \geq 0$ if and only if

$$R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{x} - S(W, \mathbf{x}) + \Delta) \geq \Delta$$

And after substituting the definition of Δ

$$R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{x} - S(W, \mathbf{y})) \geq S(W, \mathbf{x}) - S(W, \mathbf{y})$$

By proposition 5.1, this is in turn equivalent to the following:

$$R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{x} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{x})) - R(\mathbf{y} - S(W, \mathbf{y}))$$

And after simplifying

$$R(\mathbf{y} - S(W, \mathbf{y})) \geq R(\mathbf{x} - S(W, \mathbf{y}))$$

Which is what I had to prove. Now notice that the second inequality $R(\mathbf{x} - S(W, \mathbf{x})) \geq R(\mathbf{y} - S(W, \mathbf{y}))$ is equivalent to $S(W, \mathbf{x}) \geq S(W, \mathbf{y})$ by proposition 5.1. It leaves the one remaining inequality to be proved.

$\underline{S(W, \mathbf{x}) \geq S(W, \mathbf{y}) \iff R(\mathbf{y} - S(W, \mathbf{x})) \geq R(\mathbf{x} - S(W, \mathbf{x}))}$

Let $\Delta = S(W, \mathbf{y}) - S(W, \mathbf{x})$. Lemma 5 can be restated as follows:

$$R(\mathbf{x} + \Delta) \geq R(\mathbf{x}) - \Delta \iff \Delta \leq 0$$

And hence replacing $\mathbf{x}$ with $\mathbf{y} - S(W, \mathbf{y})$

$$R(\mathbf{y} - S(W, \mathbf{y})) - R(\mathbf{y} - S(W, \mathbf{y}) + \Delta) \leq \Delta$$

And after substituting the definition of Δ

$$R(\mathbf{y}-S(W,\mathbf{y}))-R(\mathbf{y}-S(W,\mathbf{x}))\leq S(W,\mathbf{y})-S(W\mathbf{x})$$

By proposition 5.1 this is equivalent to

$$R(\mathbf{y}-S(W,\mathbf{y}))-R(\mathbf{y}-S(W,\mathbf{x}))\leq R(\mathbf{y}-S(W,\mathbf{y}))-R(\mathbf{x}-S(W,\mathbf{x}))$$

And after simplifying

$$R(\mathbf{y}-S(W,\mathbf{x}))\geq R(\mathbf{x}-S(W,\mathbf{x}))$$

This finishes the proof. □

The above proposition establishes that selling price for lottery $\mathbf{x}$ is not lower than the selling price for another lottery $\mathbf{y}$ at some wealth level if and only if the riskiness measure of $\mathbf{y}-S(W,\mathbf{x})$ is not lower than $\mathbf{x}-S(W,\mathbf{x})$ and $\mathbf{y}-S(W,\mathbf{y})$ is not lower than $\mathbf{x}-S(W,\mathbf{y})$. A similar proposition is obtained for buying price for a lottery.

Proposition 5.3. *Given wealth W and two lotteries $\mathbf{x}$ and $\mathbf{y}$, if there exist wealth levels W_1, W_2 such that $B(W_1,\mathbf{x})=B(W,\mathbf{y})$ and $B(W_2,\mathbf{y})=B(W,\mathbf{x})$. Then:*

$$B(W,\mathbf{x})\geq B(W,\mathbf{y})$$
$$\Longleftrightarrow$$
$$R(\mathbf{y}-B(W,\mathbf{x}))\geq R(\mathbf{x}-B(W,\mathbf{x}))=R(\mathbf{y}-B(W,\mathbf{y}))\geq R(\mathbf{x}-B(W,\mathbf{y}))$$

Proof. As in the previous proposition, all the riskiness measures are well defined due to lemma 3 and the assumption that there exist wealth levels W_1, W_2 such that $S(W_1,\mathbf{x})=S(W,\mathbf{y})$ and $S(W_2,\mathbf{y})=S(W,\mathbf{x})$ hold.
By proposition 5.1, $R(\mathbf{x}-B(W,\mathbf{x}))=R(\mathbf{y}-B(W,\mathbf{y}))$. The two remaining inequalities can be proved by using lemma 6.

$$\begin{aligned} R(\mathbf{y}-B(W,\mathbf{x}))\geq R(\mathbf{y}-B(W,\mathbf{y})) &\iff B(W,\mathbf{x})\geq B(W,\mathbf{y}) \\ &\iff R(\mathbf{x}-B(W,\mathbf{x}))\geq R(\mathbf{x}-B(W,\mathbf{y})) \end{aligned}$$ □

The two propositions above establish a link between selling and buying price for a lottery and the riskiness measure for gambles with prices. Even if riskiness measure for a given gamble is not meaningful due to the fact that the gamble has negative expectation or no losses, it is still meaningful and well-defined for gambles with prices, i.e. for gambles constructed by subtracting buying or selling price from the original gamble.

The above two proposition can be extended beyond their local (for a given wealth level) meaning. The following corollary to these propositions states a global result on extended riskiness with prices in relation to selling (and as it will turn out also buying) price for a lottery:

Corollary 1. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be any two lotteries. Suppose that* s *is a scalar which satisfies:*

$$s \in [\max\{\min_W S(W, \mathbf{x}), \min_W S(W, \mathbf{y})\}, \min\{\max_W S(W, \mathbf{x}), \max_W S(W, \mathbf{y})\}] \quad (13)$$

Then the following equivalence holds:

$$R(\mathbf{y} - s) \geq R(\mathbf{x} - s) \iff s \in \{s : s = S(W, \mathbf{x}), W \in \mathcal{W}\}$$

where $\mathcal{W} = \{W : S(W, \mathbf{x}) \geq S(W, \mathbf{y})\}$.

First, the above corollary could alternatively be stated in terms of buying price. To see this, observe the following. If W^* is wealth level at which selling prices of $\mathbf{x}$ and $\mathbf{y}$ cross, i.e. $S(W^*, \mathbf{x}) = S(W^*, \mathbf{y}) = S^*$, then by lemma 7, it holds that: $S^* = B(W^* + S^*, \mathbf{x}) = B(W^* + S^*, \mathbf{y})$. Now, let's define an equivalent of set $\mathcal{W}$ for the case of buying price: $\mathcal{V} = \{W : B(W, \mathbf{x}) \geq B(W, \mathbf{y})\}$. The sets $\mathcal{W}$ and $\mathcal{V}$ are obviously different. However, by the argument I have just given, the sets $\{s : s = S(W, \mathbf{x}), W \in \mathcal{W}\}$ and $\{s : s = B(W, \mathbf{x}), W \in \mathcal{V}\}$ are the same. This is why the proposition above can be stated both in terms of selling price as well as in terms of buying price.

Condition (13) guarantees that s has to be in the range of both $S(W, \mathbf{x})$ and $S(W, \mathbf{y})$ as functions of wealth. In case of CRRA utility function, since proposition 2.1 establishes exactly what the range of buying and selling price is, this condition can be written explicitly. Suppose, for instance, that utility function is CRRA with coefficient of relative risk aversion greater than 1. In this case condition (13) takes the following form: $s \in [\max\{\min(\mathbf{x}), \min(\mathbf{y})\}, \min\{\mathrm{E}[\mathbf{x}], \mathrm{E}[\mathbf{y}]\}]$. Cases in which the interval in (13) is empty, are not interesting since either $\mathbf{x}$ is unambiguously better than $\mathbf{y}$[12], or the other way around. Of course, in such a case it is possible to establish a similar (to the one above) proposition, where riskiness of lotteries with different prices would be compared, i.e. the riskiness of $\mathbf{x} - s_1$ and $\mathbf{y} - s_2$, where $s_1 \neq s_2$. However, this would be a rather different exercise to the one I wish to pursue in this section.

The next two propositions inform us what can be inferred about the riskiness

[12] All the values in $\mathbf{x}$ are higher than those in $\mathbf{y}$.

measure from the global properties of selling and buying price as functions of wealth. These results are in fact special cases of the above corollary. However it is useful to state them and prove independently.

Proposition 5.4. *Given lotteries* $\mathbf{x}$ *and* $\mathbf{y}$ *and DARA utility function for which "riskiness measures"* $R(\mathbf{x})$ *and* $R(\mathbf{y})$ *are well defined, the following holds:*

$$B(W, \mathbf{y}) > B(W, \mathbf{x}) \ \ \forall W \implies R(\mathbf{x}) > R(\mathbf{y})$$

Proof. Suppose not. Then $R(\mathbf{x}) \leq R(\mathbf{y})$. By the proposition 5.1 equation (11), given any $\mathbf{x}$ for which R is defined and unique I have for $W = R(\mathbf{x})$:

$$R(\mathbf{x}) = R(\mathbf{x} - B(R(\mathbf{x}), \mathbf{x}))$$

By the uniqueness of $R(\mathbf{x})$ I get that $B(R(\mathbf{x}), \mathbf{x}) = 0$. From the fact that B is increasing in wealth for DARA utility (proposition 7.4) I have:

$$B(R(\mathbf{y}), \mathbf{y}) = 0 = B(R(\mathbf{x}), \mathbf{x}) \leq B(R(\mathbf{y}), \mathbf{x})$$

This proves that $\exists W$, such that $B(W, \mathbf{y}) \leq B(W, \mathbf{x})$. □

Proposition 5.5. *Given lotteries* $\mathbf{x}$ *and* $\mathbf{y}$ *and DARA utility function for which "riskiness measures"* $R(\mathbf{x})$ *and* $R(\mathbf{y})$ *are well defined. Then:*

$$S(W, \mathbf{y}) > S(W, \mathbf{x}) \ \ \forall W \implies R(\mathbf{x}) > R(\mathbf{y})$$

Proof. Suppose not. Then $R(\mathbf{x}) \leq R(\mathbf{y})$. From the proof of proposition 5.4 I know that given any $\mathbf{x}$ for which R is defined and unique $B(R(\mathbf{x}), \mathbf{x}) = 0$. Hence, by proposition 7.2, I know that $S(R(\mathbf{x}), \mathbf{x}) = 0$. From the the fact that S is increasing in wealth for DARA utility (corollary 2)I have:

$$S(R(\mathbf{y}), \mathbf{y}) = 0 = S(R(\mathbf{x}), \mathbf{x}) \leq S(R(\mathbf{y}), \mathbf{x})$$

This proves that $\exists W$, such that $S(W, \mathbf{y}) \leq S(W, \mathbf{x})$. □

The reverse direction in the above two propositions at the same time cannot be true, at least in the case of DARA utilities. To see it I can use the result from Lewandowski (2009a) which states that buying selling price reversal for DARA utilities is possible. That means that given any DARA utility there exist two lotteries $\mathbf{x}$ and $\mathbf{y}$ such that the following holds:

$$S(W, \mathbf{y}) > S(W, \mathbf{x}) > B(W, \mathbf{x}) > B(W, \mathbf{y})$$

If the reverse direction in both propositions 5.4 and 5.5 was true, the above pattern would not be possible. And hence the reverse direction of both propositions cannot be true. Whether the reverse direction in one case is true and in another is not or whether the reverse direction in both cases is not true remains unknown.

Below I wish to examine further connections between riskiness measure and buying and selling price for a lottery. The proposition below is an extension to Pratt (1964) famous theorem on comparative risk aversion. It shows that riskiness measure can be used along with buying price and selling price to compare risk aversion across individuals.

Proposition 5.6. *Given two CRRA utility functions U_1, U_2 with RRA coefficients α_1 and α_2, respectively, both in the interval $(\alpha^*, +\infty)$, where α^* satisfies $\phi(\frac{1}{L(\mathbf{x})}, \alpha^*) = 0$, and any non-degenerate lottery $\mathbf{x}$, such that $R_1(\mathbf{x})$ and $R_2(\mathbf{x})$ exists, the following holds:*

$$R_1(\mathbf{x}) > R_2(\mathbf{x}) \iff B_1(W, \mathbf{x}) < B_2(W, \mathbf{x}) \ \forall W \iff S_1(W, \mathbf{x}) < S_2(W, \mathbf{x}) \ \forall W$$

where R_i, B_i and S_i are, respectively, the riskiness measure, the buying price and the selling price corresponding to utility function U_i.

Proof. The second equivalence above is a special case (CRRA) of proposition 7.5 and hence was already proved there. I need to prove the first equivalence.

$(\Longleftarrow)$

I start by assuming $B_1(W, \mathbf{x}) < B_2(W, \mathbf{x}) \ \forall W$. By lemma 3 I know that $R_i(\mathbf{x} - S_i(W, \mathbf{x}))$ and $R_i(\mathbf{x} - B_i(W, \mathbf{x}))$ exist. By proposition 5.1 I know that $W = R(\mathbf{x} - B(W, \mathbf{x}))$. Furthermore, I know that $B(R(\mathbf{x}), \mathbf{x}) = 0$. Therefore:

$$\begin{aligned} R_1(\mathbf{x}) &= R_1(\mathbf{x} - B_1(R_1(\mathbf{x}), \mathbf{x})) \\ &= R_2(\mathbf{x} - B_2(R_1(\mathbf{x}), \mathbf{x})) \\ &> R_2(\mathbf{x}) \end{aligned}$$

Since $B_2(R_1(\mathbf{x}), \mathbf{x}) > B_1(R_1(\mathbf{x}), \mathbf{x}) = 0$, the last inequality follows from lemma 6. That the riskiness measure $R_2(\mathbf{x} - B_2(R_1(\mathbf{x}), \mathbf{x}))$ is well defined follows from the similar argument as in the proof of lemma 3. Since $\mathbf{x}$ was arbitrary, the above implication holds generally.

$(\Longrightarrow)$

I start by assuming $R_1(\mathbf{y}) > R_2(\mathbf{y})$ for all $\mathbf{y}$, such that R_1 and R_2 are defined. This holds in particular for lottery $\mathbf{y} = \mathbf{x} - B_1(W, \mathbf{x})$, for some W. It follows from proposition 5.1, that:

$$\begin{aligned} W = R_2(\mathbf{x} - B_2(W,\mathbf{x})) &= R_1(\mathbf{x} - B_1(W,\mathbf{x})) \\ &> R_2(\mathbf{x} - B_1(W,\mathbf{x})) \end{aligned}$$

And hence I know that $R_2(\mathbf{x} - B_2(W,\mathbf{x})) > R_2(\mathbf{x} - B_1(W,\mathbf{x}))$. By lemma 6, I conclude that $B_1(W,\mathbf{x}) < B_2(W,\mathbf{x})$. Since wealth W was arbitrary, as well as lottery $\mathbf{x}$, the proof is finished. □

I proved that one can use buying and selling price for a lottery as well as riskiness measure as equivalent ways to express absolute risk aversion. Although the proof is only valid for the CRRA case, the proposition is true whenever the existence of riskiness measure for the appropriate lotteries is guaranteed.

6 Expected utility decision-making using riskiness measure and buying and selling price for a lottery

Finally in this section, I will show how one can make decisions based on the concepts of buying and selling price for a lottery or riskiness measure. It shall come as no surprise that no matter with help of what concepts decisions are made within expected utility theory, they give rise to equivalent decision criteria.
Consider two situations.

- Case A: A decision maker with wealth $W \geq 0$ considers buying a nondegenerate lottery $\mathbf{x}$ for a price $b \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$
- Case B decision maker with wealth $W \geq 0$ participating in a nondegenerate lottery $\mathbf{x}$ considers selling lottery $\mathbf{x}$ for a price $s \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$

Proposition 6.1. *Given utility function U and lottery $\mathbf{x}$, such that the arguments of U are in the domain of U amd $S(W,\mathbf{x})$ and $B(W,\mathbf{x})$ are both well defined, the following criteria for decision making are equivalent:*

- *Expected utility criterion:*
  - *Case A: Buy $\mathbf{x}$ if $\mathrm{E}U(W + \mathbf{x} - b) \geq U(W)$, otherwise don't buy.*
  - *Case B: Sell $\mathbf{x}$ if $\mathrm{E}U(W + \mathbf{x}) \leq U(W + s)$, otherwise don't sell.*
- *Buying/selling price criterion:*
  - *Case A: Buy $\mathbf{x}$ if $B(W,\mathbf{x}) \geq b$, otherwise don't buy.*

  - *Case B: Sell* $\mathbf{x}$ *if* $S(W, \mathbf{x}) \leq s$*, otherwise don't sell.*

- *Riskiness measure criterion:*

  - *Case A: Buy* $\mathbf{x}$ *if* $W \geq R(\mathbf{x} - b)$
  - *Case B: Sell* $\mathbf{x}$ *if* $W \leq R(\mathbf{x} - s) - s$

Proof. The proof follows from the respective definitions and hence is omitted. Notice that since $b, s \in (\min(\mathbf{x}), \mathrm{E}[\mathbf{x}])$, $R(\mathbf{x} - b)$ and $R(\mathbf{x} - s)$ are well defined by lemma 3. □

7 Concluding remarks

In this paper I analyzed riskiness measure as introduced by Foster and Hart (2007). I gave simple intuition behind their result and I tried to make some steps towards extending this measure in two respects - first to define an extended riskiness measure based on DARA utility functions and derive necessary and sufficient conditions for existence and uniqueness of such measure for DARA and CRRA class of utility functions. Obviously, for the more specialized case of CRRA utility functions more exact conditions are obtained than for the more general case of DARA utilities. I also tried to extend the domain of riskiness measure. For gambles with non-positive expectation or no losses I proposed a way to compare their riskiness by subtracting prices from them. If the riskiness ordering is unchanged over the whole range of prices for which the lottery minus the price exists is unchanged, something can be inferred about the riskiness of a gamble without prices. To this end a number of useful properties relating buying and selling price for a lottery and riskiness measure were established and should be useful also for their own sake. An extension of Pratt (1964) famous result on comparative risk aversion involving riskiness measure along with buying and selling price for a lottery was stated and proved. Finally a simple link between decision-making using riskiness measure and decision-making using buying and selling price was developed.

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Appendix

Lemma 7. *Given any lottery* $\mathbf{x}$ *and wealth level* W, *the following three relations between buying price and selling price hold:*

$$
\begin{aligned}
S[W, \mathbf{x} - B(W, \mathbf{x})] &= 0 && (14)\\
S[W - B(W, \mathbf{x}), \mathbf{x}] &= B(W, \mathbf{x}) && (15)\\
B[W + S(W, \mathbf{x}), \mathbf{x}] &= S(W, \mathbf{x}) && (16)
\end{aligned}
$$

Proposition 7.1. *For any non-degenerate lottery* $\mathbf{x}$ *and any wealth* W *such that buying and selling price exist,* $S(W, \mathbf{x})$ *and* $B(W, \mathbf{x})$ *lie in the interval* $(\min(\mathbf{x}), \mathrm{E}(\mathbf{x}))$. *For a degenerate lottery* $\mathbf{x}$, $S(W, \mathbf{x}) = B(W, \mathbf{x}) = x$.

The following is a corollary to Pratt (1964) famous theorem of comparative risk aversion.

Corollary 2. *For a strictly increasing and twice differentiable utility function* U *with continuous second derivative, the following holds:*

- $S(W, \mathbf{x})$ *is increasing/constant/decreasing in* W *for every* $\mathbf{x}$ *iff* $A(W)$ *is decreasing/constant/increasing in* W

Proposition 7.2. *For any lottery* $\mathbf{x}$ *and any wealth* W, *for utilities with decreasing absolute risk aversion (DARA) the following equivalence holds:*

$$B(W, \mathbf{x}) > 0 \iff B(W, \mathbf{x}) < S(W, \mathbf{x})$$

Proposition 7.3. *For any lottery* $\mathbf{x}$ *and any wealth level* W *and for* $\Delta \in \mathbb{R}$, *the following holds:*

$$
\begin{aligned}
B(W, \mathbf{x} + \Delta) &= B(W, \mathbf{x}) + \Delta && (17)\\
S(W, \mathbf{x} + \Delta) &= S(W + \Delta, \mathbf{x}) + \Delta && (18)
\end{aligned}
$$

Notice that for DARA utility function and $B(W, \mathbf{x}) > 0$ the above result together with proposition 7.2 implies the following:

$$S(W, \mathbf{x} + \Delta) - B(W, \mathbf{x} + \Delta) = S(W + \Delta, \mathbf{x}) - B(W, \mathbf{x}) > S(W, \mathbf{x}) - B(W, \mathbf{x})$$

Proposition 7.4. *For a strictly increasing and twice differentiable utility function* U *with continuous second derivative, the following holds:*

- $B(W, \mathbf{x})$ *is increasing/constant/decreasing in* W *for every* $\mathbf{x}$ *iff* $A(W)$ *is decreasing/constant/increasing in* W

Lemma 8. *For differentiable DARA utility functions, given any n-dimensional non-degenerate lottery* $\mathbf{x}$ *and any wealth level* W*, the following holds:*

- $\mathrm{E}U'(W+\mathbf{x}) - U'(W+S(W,\mathbf{x})) > 0$
- $\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x})) - U'(W) > 0$
- $\mathrm{E}U'(R(\mathbf{x})+\mathbf{x}) - U'(R(\mathbf{x})) > 0$
- $0 < \frac{\partial B(W,\mathbf{x})}{\partial W} < 1$

Proof. From the definition of buying, selling price and the fact that they are both increasing in wealth, it follows that:

$$\frac{\partial S(W,\mathbf{x})}{\partial W} = \frac{\mathrm{E}U'(W+\mathbf{x}) - U'(W+S(W,\mathbf{x}))}{U'(W+S(W,\mathbf{x}))} > 0$$
$$\frac{\partial B(W,\mathbf{x})}{\partial W} = \frac{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x})) - U'(W)}{\mathrm{E}U'(W+\mathbf{x}-B(W,\mathbf{x}))} > 0$$

All of the properties above follow immediately. □

Proposition 7.5. *For two different utility functions* U_1 *and* U_2*, any wealth level* W *and any n-dimensional non-degenerate random variable* $\mathbf{x}$ *with bounded values, I define corresponding selling and buying prices* $S_1(W,\mathbf{x})$*,* $B_1(W,\mathbf{x})$ *and* $S_2(W,\mathbf{x})$*,* $B_2(W,\mathbf{x})$*. The following equivalence holds:*

$$\forall W\ \forall \mathbf{x}:\ \exists\ \delta > 0\ |x_i| < \delta\ \forall i \in \{1,...,n\}$$
$$S_1(W,\mathbf{x}) > S_2(W,\mathbf{x}) \iff B_1(W,\mathbf{x}) > B_2(W,\mathbf{x})$$

Proposition 7.6. *The following two statements are equivalent:*

i. *Bernoulli utility function exhibits CRRA*

ii. *buying and selling price for any lottery are homogeneous of degree one i.e.*

$$S(\lambda W, \lambda \mathbf{x}) = \lambda S(W,\mathbf{x}),\ \forall \lambda > 0$$
$$B(\lambda W, \lambda \mathbf{x}) = \lambda B(W,\mathbf{x}),\ \forall \lambda > 0$$