Perfect or Imperfect Competitors? Market Risk Perception Makes the Difference*
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Abstract

We study a model of risky asset pricing in an incomplete market with asymmetrically informed risk-averse rational investors, originally introduced by Wang [35] (1993). In his paper, adopting a perfect competitive rational expectation equilibrium perspective, Wang focuses on the existence and properties of an informationally efficient equilibrium of the model. The existence of equilibria other than the revealed one is not addressed. By contrast, adopting an imperfect competitive Bayesian-Nash approach, besides the rediscovery of Wang's equilibria, we reveal the existence of a large number of strategic equilibrium candidates, characterized by some extent of informationally inefficiency. Moreover, our computational procedures indicate that while in case of "low" market noise volatility and "low" subjective risk aversion Wang's equilibrium Pareto dominates the strategic candidates, with some exceptions, in case of "high" market noise volatility or "high" subjective risk aversion strategic equilibrium candidates prevail, with no exceptions. We are intrigued by the following economic interpretation: increasing perception of market risk turns rational investors from perfect to imperfect competitors and causes prices of the risky assets to lose informational efficiency.

Keywords: Asymmetric information; Insider trading; Linear filtering; Stochastic optimization

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1 Introduction

We analyze a model of dynamic equilibrium risky asset pricing in an incomplete market with asymmetrically informed risk-averse rational investors, which was originally introduced by Wang in his seminal paper [35] (1993). The only relevant difference between Wang's approach and ours is the equilibrium perspective. However, this has a rather significant impact both on the analysis and outcomes of the model. Wang adopts a perfect competitive rational expectation equilibrium (PCREE) perspective, which allows him to prove the existence of a single semi-strong efficient equilibrium candidate, depending on the exogenous parameters of the model. Instead, we pursue a strategic Bayesian-Nash (BN) approach, revealing that, under the same exogenous parameters, multiple strategic equilibrium candidates occur, most of which are informationally inefficient. Moreover, such informationally inefficient equilibrium candidates Pareto dominate Wang's one on the increasing of the exogenous parameters of the model related to market risk. Loosely speaking, as risk averse investors perceive a low [resp. high] exposure to market risk, they rationally trade as perfect [imperfect] competitors, which leads to informationally efficient [inefficient] equilibria.

The PCREE concept, developed by Lucas [28] (1972), Green [13] (1973), Grossman [14] (1976), and Kreps [22] (1977), has been widely exploited in literature. However, as discussed by Hellwig [18] (1980) (see also Back [3] (2004)), in PCREE perspective investors are assumed to behave as price takers, since they postulate the equilibrium price of the risky asset when choosing their optimal demands. As a consequence, investors' activity should not affect the price. On the other hand, when used to model trading activity on the basis of private information, this is quite unsatisfactory because investors, who formulate consistent beliefs, are led to assume that the putative price of the risky asset reflects also the private information. Hence, privately informed investors end up with trading on account of their private information. But how can this be reconciled with the story that trading activity does not affect the price? Quoting Kyle [24] (1989) a PCREE perspective requires informed investors to behave "schizofrenically". Instead, it is more likely that informed investors would try to influence the market price by suitably exploiting their trading strategies thanks to their superior information. In turn, the efficient market hypothesis (EMH), formulated by Fama [9] (1965), is one of the most influential and controversial topic in modern finance. Exploited in PCREE models, EMH leads to equilibria which are necessarily characterized by some extent of informational efficiency. This because EMH enters PCREE models by the more or less explicit assumption that the price of the risky asset is set by a risk neutral market maker as the expected present value of the future dividends, given her information. Nevertheless, despite Jensens' statement: "there is no other proposition in Economics which has more solid empirical evidence supporting it than EMH " (see Jensen [20] (1978)), lately an increasing number of empirical studies supporting EMH have been challenged and even reversed. In a PCREE-EMH setting, some relevant phenomena, commonly known as "market anomalies", such as the equity premium puzzle (see Mehra & Prescott [29] (1985)), the excess volatility in stock returns and price-dividend ratios (see Grossman & Shiller [16] (1981), LeRoy & Porter [26] (1981), Shiller [33] (1981)), the predictability of stock returns (see Poterba & Summers [31] (1988), Fama & French [10] (1989), see also Campbell & Shiller [6] (1988)), can find no housing. Even a new approach to Finance, known as "Behavioral Finance" (see Shleifer [32] (2000)), which relaxes both the assumptions of individual rationality and consistent beliefs, has been developed to show how the trading activity of boundedly rational investors may significantly deviate the prices of the risky assets from their fundamental values, proposing a possible explanation of the above mentioned phenomena (see e.g. DeLong & als [7] (1990), Benartzi & Thaler [4] (1995), Timmermann [34] (1996)). However, our results suggest that, still in a rational expectation setting, in which risk averse investors formulate consistent beliefs, the investors' perception of market risk may lead them to pursue a strategic trading which cause significant deviations from market efficiency.

In Wang's paper, as well as in ours, the incompleteness of the market is modeled by introducing a stochastic shock on the total supply of the risky asset. Asymmetric information is realized by enabling a group of rational investors to hold a private information on the growth rate of the dividend flow payed by the risky asset. These privately informed investors can also observe the shock on the risky asset supply, thereby ending up with a complete information. The other rational investors can directly observe neither the private information nor the shock on the risky asset supply. They rationally extract their missing information from the market public information, that is the history of the dividends and the risky asset price. The rational investors of each group have the same constant absolute risk aversion and maximize the utility of their intertemporal consumption over an infinite horizon. In this asymmetric information setting, trading equilibria are possible thanks to the market incompleteness (see Grossman & Stiglitz [17] (1980)). In Wang's PCREE equilibrium perspective, both the informed and uninformed rational investors, trading as perfect competitors, determine an optimal demand schedule in terms of a putative equilibrium price of the risky asset and submit it to an implicit Walrasian auctioneer. The latter aggregates the investors' demands and sets the actual equilibrium price via the market clearing condition. Wang's putative equilibrium price is assumed to be a perturbation of the risky asset fundamental value[1] by a linear combination of the other variables of the economy with misspecified coefficients. This with reference to Campbell & Kyle [5] (1993), who argued that, in an incomplete market with completely informed risk-averse rational investors, the price of a risky asset should be deviated from its fundamental value by a discount term accounting for investors' risk aversion and a linear term expressing the sensitivity of the price to the supply shocks. In turn, Wang's putative equilibrium price differs from Campbell & Kyle's price by an additional linear term to reflect the information asymmetry. Accordingly, Wang's equilibrium price is informationally efficient in the semi-strong form. Moreover, the private information is revealed over time so that the strong efficiency is achieved in the long run. As a consequence, the optimal demands for the risky asset of the two groups of investors turn out to be uncorrelated with the variables of the model conveying both the public and private information. Furthermore, the informed investors' strategies are positively correlated with both the risky asset supply shocks and the uninformed investors' estimation errors of the private information, while the uninformed investors' strategies are positively correlated with their estimate of the risky asset supply shocks[2]. Hence, the trading activity of the rational investors confirms that the investors trade as perfect competitors. Wang does not address the possibility that his model allows equilibria other than the revealed one and the uniqueness of the determined equilibrium is left pending[3].

By contrast, in the spirit of Kyle [24] (1989), our asymmetrically informed investors are imperfect competitors who do not trade as price takers. In fact, we assume that the informed [resp. uninformed] investors postulate the structure of the uninformed [resp. informed] investors' demand for the risky asset before choosing theirs. To the extent to which the two postulates are true, the market clearing condition determines a putative equilibrium price of the risky asset, which depends on the investors' demands. Hence, on the ground of the putative equilibrium price, the investors choose the demand that allows them to maximize their expected utility. The actual equilibrium is then achieved under the condition that the investors' optimal demands coincide with the postulated ones. As in Kyle [24] (1989), the schizofrenia problem is avoided since the actual equilibrium price is determined after the optimal demands are chosen and the effect that this choice has on the risky asset price is taken into account. Our approach allows to discover a large class of strategic equilibria which may differ from Wang's perfect competitive equilibria for the impact of private information on the coefficients of both

[1]That is the expected present value of the cumulative dividend payments under complete information.

[2]The informed investors, thanks to their complete information, are in a position to detect the uninformed investors' estimate of the private information, while the uninformed investors' estimate of the private information is equivalent to their estimate of the risky asset supply shocks.

[3]See Wang [35] (1993) note 19.

the risky asset price and the investors' strategies. The equilibrium price no longer needs to be a linear perturbation of the fundamental value, thereby losing information efficiency. The equilibrium strategies may be correlated with some information source. However, since we are primarily interested in studying the investors' behavior in response to an increase of the risk asset supply volatility or subjective risk aversion, we have also replied genuine Wang's approach to allow the analysis of several specifications of his model characterized by different sets of exogenous parameters.

As a first result, we have found that, while Wang's equilibrium of the model is revealed via our BN approach[4], BN equilibrium candidates exist which cannot be achieved via Wang's approach. However, this is not so surprising because seeking BN equilibria ultimately leads to a relaxation of the Campbell-Kyle-Wang constraints on the coefficients of the putative equilibrium risky asset price, yielding the latter as a linear perturbation of the fundamental value of the risky asset. Indeed, from a mathematical viewpoint, the newly discovered equilibrium candidates could also be obtained via a Walrasian auctioneer approach, which directly generalizes Wang's one, just allowing the coefficients of the putative equilibrium risky asset price to be all misspecified. Nevertheless, we think that our BN approach is better suited to give the new candidates a strategic flavor.

As a second result, somewhat surprising, we have discovered that genuine Wang's approach itself allows to reveal multiple equilibrium candidates. Some of these candidates are clearly to be rejected, on the basis of their lack of Pareto efficiency, since they are characterized by a lower expected utility for both the groups of competitors with respect to the benchmark of Wang's equilibria. However, others are characterized by a lower expected utility for a group of investors, while the expected utility of the investors of the other group increases. Despite neither the Pareto efficiency criterium nor a Walrasian approach does help with dealing with these cases, we think that a Bayesian Nash approach may do. Indeed, in a model allowing both perfect competitive and strategic equilibrium candidates, it does not seem unreasonable to us to interpret the latter as deviations from the former. Actually, a computational procedure of ours obtains the discovered strategic equilibrium candidates as deviations from Wang's perfect competitive benchmark: a group of investors postulate that their competitors follow Wang's perfect competitive equilibrium strategy and try to deviate from their own equilibrium strategy to increase their expected utility. Such a deviation originates a sequential bargaining procedure, which most frequently leads back to Wang's equilibrium, but may also lead to a strategic equilibrium candidate. On the other hand, a strategic deviation from an equilibrium position which ends up with

[4]The finding of Wang's equilibria via our approach is not anticipated in any assumption on the putative equilibrium price of the model, though.

a lower expected utility seems to us to miss the Bayesian Nash flavor. Therefore, we would propose the rejection of the equilibrium candidates in which the expected utility of strategic investors lessens. However, the rejection of some other equilibrium candidates seems to us more questionable. In particular, we have discovered the existence of equilibrium candidates in which a small minority of imperfect competitors of one group, informed or uninformed, achieve a higher expected utility, with respect to Wang's equilibria, trading against a large majority of perfect competitors of the other group, whose expected utility lowers[5]. Moreover, still via genuine Wang's approach, we have discovered that, on the increasing of risky asset supply volatility or investors' risk aversion, perfect competitive equilibrium candidates appear in which alternatively one of the two groups of traders achieve a higher expected utility, while the expected utility of the investors of the other group lowers.

As a third and main result we have discovered that risky asset supply volatility or investors' risk aversion matters in Pareto ranking the equilibrium candidates. In fact, while under "low" risky asset supply volatility and investors' risk aversion both Wang's approach and ours lead to the same equilibria, under "high" risky asset supply volatility or investors' risk aversion, equilibrium candidates occur in which the investors of the two groups trading both strategically achieve both a higher utility. More specifically, our numerical simulations suggest the introduction of a parameter, which we propose to call "market risk perception", defined in terms of the market noise volatility, the investors' subjective risk aversion and the proportion of different investors. Perfect [resp. imperfect] competitive equilibria Pareto dominate as the market risk perception takes values below [resp. above] a certain level.

As we will show in more details in Subsection 9.2, the new equilibrium prices we have discovered always exhibit a much stronger discount for holding the risky asset and a much stronger sensitivity to the supply shocks on the risky asset than the corresponding Wang's perfect competitive prices do. In addition, they may be characterized by informational inefficiency, which persists even in the long run.

Besides our preference toward a BN approach in dealing with a model revealing the existence of strategic equilibria, two results of our analysis seem to us the most intriguing for their economic implications: the possibility that a small minority of investors trading strategically against a large majority of perfect competitors may achieve a higher expected utility than trading as perfect competitors; the possibility that, under high market risk perception, a strategic trading gives investors a higher expected utility than a perfect competitive trading. The latter possibility might even suggest a rough "equilibrium explanation" to the alternate of bull and bear market periods: changes in investors' perception of

[5] According to our computational analysis, these candidates likely disappear on the vanishing of the minority condition.

market risk might lead them to prefer imperfect competitive strategies to perfect competitive ones so that the equilibrium prices of the risky assets fall from the perfect competitive values to the strategic ones, while prices volatility increases and their informational efficiency reduces.

Following Wang [35] (1993), we also consider an infinite-horizon economy with a single consumption good, where a risk free asset and a risky asset are traded continuously in time. The risk free asset rewards with a constant continuous rate of return, while the risky asset pays a continuous flow of dividends growing at a stochastic rate. Current dividend payments and risky asset prices are public knowledge. In the economy, two groups of rational investors, both with constant absolute risk aversion, are each endowed with the same preferences and information. Hence, it is possible to deal with each group as it is a single agent. Despite the structure of the economy as well as the values of all exogenous parameters are publicly known, a private information allows one group of investors to have a sharper knowledge of the future growth rate of dividends than the other one. These informed investors will anticipate more accurate expected returns from investing in the risky asset. On the other hand, since the growth rate of dividends determines the rate of appreciation of stock prices, changes of prices provide signals about the future growth of dividends. Therefore, uninformed investors will extract information about the state of the economy from prices as well as dividends. Nevertheless, the observed signals do not fully reveal the true values of all the state variables of the economy, because the market is incomplete, due to stochastic shocks on the total supply of the risky asset, as a possible consequence of the trading activity of non rational investors. The basic differences between Wang's approach and ours are the equilibrium perspective and the focus on the search and classification of the equilibrium candidates, rather than the analysis of a single perfect competitive equilibrium. However, our different approach reflects on both the development of the analytic structure of the model, characterized by a large nonlinear system of equation, and a computational sequential bargaining procedure, built in Wolfram Mathematica®environment, leading to the equilibrium. According to this procedure, each group of investors progressively adjust their demand for the risky asset, as an optimal response to the demand of the other group[6]. The risky asset price moves accordingly. The equilibrium is declared when no further significant adjustments occur. Such a computational procedure allows to achieve the equilibrium in all specifications of the exogenous parameters of the model considered.

The paper is organized as follows. In Section 2 we describe the model. In Sections 3 and 4 we present our BN approach to the uniformed and informed investors' optimization problem, respectively.

[6]We have initialized our procedure either with a random demand or Wang's perfect competitive equilibrium demand with no meaningful differences in the achieved equilibrium candidates but their occurrence frequency.

Section 5 is devoted to BNE characterization. In Section 6 we briefly reply the RE approach, with particular reference to Wang's one, as it is a benchmark for ours. Section 7 analyzes the cases in which the rational investors are all informed or uninformed. In Section 8 we establish the formulas to compare our approach with Wang's one. Section 9 presents the achieved results. Section 10 concludes. All proofs are given in the Appendix.

A final comment on notation is in order. To make the reader easier to compare our approach with Wang's one, we have used his notations but some slight modification. More specifically, we have used upper-case letters, Greek or Roman, for the state variables of the economy and lower-case letters for the exogenous parameters of the model. The upper-case Roman letters denote the publicly observable variables, while the upper-case Greek letters denote the variables conveying private information. Throughout the paper, $\sigma(X(s), Y(s), \ldots;\ s \le t)$ stands for the σ-field generated by the processes $X(s), Y(s), \ldots$ and I_j is the j-th order identity matrix, for any $j = 1, 2, \ldots$.

2 The Model

We consider an infinite-horizon economy with a single consumption good, where a risk free asset and a risky asset are traded continuously in time in a frictionless market. The risk free asset rewards with a constant continuous rate of return $r > 0$, while the risky one, with price $P(t)$, yields a continuous dividend rate $D(t)$, whose history is publicly observable. The dynamics of $D(t)$ is described by the equation

$$dD(t) = (\Pi(t) - \alpha_D D(t))dt + \sigma_{D,D}\, dw_D(t) + \sigma_{D,\Pi}\, dw_\Pi(t), \tag{1}$$

characterizing a mean reverting process towards the level $\Pi(t)$, which follows in turn the null mean reverting process

$$d\Pi(t) = -\alpha_\Pi \Pi(t)dt + \sigma_\Pi\, dw_\Pi(t). \tag{2}$$

In (1) and (2), the terms $w_D(t)$ and $w_\Pi(t)$ are independent standard Wiener processes, the positive parameter α_D [resp. α_Π] is the constant mean speed of reversion of the process $D(t)$ [resp. $\Pi(t)$] around its long-run level, the differential $\sigma_{D,D} dw_D(t) + \sigma_{D,\Pi} dw_\Pi(t)$ [resp. $\sigma_\Pi dw_\Pi$], for constant $\sigma_{D,D}$, $\sigma_{D,\Pi}$ [resp. σ_Π], constitutes the innovation in $D(t)$ [resp. $\Pi(t)$], and the quantity $\sigma_{D,D}^2 + \sigma_{D,\Pi}^2 \equiv \sigma_D^2$ [resp. σ_Π^2] is the innovation variance of $D(t)$ [resp. $\Pi(t)$]. The choice of a positive [resp. negative] $\sigma_{D,\Pi}$ causes a positive [resp. negative] correlation between changes in dividend rate $D(t)$ and the signal $\Pi(t)$. Setting $\sigma_{D,\Pi} = 0$ makes independent innovations in $D(t)$ and $\Pi(t)$. Note that the bivariate process $(D(t), \Pi(t))$

constitutes a Gaussian-Markov system.

The interpretation of $\Pi(t)$ as a private information on $D(t)$ follows by remarking that the quantity

$$\mathbf{E}[\Pi(t)] - \alpha_D D(t) \qquad [\text{resp.} \quad \Pi(t) - \alpha_D D(t)]$$

approximates the growth rate of dividend rate process at time $t + \Delta t$, which is expected by an investor whose information up to t is restricted to the only history of dividend rate process itself [resp. by an investor who can observe both the histories of dividend rate and the informative signal]. The mean reversion of $\Pi(t)$ to zero models the natural decaying of private information.

Following Wang, we also assume that the total supply of the risky asset is stochastic with a long-run stationary level normalized to 1. More specifically, to model the deviation of the current risky asset supply from its long-run stationary level, we introduce a process $\Theta(t)$ driven by a null mean reverting process independent of the system $(D(t), \Pi(t))$, that is

$$d\Theta(t) = -\alpha_\Theta \Theta(t)dt + \sigma_\Theta \, dw_\Theta(t), \tag{3}$$

where $w_\Theta(t)$ is a standard Wiener process, which is independent of both $w_D(t)$ and $w_\Pi(t)$. The positive parameter α_Θ expresses the constant mean speed of reversion of the processes $\Theta(t)$ towards its long-run null level, and σ_Θ^2 is the constant innovation variance of $\Theta(t)$. It is worth noting that the stochastic supply of the risky asset can be equivalently interpreted as the presence in the market of non rational investors, trading only for liquidity reason, whose demand for the risky asset, $-\Theta(t)$, introduces a noise component in the aggregate market demand.

There are two groups of rational agents, with constant absolute risk aversion, participating in the market: informed investors and uninformed investors, identified in the sequel by the indices i and u, respectively. All the informed [resp. uninformed] investors are endowed with the same preferences and information. Therefore, it is possible to deal with them as they were a single informed [resp. uninformed] agent, whose inventory at time t, that is the holding of the risky asset, we denote by $\Psi_k(t)$, for $k = i, u$. Hence, in equilibrium, at any time t, the total supply of the risky asset in the market satisfies the market clearing condition

$$(1 - \omega)\Psi_i(t) + \omega\Psi_u(t) = 1 + \Theta(t), \tag{4}$$

where $\omega \in [0,1]$ is a parameter modeling the fraction of the uninformed investors in the market: when $\omega = 0$ [resp. $\omega = 1$] the rational investors are all informed [resp. all uninformed]. All the rational investors can observe the history of the price $P(t)$ and the dividend rate $D(t)$, but only the informed investors can observe the history of the signal $\Pi(t)$. In addition, the informed [resp. uninformed] investors can observe the part of the risky asset supply

$$O_i(t) \equiv 1 + \Theta(t) - \omega\Psi_u(t) \qquad [\text{resp.} O_u(t) \equiv 1 + \Theta(t) - (1-\omega)\Psi_i(t)],$$

which is complementary with respect to their own inventory $\Psi_i(t)$ [resp. $\Psi_u(t)$]. However, as we will show below, while the informed investors can distinguish between the components $\Theta(t)$ and $\Psi_u(t)$ of $O_i(t)$, the uninformed ones cannot disaggregate the observed $O_u(t)$.

In light of their information, both the two groups of investors maximize the expected value of the discounted utility of their consumption over the infinite time-horizon, by controlling their consumption policy $c_k(t)$ and their inventory $\Psi_k(t)$. Formally, the k-investors' objective function can be written as

$$\max_{c_k(\cdot),\Psi_k(\cdot)} \left\{ \mathbf{E}\left[\int_t^{+\infty} -e^{-(\rho_k s + \varphi_k c_k(s))}\, ds | \mathfrak{F}_k(t) \right] \right\}, \quad k = i, u \tag{5}$$

where ρ_k is the k-investors' subjective rate of time preference, φ_k is the coefficient of their absolute risk aversion, $\mathfrak{F}_k(t)$ stands for the σ-field representing the k-investors' information up to the current instant t, and $\mathbf{E}[\,\cdot\,|\mathfrak{F}_k(t)]$ is the conditional expectation operator given $\mathfrak{F}_k(t)$. In turn, $c_k(t)$ and $\Psi_k(t)$ are subject to the dynamics of the k-investors' wealth, $W_k(t)$, and the other variables of the economy. More specifically, $W_k(t)$ is a solution to the stochastic differential equation

$$dW_k(t) = (rW_k(t) - c_k(t))\, dt + \Psi_k(t) dQ(t), \quad k = i, u \tag{6}$$

where $Q(t)$ is the instantaneous excess return to one share of risk asset, satisfying

$$dQ(t) = (D(t) - rP(t))\, dt + dP(t), \tag{7}$$

and the price $P(t)$ of the risky asset is obtained via the market clearing condition (4).

In an imperfect competitive Bayesian-Nash linear equilibrium perspective, we assume the informed investors postulate the uninformed investors' inventory is linear in the variables of the economy which are observed by the latter, that is

$$\Psi_u(t) = \psi_u + \psi_{u,D} D(t) + \psi_{u,\hat{\Pi}} \hat{\Pi}(t) + \psi_{u,\hat{\Theta}} \hat{\Theta}(t) - \psi_{u,P} P(t), \tag{8}$$

for suitable coefficients ψ_u, $\psi_{u,D}$, $\psi_{u,\hat{\Pi}}$, $\psi_{u,\hat{\Theta}}$, $\psi_{u,P}$, where $\hat{\Pi}(t) \equiv \mathbf{E}[\Pi(t) \,|\mathfrak{F}_u(t)]$ and $\hat{\Theta}(t) \equiv \mathbf{E}[\Theta(t) \,|\mathfrak{F}_u(t)]$. In turn, the uninformed investors postulate an informed investors' linear inventory accounting of the information held by the latter, that is

$$\Psi_i(t) = \psi_i + \psi_{i,D} D(t) + \psi_{i,\Pi} \Pi(t) + \psi_{i,\check{\Theta}} \check{\Theta}(t) - \psi_{i,P} P(t), \tag{9}$$

for suitable coefficients ψ_i, $\psi_{i,D}$, $\psi_{i,\Pi}$, $\psi_{i,\check{\Theta}}$, $\psi_{i,P}$, where $\check{\Theta}(t) \equiv \mathbf{E}[\Theta(t) \,|\mathfrak{F}_i(t)]$. To the extent to which both the investors' postulates are true, the market clearing condition (4) yields the putative equilibrium price of the risky asset in the form

$$P(t) = p + p_D D(t) + p_\Pi \Pi(t) + p_\Theta \Theta(t) + p_{\check{\Theta}} \check{\Theta}(t) + p_{\hat{\Pi}} \hat{\Pi}(t) + p_{\hat{\Theta}} \hat{\Theta}(t), \tag{10}$$

where

$$\begin{gathered}
p \equiv \frac{(1-\omega)\psi_i + \omega\psi_u - 1}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \qquad p_D \equiv \frac{(1-\omega)\psi_{i,D} + \omega\psi_{u,D}}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \\
p_\Pi \equiv \frac{(1-\omega)\psi_{i,\Pi}}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \qquad p_\Theta \equiv \frac{-1}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \\
p_{\hat{\Pi}} \equiv \frac{\omega\psi_{u,\hat{\Pi}}}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \qquad p_{\hat{\Theta}} \equiv \frac{\omega\psi_{u,\hat{\Theta}}}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}, \\
p_{\check{\Theta}} \equiv \frac{(1-\omega)\psi_{i,\check{\Theta}}}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}}.
\end{gathered} \tag{11}$$

This follows combining (8) and (9) with (4) and solving with respect to $P(t)$. Hence, in equilibrium, the uninformed investors' observation of $P(t)$ is equivalent to the observation of the signal

$$\check{S}(t) \equiv p_\Pi \Pi(t) + p_\Theta \Theta(t) + p_{\check{\Theta}} \check{\Theta}(t). \tag{12}$$

On the other hand, on account of (9) and (11), we have

$$O_u(t) = 1 - (1-\omega)\psi_i - (1-\omega)\psi_{i,D} D(t) - ((1-\omega)\psi_{i,P} + \omega\psi_{u,P})\check{S}(t) + (1-\omega)\psi_{i,P} P(t).$$

Therefore, the observation of $O_u(t)$ conveys the uninformed investors no further information than that already in their hands. This implies that, in equilibrium, we can write

$$\mathfrak{F}_u(t) \equiv \sigma(D(s), P(s), O_u(t);\ s \leq t) = \sigma(D(s), P(s);\ s \leq t).$$

Now, the informed investors can observe $P(t)$, $D(t)$, $\Pi(t)$, and $O_i(t)$. Therefore,

$$\mathfrak{F}_i(t) \equiv \sigma(D(s), P(s), \Pi(t), O_i(t);\ s \leq t) \supseteq \sigma(D(s), P(s);\ s \leq t) = \mathfrak{F}_u(t).$$

Hence, the informed investors hold a superior information. As a consequence, the informed investors can also observe the uninformed investors' estimates $\hat{\Pi}(t)$ and $\hat{\Theta}(t)$. It then follows that, in equilibrium, the informed investors observation of $P(t)$ is equivalent to the observation of $\Theta(t)$, that is

$$\check{\Theta}(t) = \Theta(t).$$

Therefore, the putative equilibrium price of the risky asset takes the form

$$P(t) = p + p_D D(t) + p_\Pi \Pi(t) + p_\Theta \Theta(t) + p_{\hat{\Pi}} \hat{\Pi}(t) + p_{\hat{\Theta}} \hat{\Theta}(t), \tag{13}$$

where p, p_D, p_Π, $p_{\hat{\Pi}}$, $p_{\hat{\Theta}}$ are given by (11) and

$$p_\Theta \equiv \frac{(1-\omega)\psi_{i,\Theta} - 1}{(1-\omega)\psi_{i,P} + \omega\psi_{u,P}} \tag{14}$$

for $\psi_{i,\Theta} \equiv \psi_{i,\check{\Theta}}$. As a consequence, the uniformed investors' observation of $P(t)$ is equivalent to the observation of the signal [7]

$$S(t) \equiv p_\Pi \Pi(t) + p_\Theta \Theta(t). \tag{15}$$

The observability of $S(t)$ and the linearity of the conditional expectation operator imply

$$p_\Pi \hat{\Pi}(t) + p_\Theta \hat{\Theta}(t) = p_\Pi \Pi(t) + p_\Theta \Theta(t). \tag{16}$$

The effective imperfect competitive Bayesian-Nash equilibrium is then achieved when the uninformed [resp. informed] investors maximize their utility (5), under the price (13), by virtue of an optimal

[7] Since $S(t)$ is publicly observable, we prefer to modify Wang's notation $\Lambda(t)$.

inventory which is exactly the one postulated by the informed [resp. uninformed] investors.

3 Uninformed Investors' Filtering-Optimization Problem

To solve their maximization problem (5), subject to (6), the uninformed investors need to write Equation (7) in terms of the variables and noises of the economy they can observe. On the other hand, in the putative equilibrium, on account of (13) and (15), they can write

$$dQ(t) = (D(t) - rP(t))\, dt + p_D dD(t) + dS(t) + p_{\hat{\Pi}} d\hat{\Pi}(t) + p_{\hat{\Theta}} d\hat{\Theta}(t). \tag{17}$$

Therefore, as a first step, the uninformed investors need to write the equations driving the dynamics of their estimates $\hat{\Pi}(t)$ and $\hat{\Theta}(t)$. To this task, we consider the filtering problem for the unobservable processes characterized by Equations (2) and (3), given the dividend process $D(t)$ solution to (1) and the signal $S(t)$ satisfying

$$dS(t) = -\alpha_\Pi p_\Pi \Pi(t)dt - \alpha_\Theta p_\Theta \Theta(t)dt + \sigma_\Pi p_\Pi dw_\Pi(t) + \sigma_\Theta p_\Theta dw_\Theta(t). \tag{18}$$

By a standard linear filtering procedure (see e.g. Liptser & Shiryayev 2001 [27, Vol. I, Thm 10.3, p. 392]), we have

Proposition 1 *The uninformed trader's estimates $\hat{\Pi}(t)$ and $\hat{\Theta}(t)$ satisfy the system*

$$\begin{pmatrix} d\hat{\Pi}(t) \\ d\hat{\Theta}(t) \end{pmatrix} = \begin{pmatrix} -\alpha_\Pi \hat{\Pi}(t) \\ -\alpha_\Theta \hat{\Theta}(t) \end{pmatrix} dt + \begin{pmatrix} h_{\hat{\Pi},D}(t) & h_{\hat{\Pi},S}(t) \\ h_{\hat{\Theta},D}(t) & h_{\hat{\Theta},S}(t) \end{pmatrix} \begin{pmatrix} \sigma_D^2 & \sigma_{D,\Pi}\sigma_\Pi p_\Pi \\ \sigma_{D,\Pi}\sigma_\Pi p_\Pi & \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 \end{pmatrix}^{1/2} d\tilde{w}(t), \quad t \geq t_0, \tag{19}$$

where the process $\tilde{w}(t) \equiv (\tilde{w}_D(t), \tilde{w}_S(t))^T$ defined by

$$d\tilde{w}(t) \equiv \begin{pmatrix} \sigma_D^2 & \sigma_{D,\Pi}\sigma_\Pi p_\Pi \\ \sigma_{D,\Pi}\sigma_\Pi p_\Pi & \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 \end{pmatrix}^{-1/2} \begin{pmatrix} dD(t) - (\hat{\Pi}(t) - \alpha_D D(t))dt \\ dS(t) + \alpha_\Pi p_\Pi \hat{\Pi}(t)dt + \alpha_\Theta p_\Theta \hat{\Theta}(t)dt \end{pmatrix}, \quad t \geq t_0, \tag{20}$$

is Wiener with respect to the generated information structure, which is equivalent to $\mathfrak{F}_u(t)$*, and*

$$
\begin{aligned}
h_{\hat{\Pi},D}(t) &\equiv \frac{\sigma_\Pi \sigma_{D,\Pi}\sigma_\Theta^2 p_\Theta^2 + \left((\alpha_\Pi - \alpha_\Theta)\sigma_\Pi\sigma_{D,\Pi} + \sigma_\Pi^2 + \sigma_\Theta^2 \frac{p_\Theta^2}{p_\Pi^2}\right) p_\Pi^2 \sigma(t_0;t)}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2}, \qquad (21)\\
h_{\hat{\Pi},S}(t) &\equiv \frac{(\sigma_{D,D}^2\sigma_\Pi^2 - \left((\alpha_\Pi - \alpha_\Theta)\sigma_D^2 + \sigma_\Pi\sigma_{D,\Pi}\right)\sigma(t_0;t))p_\Pi}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2},\\
h_{\hat{\Theta},D}(t) &= -\frac{p_\Pi}{p_\Theta} h_{\hat{\Pi},D}(t),\\
h_{\hat{\Theta},S}(t) &= \frac{1 - p_\Pi h_{\hat{\Pi},S}(t)}{p_\Theta},
\end{aligned}
$$

for

$$
\sigma(t_0,t) = -\frac{b}{a^2} + \frac{\sqrt{b^2+a^2c^2}}{a^2}\tanh\left(\sqrt{b^2+a^2c^2}(t-t_0) + sech^{-1}\left(\frac{ac}{\sqrt{b^2+a^2c^2}}\right)\right), \quad t \geq t_0, \qquad (22)
$$

where

$$
\begin{aligned}
a^2 &\equiv \frac{\left((\alpha_\Pi - \alpha_\Theta)^2\sigma_{D,D}^2 + ((\alpha_\Pi - \alpha_\Theta)\sigma_{D,\Pi} + \sigma_\Pi)^2\right) p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2}, \qquad (23)\\
b &\equiv \frac{\alpha_\Theta\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + (\alpha_\Pi\sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi)\sigma_\Theta^2 p_\Theta^2}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2},\\
c^2 &\equiv \frac{\sigma_{D,D}^2\sigma_\Pi^2\sigma_\Theta^2 p_\Theta^2}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2}.
\end{aligned}
$$

In particular, in the stationary case, letting t_0 *go to* $-\infty$*, we have*

$$
\sigma \equiv \lim_{t_0 \to -\infty} \sigma(t_0, 0) = \frac{-b + \sqrt{b^2 + a^2c^2}}{a^2}. \qquad (24)
$$

Now, in the stationary case, combining (13) with (16) and (19), the uninformed investors can write

$$
\begin{aligned}
dP(t) = -p_D\alpha_D D(t)dt - (\alpha_\Pi(p_\Pi + p_{\hat{\Pi}}) - p_D)\hat{\Pi}(t)dt - \alpha_\Theta(p_\Theta + p_{\hat{\Theta}})\hat{\Theta}(t)dt \qquad (25)\\
+\tilde{\sigma}_{P,D}d\tilde{w}_D(t) + \tilde{\sigma}_{P,S}d\tilde{w}_S(t), \quad t \geq 0,
\end{aligned}
$$

where

$$\tilde{\sigma}_{P,D} \equiv b_{1,1}(p_D + (p_\Pi + p_{\hat{\Pi}})h_{\hat{\Pi},D} + (p_\Theta + p_{\hat{\Theta}})h_{\hat{\Theta},D}) + b_{2,1}((p_\Pi + p_{\hat{\Pi}})h_{\hat{\Pi},S} + (p_\Theta + p_{\hat{\Theta}})h_{\hat{\Theta},S}),$$
$$\tilde{\sigma}_{P,S} \equiv b_{1,2}(p_D + (p_\Pi + p_{\hat{\Pi}})h_{\hat{\Pi},D} + (p_\Theta + p_{\hat{\Theta}})h_{\hat{\Theta},D}) + b_{2,2}((p_\Pi + p_{\hat{\Pi}})h_{\hat{\Pi},S} + (p_\Theta + p_{\hat{\Theta}})h_{\hat{\Theta},S}).$$

for

$$\begin{pmatrix} \sigma_D^2 & \sigma_{D,\Pi}\sigma_\Pi p_\Pi \\ \sigma_{D,\Pi}\sigma_\Pi p_\Pi & \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 \end{pmatrix}^{1/2} \equiv \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}, \tag{26}$$

and $h_{J,K}$ is given by (21) on account of (24), for $J = \hat{\Pi}, \hat{\Theta}$, $K = D, S$.

Equation (25), combined with (7), allows the uninformed investors to write their wealth equation in terms of only observed variables and the Wiener process $\tilde{w}(t)$. In addition, from (20) we have

$$dD(t) = (\hat{\Pi}(t) - \alpha_D D(t))dt + b_{1,1}d\tilde{w}_D(t) + b_{1,2}d\tilde{w}_S(t), \quad t \geq 0. \tag{27}$$

Hence, subject to (6), (19), (25), and (27), which define a Markov multivariate process with respect to the noise $\tilde{w}(t)$, on account of the Separation Principle (see e.g. Fleming & Rishel (1975) [11]), the uninformed investors' objective function becomes

$$\max_{\Psi_u(\cdot),c_u(\cdot)} \left\{ \mathbf{E}_{t,D,\hat{\Pi},\hat{\Theta},P,W_u} \left[\int_t^{+\infty} -e^{-(\rho s + \varphi_u c_u(s))}\, ds \right] \right\}, \quad t \geq 0, \tag{28}$$

where $\mathbf{E}_{t,D,\hat{\Pi},\hat{\Theta},P,W_u}[\cdot]$ is the conditional expectation operator given the state of the random variables D, $\hat{\Pi}$, $\hat{\Theta}$, P, W_u at time t. More specifically, setting $Z_u(t) \equiv (1, D(t), \hat{\Pi}(t), \hat{\Theta}(t), P(t))^T$, we have

$$dZ_u(t) = A_u Z_u(t)dt + Q_u^{1/2} d\tilde{w}(t), \quad t \geq 0, \tag{29}$$

where

$$A_u \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_D & 1 & 0 & 0 \\ 0 & 0 & -\alpha_\Pi & 0 & 0 \\ 0 & 0 & 0 & -\alpha_\Theta & 0 \\ 0 & -p_D\alpha_D & p_D - \alpha_\Pi(p_\Pi + p_{\hat{\Pi}}) & -\alpha_\Theta(p_\Theta + p_{\hat{\Theta}}) & 0 \end{pmatrix},$$

and

$$Q_u^{1/2} \equiv \begin{pmatrix} 0 & 0 \\ b_{1,1} & b_{1,2} \\ b_{1,1}h_{\hat{\Pi},D} + b_{2,1}h_{\hat{\Pi},S} & b_{1,2}h_{\hat{\Pi},D} + b_{2,2}h_{\hat{\Pi},S} \\ b_{1,1}h_{\hat{\Theta},D} + b_{2,1}h_{\hat{\Theta},S} & b_{1,2}h_{\hat{\Theta},D} + b_{2,2}h_{\hat{\Theta},S} \\ \tilde{\sigma}_{P,D} & \tilde{\sigma}_{P,S} \end{pmatrix}.$$

Moreover,

$$dW_u(t) = (rW_u(t) - c_u(t))dt - \Psi_u(t)B_u Z_u(t)dt + \Psi_u(t)R^{1/2}d\tilde{w}(t), \quad t \geq 0, \tag{30}$$

where

$$B_u = \begin{pmatrix} 0 & \alpha_D p_D - 1 & \alpha_\Pi(p_\Pi + p_{\hat{\Pi}}) - p_D & \alpha_\Theta(p_\Theta + p_{\hat{\Theta}}) & r \end{pmatrix}, \qquad R_u^{1/2} = \begin{pmatrix} \tilde{\sigma}_{P,D} & \tilde{\sigma}_{P,S} \end{pmatrix}.$$

In the above setting, the uninformed investors' optimization problem is solved as follows.

Proposition 2 *The objective function (28) is given by*

$$V(t, Z_u, W_u) = -e^{-(\rho_u t + \frac{1}{2}Z_u^T G Z_u + r\varphi_u W_u + \gamma)}, \tag{31}$$

where $G \equiv (g_{j,k})_{j,k=1}^5$ *is a symmetric solution of the algebraic Riccati equation*

$$GU_uG - GV_u - V_u^T G - W_u = 0, \tag{32}$$

with coefficients

$$U_u \equiv Q_u^{1/2}\left(R_u I_j - \left(R_u^{1/2}\right)^T R_u^{1/2}\right)\left(Q_u^{1/2}\right)^T, \quad V_u \equiv Q_u^{1/2}\left(R_u^{1/2}\right)^T B_u + R_u\left(A_u - \tfrac{1}{2}rI_k\right), \quad W_u \equiv B_u^T B_u, \tag{33}$$

for $j = 2, k = 5$*, and* γ *is a real number satisfying*

$$r(1 + \gamma - \log(r)) - \rho_u - \frac{1}{2}\mathbf{tr}\left(\left(Q_u^{1/2}\right)^T G \left(Q_u^{1/2}\right)\right) = 0. \tag{34}$$

In addition, the uninformed investors' optimal demand for the stock and consumption are given by

$$\mathring{\Psi}_u(t) = -\frac{R_u^{1/2}\left(Q_u^{1/2}\right)^T G + B_u}{r\varphi_u R_u}\mathring{Z}_u(t) \tag{35}$$

and

$$\mathring{c}_u(t) = \frac{\frac{1}{2}\mathring{Z}_u^T(t)G\mathring{Z}_u(t) + r\varphi_u\mathring{W}_u(t) + \gamma - \ln(r)}{\varphi_u}, \tag{36}$$

respectively, where $(\mathring{Z}_u(t), \mathring{W}_u(t))$ *is the solution of (29), (30), corresponding to the choice of the optimal control* $(\mathring{\Psi}_u(t), \mathring{c}_u(t))$ *and the initial state of the random variables* D, $\hat{\Pi}$, $\hat{\Theta}$, P, W_u.

4 Informed Investors' Optimization Problem

Similarly to the uninformed investors, also the informed investors need to write Equation (7) in terms of the variables and noises of the economy they can observe. On the other hand, they hold complete information. Hence, with reference to Equation (13), they are in a position to replicate the uninformed investors' estimates and write the equations for $\hat{\Pi}(t)$ and $\hat{\Theta}(t)$. In the stationary case, by a straightforward computation, Equations (19), (20) yield

$$\begin{aligned} d\hat{\Pi}(t) &= (h_{\hat{\Pi},D} - \alpha_\Pi p_\Pi h_{\hat{\Pi},S})\Pi(t)dt - \alpha_\Theta p_\Theta h_{\hat{\Pi},S}\Theta(t)dt \\ &\quad - (h_{\hat{\Pi},D} + \alpha_\Pi(1 - p_\Pi h_{\hat{\Pi},S}))\hat{\Pi}(t)dt + \alpha_\Theta p_\Theta h_{\hat{\Pi},S}\hat{\Theta}(t)dt \\ &\quad + \sigma_{D,D}h_{\hat{\Pi},D}dw_D(t) + (\sigma_{D,\Pi}h_{\hat{\Pi},D} + \sigma_\Pi p_\Pi h_{\hat{\Pi},S})dw_\Pi(t) + \sigma_\Theta p_\Theta h_{\hat{\Pi},S}dw_\Theta(t), \quad t \geq 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} d\hat{\Theta}(t) &= (h_{\hat{\Theta},D} - \alpha_\Pi h_{\hat{\Theta},S}p_\Pi)\Pi(t)dt - \alpha_\Theta h_{\hat{\Theta},S}p_\Theta\Theta(t)dt \\ &\quad - (h_{\hat{\Theta},D} - h_{\hat{\Theta},S}\alpha_\Pi p_\Pi)\hat{\Pi}(t)dt - \alpha_\Theta(1 - h_{\hat{\Theta},S}p_\Theta)\hat{\Theta}(t)dt \\ &\quad + h_{\hat{\Theta},D}\sigma_{D,D}dw_D(t) + (h_{\hat{\Theta},D}\sigma_{D,\Pi} + h_{\hat{\Theta},S}\sigma_\Pi p_\Pi)dw_\Pi(t) + h_{\hat{\Theta},S}\sigma_\Theta p_\Theta dw_\Theta(t), \quad t \geq 0, \end{aligned} \tag{38}$$

In addition, combining Equations (13) and (16), we obtain

$$\hat{\Pi}(t) = \frac{pp_\Theta + p_D p_\Theta D(t) - p_\Theta P(t) + p_\Pi(p_\Theta + p_{\hat{\Theta}})\Pi(t) + p_\Theta(p_\Theta + p_{\hat{\Theta}})\Theta(t)}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}, \tag{39}$$

and

$$\hat{\Theta}(t) = \frac{-pp_\Pi - p_D p_\Pi D(t) + p_\Pi P(t) - p_\Pi(p_\Pi + p_{\hat{\Pi}})\Pi(t) - p_\Theta(p_\Pi + p_{\hat{\Pi}})\Theta(t)}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}. \tag{40}$$

Hence, replacing (39) and (40) into (37) and (38), the informed investors can write

$$\begin{aligned}
d\hat{\Pi}(t) = & -\frac{(h_{\hat{\Pi},D} + \alpha_\Pi - (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Pi},S}p_\Pi)pp_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}dt \\
& - \frac{(h_{\hat{\Pi},D} + \alpha_\Pi - (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Pi},S}p_\Pi)p_D p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}D(t)dt \\
& + \frac{(h_{\hat{\Pi},D} + \alpha_\Pi - (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Pi},S}p_\Pi)p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}P(t)dt \\
& - \alpha_\Pi - \frac{(h_{\hat{\Pi},D} + \alpha_\Pi - (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Pi},S}p_\Pi)(p_\Pi + p_{\hat{\Pi}})p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}\Pi(t)dt \\
& - \frac{(h_{\hat{\Pi},D} + \alpha_\Pi - (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Pi},S}p_\Pi)(p_\Theta + p_{\hat{\Theta}})p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}\Theta(t)dt \\
& + h_{\hat{\Pi},D}\sigma_{D,D}dw_D(t) + (h_{\hat{\Pi},D}\sigma_{D,\Pi} + h_{\hat{\Pi},S}\sigma_\Pi p_\Pi)dw_\Pi(t) + h_{\hat{\Pi},S}\sigma_\Theta p_\Theta dw_\Theta(t), \quad t \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
d\hat{\Theta}(t) = & \frac{(\alpha_\Theta p_\Pi - h_{\hat{\Theta},D}p_\Theta + (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Theta},S}p_\Pi p_\Theta)p}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}dt \\
& + \frac{(\alpha_\Theta p_\Pi - h_{\hat{\Theta},D}p_\Theta + (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Theta},S}p_\Pi p_\Theta)p_D}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}D(t)dt \\
& - \frac{\alpha_\Theta p_\Pi - h_{\hat{\Theta},D}p_\Theta + (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Theta},S}p_\Pi p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}P(t)dt \\
& + \frac{(\alpha_\Theta p_\Pi - h_{\hat{\Theta},D}p_\Theta + (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Theta},S}p_\Pi p_\Theta)(p_\Pi + p_{\hat{\Pi}})}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}\Pi(t)dt \\
& - \alpha_\Theta + \frac{(\alpha_\Theta p_\Pi - h_{\hat{\Theta},D}p_\Theta + (\alpha_\Pi - \alpha_\Theta)h_{\hat{\Theta},S}p_\Pi p_\Theta)(p_\Theta + p_{\hat{\Theta}})}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}\Theta(t)dt \\
& + h_{\hat{\Theta},D}\sigma_{D,D}dw_D(t) + (h_{\hat{\Theta},D}\sigma_{D,\Pi} + h_{\hat{\Theta},S}\sigma_\Pi p_\Pi)dw_\Pi(t) + h_{\hat{\Theta},S}\sigma_\Theta p_\Theta dw_\Theta(t), \quad t \geq 0.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
dP(t) = & -kpdt - (\alpha_D + k)\, p_D D(t)dt - \big((\alpha_\Pi + k)\,(p_\Pi + p_{\hat{\Pi}}) - p_D\big)\,\Pi(t)dt \\
& - (\alpha_\Theta + k)\,(p_\Theta + p_{\hat{\Theta}})\Theta(t)dt + kP(t)dt \\
& + \sigma_{P,D}dw_D(t) + \sigma_{P,\Pi}dw_\Pi(t) + \sigma_{P,\Theta}dw_\Theta(t), \quad t \geq 0,
\end{aligned} \tag{41}$$

and

$$dQ(t) = kpdt + ((\alpha_D + k)\, p_D - 1)D(t)dt + \big((\alpha_\Pi + k)\,(p_\Pi + p_{\hat{\Pi}}) - p_D\big)\,\Pi(t)dt \tag{42}$$
$$+ (\alpha_\Theta + k)\,(p_\Theta + p_{\hat{\Theta}})\Theta(t)dt + (r - k)P(t)dt$$
$$+ \sigma_{P,D}dw_D(t) + \sigma_{P,\Pi}dw_\Pi(t) + \sigma_{P,\Theta}dw_\Theta(t), \quad t \geq 0,$$

where

$$\sigma_{P,D} \equiv (p_D + h_{\hat{\Pi},D}p_{\hat{\Pi}} + h_{\hat{\Theta},D}p_{\hat{\Theta}})\sigma_{D,D},$$
$$\sigma_{P,\Pi} \equiv ((p_D + h_{\hat{\Pi},D}p_{\hat{\Pi}} + h_{\hat{\Theta},D}p_{\hat{\Theta}})\sigma_{D,\Pi} + (1 + h_{\hat{\Pi},S}p_{\hat{\Pi}} + h_{\hat{\Theta},S}p_{\hat{\Theta}})p_\Pi\sigma_\Pi),$$
$$\sigma_{P,\Theta} \equiv (1 + h_{\hat{\Pi},S}p_{\hat{\Pi}} + h_{\hat{\Theta},S}p_{\hat{\Theta}})p_\Theta\sigma_\Theta,$$

and

$$k \equiv \frac{\alpha_\Pi p_\Theta p_{\hat{\Pi}} - \alpha_\Theta p_\Pi p_{\hat{\Theta}} + (h_{\hat{\Pi},D}p_{\hat{\Pi}} + h_{\hat{\Theta},D}p_{\hat{\Theta}})p_\Theta - (\alpha_\Pi - \alpha_\Theta)(h_{\hat{\Pi},S}p_{\hat{\Pi}} + h_{\hat{\Theta},S}p_{\hat{\Theta}})p_\Pi p_\Theta}{p_\Pi p_{\hat{\Theta}} - p_\Theta p_{\hat{\Pi}}}.$$

Hence, setting $Z_i(t) \equiv (1, D(t), \Pi(t), \Theta(t), P(t))^T$, on account of (41) and (42), it is not difficult to check that the informed investors' optimization problem is subject to the state equation

$$dZ_i(t) = A_i Z_i(t)dt + Q_i^{1/2}dw(t), \quad t \geq 0, \tag{43}$$

and to the wealth equation

$$dW_i(t) = (rW_i(t) - c_i(t))\,dt - \Psi_i(t)B_iZ_i(t) + \Psi_i(t)R_i^{1/2}dw(t), \quad t \geq 0, \tag{44}$$

where

$$A_i \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha_D & 1 & 0 & 0 \\ 0 & 0 & -\alpha_\Pi & 0 & 0 \\ 0 & 0 & 0 & -\alpha_\Theta & 0 \\ -kp & -(\alpha_D + k)p_D & -((\alpha_\Pi + k)(p_\Pi + p_{\hat{\Pi}}) - p_D) & -(\alpha_\Theta + k)\,(p_\Theta + p_{\hat{\Theta}}) & k \end{pmatrix},$$

$$Q_i^{1/2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \sigma_{D,D} & \sigma_{D,\Pi} & 0 \\ 0 & \sigma_\Pi & 0 \\ 0 & 0 & \sigma_\Theta \\ \sigma_{P,D} & \sigma_{P,\Pi} & \sigma_{P,\Theta} \end{pmatrix},$$

$$B_i = \begin{pmatrix} kp & (\alpha_D + k)\, p_D - 1 & (\alpha_\Pi + k)\,(p_\Pi + p_{\hat{\Pi}}) - p_D & (\alpha_\Theta + k)\,(p_\Theta + p_{\hat{\Theta}}) & r - k \end{pmatrix},$$

and

$$R_i^{1/2} \equiv \begin{pmatrix} \sigma_{P,D} & \sigma_{P,\Pi} & \sigma_{P,\Theta} \end{pmatrix}.$$

Now, since the process $(Z_i(t), W_i(t))$ is Markov, the informed investors' objective function(5) can be rewritten as

$$\max_{\Psi_i(\cdot), c_i(\cdot)} \left\{ \mathbf{E}_{t,D,\Pi,\Theta,P,W_i} \left[\int_t^{+\infty} -e^{-(\rho_i s + \varphi_i c_i(s))}\, ds \right] \right\}, \quad t \geq 0, \tag{45}$$

subject to (43) and (44), where $\mathbf{E}_{t,D,\Pi,\Theta,P,W_i}[\cdot]$ is the conditional expectation operator given the state of the random variables D, Π, Θ, P, W_i at time t. We then have

Proposition 3 *The objective function (45) is given by*

$$V(t, Z_i, W_i) = -e^{-(\rho_i t + \frac{1}{2} Z_i^T L Z_i + r\varphi_i W_i + \lambda)}, \tag{46}$$

where $L \equiv (\ell_{j,k})_{j,k=1}^5$ is a symmetric solution of the algebraic Riccati equation

$$L U_i L - L V_i - V_i^T L - W_i = 0, \tag{47}$$

with coefficients

$$U_i \equiv Q_i^{1/2} \left(R_i I_j - \left(R_i^{1/2}\right)^T R_i^{1/2} \right) \left(Q_i^{1/2}\right)^T, \quad V_i \equiv Q_i^{1/2} \left(R_i^{1/2}\right)^T B_i + R_i \left(A_i - \frac{1}{2} r I_k \right), \quad W_i \equiv B_i^T B_i, \tag{48}$$

for $j = 3, k = 5$, and λ is a real number satisfying

$$r(1 + \lambda - \log(r)) - \rho_i - \frac{1}{2} \mathbf{tr} \left(\left(Q_i^{1/2}\right)^T L \left(Q_i^{1/2}\right) \right) = 0. \tag{49}$$

In addition, the informed investors' optimal demand for the risky asset and their optimal consumption *are given by*

$$\mathring{\Psi}_i(t) = -\frac{R_i^{1/2}\left(Q_i^{1/2}\right)^T L + B_i}{r\varphi_i R_i}\mathring{Z}_i(t) \tag{50}$$

and

$$\mathring{c}_i(t) = \frac{\frac{1}{2}\mathring{Z}_i^T(t)L\mathring{Z}_i(t) + r\varphi_i\mathring{W}_i(t) + \lambda - \ln(r)}{\varphi_i}, \tag{51}$$

respectively, where $(\mathring{Z}_i(t), \mathring{W}_i(t))$ *is the solution of (43) and (44), corresponding to the choice of the optimal control* $(\mathring{\Psi}_i(t), \mathring{c}_i(t))$ *and the initial state of the random variables* D, Π, Θ, P, W_i.

5 Bayesian Nash Stationary Equilibria

As already discussed, a Bayesian-Nash stationary equilibrium of the model is achieved by solving the investors' optimization problem, under the conditions that the informed investors' optimal demand for the risky asset coincides with the one postulated by the uninformed investors and, conversely, the uninformed investors' optimal demand coincides with the one postulated by the informed investors. On account of (35) and (50), the equilibrium condition leads to the set of equations

$$\begin{pmatrix} \psi_u & \psi_{u,D} & \psi_{u,\hat{\Pi}} & \psi_{u,\hat{\Theta}} & -\psi_{u,P} \end{pmatrix} = -\frac{R_u^{1/2}\left(Q_u^{1/2}\right)^T G + B_u}{r\varphi_u R_u}, \tag{52}$$

and

$$\begin{pmatrix} \psi_i & \psi_{i,D} & \psi_{i,\Pi} & \psi_{i,\Theta} & -\psi_{i,P} \end{pmatrix} = -\frac{R_i^{1/2}\left(Q_i^{1/2}\right)^T L + B_i}{r\varphi_i R_i}. \tag{53}$$

These have to be coupled with Equations (32) and (47), yielding the matrices G and L respectively, with the equations for the price coefficients, derived from (11) and (14), and finally with the equations characterizing the filtering procedure given by (21), (10), (24), and (26). The coupling produces a large number of nonlinear equations yielding the equilibria of the model, which we could manage only by numerical procedures. More specifically, in Wolfram Mathematica®environment, we have written two procedures, both leading to the same results, which tackle the achievement of the equilibria in two different approaches: a straightforward one of Walrasian flavor and an iterative one of strategic flavor. In the first approach, numerical solutions to the system of the equation characterizing the equilibrium are found simultaneously, in the Walrasian spirit of the aggregation of the optimal demands of the two different types of traders. In the second approach, a group of investors, let us say the informed ones, postulate a demand for the risky asset of the investors of the other group[8]. On the basis of this demand, they compute their optimal response and submit it to their competitors. In turn, on the basis of the submitted demand, the latter compute their best response and submit back to former. The procedure stops when no further meaningful modifications of the submitted demands occur[9] or the number of the iterations exceeds a prescribed maximum[10]. Somewhat pleasantly, hardly ever the latter circumstance occurs and such a bargaining procedure almost always allows to achieve the equilibrium.

The exploited procedures produce multiple mathematical equilibria, each of which characterized by a quintuple $(P, \Psi_i, V_i, \Psi_u, V_u)$ whose entries consist of the market price of the risky asset and the optimal strategy and value function of the investors of the two groups. We will tackle the problem of their selection and ranking in terms of their economical meaning in Section 9.

6 Benchmark Case: Wang's REE Approach

Wang's results constitute a crucial benchmark to test ours. Moreover, we got also interested in exploring the consequences of both a "high" market risk perception and different investors' risk aversion that have not been addressed by Wang. This has led us to exploit Wang's model in all details. Hence, for the convenience of the reader, in this section we give a brief overview of Wang's approach (see [35, Theorem 3.1 p. 254 and Proposition p. 256] (1993)).

Wang assumes that the putative equilibrium price of the risky asset has the functional form[11]

$$P(t) = p^W + p_D^W D(t) + p_\Pi^W \Pi(t) + p_\Theta^W \Theta(t) + p_\Delta^W \Delta(t), \tag{54}$$

where $p_D^W \equiv \frac{1}{r+\alpha_D}$[12], $p_\Pi^W \equiv \frac{1}{(r+\alpha_D)(r+\alpha_\Pi)}$, the coefficients p^W, p_Θ^W, and p_Δ^W are misspecified, and $\Delta(t) \equiv \hat{\Pi}(t) - \Pi(t)$ is the uninformed investors' estimation error of the informed investors' private information. Note that, since the informed risk-neutral price of the risky asset is given by

$$\mathbf{E}\left[\int_t^{+\infty} e^{-rs} D(s)\, ds | \mathfrak{F}_1(t)\right] = \frac{1}{r+\alpha_D} D(t) + \frac{1}{(r+\alpha_D)(r+\alpha_\Pi)} \Pi(t), \tag{55}$$

[8] Our procedure is initialized by a random demand or a random perturbation of Wang's equilibrium demand (see Section 9)

[9] A modification is declared to be non meaningful if it is smaller than 10^{-6}.

[10] The maximum number of allowed iterations is set to 100.

[11] Except for the suppression of the constant additive term ϕ, which depends on a minor difference in the structure of Wang's informative process $\Pi(t)$ and plays no role in the following analysis.

[12] In Wang's notation $\alpha_D \equiv k$.

where $\mathfrak{F}(t) = \sigma(D(s), \Pi(s); \quad s \leq t)$, we can rewrite (54) as

$$P(t) = p^W + \mathbf{E}\left[\int_t^{+\infty} D(s)ds|\mathfrak{F}(t)\right] + p_\Theta^W \Theta(t) + p_\Delta^W \Delta(t).$$

Hence, Wang's price turns out to be a linear perturbation of the informed risk-neutral price accounting for market noise, investors' risk aversion, and information asymmetry[13].

Now, by applying the operator $\mathbf{E}[\cdot|\mathfrak{F}_U(t)]$ to both sides of (54), from the uninformed investors' point of view, the putative price of the risky asset has the functional form

$$P(t) = p^W + p_D^W D(t) + p_\Pi^W \hat{\Pi}(t) + p_\Theta^W \hat{\Theta}(t), \tag{56}$$

where the variables $\hat{\Pi}(t)$ and $\hat{\Theta}(t)$ are still driven by System (19), because in our approach the uninformed investors' information has the same structure as Wang's one. Hence, on account of (56), (27), and (19), the instantaneous excess return to one share of the risky asset satisfies the equation

$$dQ(t) = -rp^W dt - (r + \alpha_\Theta) p_\Theta^W \hat{\Theta}(t) dt + \tilde{\sigma}_{W,Q,D} d\tilde{w}_{Q,D}(t) + \tilde{\sigma}_{W,Q,S} d\tilde{w}_S(t), \quad t \geq 0, \tag{57}$$

where

$$\begin{aligned}
\tilde{\sigma}_{W,Q,D} &\equiv b_{1,1}(p_D^W + p_\Pi^W h_{\hat{\Pi},D} + p_\Theta^W h_{\hat{\Theta},D}) + b_{2,1}(p_\Pi^W h_{\hat{\Pi},S} + p_\Theta^W h_{\hat{\Theta},S}), \\
\tilde{\sigma}_{W,Q,S} &\equiv b_{1,2}(p_D^W + p_\Pi^W h_{\hat{\Pi},D} + p_\Theta^W h_{\hat{\Theta},D}) + b_{2,2}(p_\Pi^W h_{\hat{\Pi},S} + p_\Theta^W h_{\hat{\Theta},S}).
\end{aligned}$$

The structure of (57) leads to introduce the vector variable $Z_u(t) \equiv (1, \hat{\Theta}(t))^T$ and the matrices

$$A_u \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_\Theta \end{pmatrix}, \quad Q_u^{1/2} \equiv \begin{pmatrix} 0 & 0 \\ b_{1,1}h_{\hat{\Theta},D} + b_{2,1}h_{\hat{\Theta},S} & b_{1,2}h_{\hat{\Theta},D} + b_{2,2}h_{\hat{\Theta},S} \end{pmatrix},$$

$$B_u \equiv \begin{pmatrix} rp^W & (r + \alpha_\Theta)p_\Theta^W \end{pmatrix}, \quad R_u^{1/2} \equiv \begin{pmatrix} \tilde{\sigma}_{W,Q,D} & \tilde{\sigma}_{W,Q,S} \end{pmatrix},$$

by which to rewrite the state equation (29) and the wealth equation (30). As a consequence, the optimization procedure still yields the informed investors' objective function in the form (31), where $G \equiv (g_{j,k})_{j,k=1}^3$ is a symmetric solution of (32) with coefficients given by (33), for $j = k = 2$, and γ

[13]Wang's price generalizes the price proposed by Campbell & Kyle in the celebrate paper [5] (1993)), which addresses only market noise and investors' risk aversion

satisfies (34). Moreover, the uninformed investors' optimal demand for the stock and consumption are given by (35) and (36). On the other hand, since $\Delta(t)$ is driven by

$$d\Delta(t) = -\alpha_\Delta \Delta(t)dt + \sigma_{\Delta,D} dw_D(t) + \sigma_{\Delta,\Pi} dw_\Pi(t) + \sigma_{\Delta,\Theta} dw_\Theta(t), \quad t \geq 0,$$

where

$$\alpha_\Delta \equiv \alpha_\Pi + h_{\hat{\Pi},D} - (\alpha_\Pi - \alpha_\Theta)(p_\Pi^W - p_\Delta^W) h_{\hat{\Pi},S},$$

and

$$\sigma_{\Delta,D} \equiv \sigma_{D,D} h_{\hat{\Pi},D}, \qquad \sigma_{\Delta,\Pi} \equiv (\sigma_{D,\Pi} h_{\hat{\Pi},D} + \sigma_\Pi((p_\Pi^W - p_\Delta^W) h_{\hat{\Pi},S} - 1)), \qquad \sigma_{\Delta,\Theta} \equiv \sigma_\Theta p_\Theta^W h_{\hat{\Pi},S}.$$

Therefore, from the informed investors' point of view, the instantaneous excess return to one share of the risky asset satisfies the equation

$$\begin{aligned} dQ(t) = &-rp^W\, dt - (r + \alpha_\Theta) p_\Theta^W \Theta(t)\, dt - (r + \alpha_\Delta) p_\Delta^W \Delta(t)\, dt \\ &+ \sigma_{Q,D} dw_D(t) + \sigma_{Q,\Pi} dw_\Pi(t) + \sigma_{Q,\Theta} dw_\Theta(t), \quad t \geq 0, \end{aligned} \tag{58}$$

where

$$\sigma_{Q,D} \equiv \sigma_{D,D} p_D^W + \sigma_{\Delta,D} p_\Delta^W, \qquad \sigma_{Q,\Pi} \equiv \sigma_\Pi p_\Pi^W + \sigma_{\Delta,\Pi} p_\Delta^W, \qquad \sigma_{Q,\Theta} \equiv \sigma_\Theta p_\Theta^W + \sigma_{\Delta,\Theta} p_\Delta^W.$$

The structure of (58) leads to introduce the vector variable $Z_i(t) \equiv (1, \Theta(t), \Delta(t))^T$ and the matrices

$$A_i \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha_\Theta & 0 \\ 0 & 0 & -\alpha_\Delta \end{pmatrix}, \quad Q_i^{1/2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_\Theta \\ \sigma_{\Delta,D} & \sigma_{\Delta,\Pi} & \sigma_{\Delta,\Theta} \end{pmatrix},$$

$$B_i \equiv \begin{pmatrix} rp^W & (r+\alpha_\Theta) p_\Theta^W & (r+\alpha_\Delta) p_\Delta^W \end{pmatrix}, \qquad R_{W,i}^{1/2} \equiv \begin{pmatrix} \sigma_{Q,D} & \sigma_{Q,\Pi} & \sigma_{Q,\Theta} \end{pmatrix},$$

by which to rewrite the state equation (43) and the wealth equation (44). As a consequence, the optimization procedure still yields the informed investors' objective function in the form (46), where $L \equiv (\ell_{j,k})_{j,k=1}^3$ is a symmetric solution of (47) with coefficients given by (48), for $j = k = 3$, and λ satisfies (49). In addition, the informed investors' optimal demand for the stock and the optimal consumption are given by (50) and (51).

Finally, setting

$$\mathring{\Psi}_i(t) \equiv (\psi_i^W, \psi_{i,\Theta}^W, \psi_{i,\Delta}^W)\mathring{Z}_i(t), \qquad \mathring{\Psi}_u(t) \equiv (\psi_u^W, \psi_{u,\hat{\Theta}}^W)\mathring{Z}_{u,W}(t),$$

the market clearing condition (4) yields

$$\begin{aligned}(1-\omega)\psi_i^W + \omega\psi_u^W - 1 &= 0\\ (1-\omega)\psi_{i,\Theta}^W + \omega\psi_{u,\hat{\Theta}}^W - 1 &= 0\\ (1-\omega)\psi_{i,\Delta}^W - \omega(p_\Theta^W)^{-1}(p_\Pi^W - p_\Delta^W)\psi_{u,\hat{\Theta}}^W &= 0.\end{aligned}$$

7 Degenerate Cases

In his paper, Wang highlights the cases in which the rational investors are either all informed or all uninformed, corresponding to the values 0 and 1 of the parameter ω, respectively. These cases, characterized by the lack of competition between asymmetrically informed rational investors, support Wang's analysis of the general case and are also interesting for our approach because they provide the early insight on existence of equilibria other than Wang's RE ones. Nevertheless, it seems to us worth noting that these cases constitute a degeneration of the model. Indeed, both in Wang's approach and in ours, the lack of competition prevents any choice of the rational investors' equilibrium demand for the risky asset. In equilibrium, the market clearing condition constrains the rational investors' demand for the stock to respond to the stochastic supply. Hence, the rational investors' optimization problem degenerates and should be more appropriately interpreted as the determination of the risky asset price, which makes the rational investors' equilibrium demand for the stock optimal. As a consequence, when the rational investors are all informed, we assume that the putative equilibrium price of the risky asset takes the form

$$P(t) = p + p_D D(t) + p_\Pi \Pi(t) + p_\Theta \Theta(t), \tag{59}$$

where the coefficients p, p_D, p_Π, p_Θ are all misspecified. The equation for the instantaneous excess return to one share of stock is then given by

$$\begin{aligned}dQ(t) = {}& -rpdt - ((r+\alpha_D)p_D - 1)D(t)dt - ((r+\alpha_\Pi)p_\Pi - p_D)\Pi(t)dt - (r+\alpha_\Theta)p_\Theta\Theta(t)dt \qquad (60)\\ &+ \sigma_{D,D}p_D dw_D(t) + (\sigma_{D,\Pi}p_D + \sigma_\Pi p_\Pi)dw_\Pi(t) + \sigma_\Theta p_\Theta dw_\Theta(t).\end{aligned}$$

Therefore, setting $Z_i(t) \equiv (1, D(t), \Pi(t), \Theta(t))$, and introducing the matrices

$$A_i \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha_D & 1 & 0 \\ 0 & 0 & -\alpha_\Pi & 0 \\ 0 & 0 & 0 & -\alpha_\Theta \end{pmatrix}, \quad Q_i^{1/2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ \sigma_{D,D} & \sigma_{D,\Pi} & 0 \\ 0 & \sigma_\Pi & 0 \\ 0 & 0 & \sigma_\Theta \end{pmatrix}.$$

$$B_i = \begin{pmatrix} rp & -(1-(r+\alpha_D)p_D) & -(p_D-(r+\alpha_\Pi)p_\Pi) & (r+\alpha_\Theta)p_\Theta \end{pmatrix},$$

$$R_i^{1/2} = \begin{pmatrix} \sigma_{D,D}p_D & \sigma_{D,\Pi}p_D + \sigma_\Pi p_\Pi & \sigma_\Theta p_\Theta \end{pmatrix},$$

we are again in a position to rewrite the state equation (43) and the wealth equation (44). In turn, the uninformed investors' value function takes again the form (46), where $L \equiv (\ell_{j,k})_{j,k=1}^4$ is still a symmetric solution of the Riccati equation (47), with coefficients given by (48), for $j = 2$, $k = 4$ and λ is still a real number satisfying (49). In addition, the informed investors' optimal demand for the stock and the optimal consumption still fulfill (50) and (51), repsctively.

Now, the constraint

$$\mathring{\Psi}_i(t) = 1 + \Theta(t), \tag{61}$$

implies that the coefficients of $\mathring{\Psi}_i(t)$ must satisfy the equalities

$$\psi_i = 1, \quad \psi_{i,D} = 0, \quad \psi_{i,\Pi} = 0, \quad \psi_{i,\Theta} = 1. \tag{62}$$

Combining Equation (**??**) with (47), on account of (62), we obtain a non-linear system yielding p, p_D, p_Π, p_Θ and the entries $\ell_{j,k},\ j,k = 1,\dots,4$. The solution of this system requires a numerical approach. The results will be presented in Section 9.

Dealing with the case in which the rational investors are all informed, Wang's REE approach turns out to be much simpler and can be analytically pursued much further. Indeed, Wang assumes the putative equilibrium price has the functional form[14]

$$P(t) = p^W + p_D^W D(t) + p_\Pi^W \Pi(t) + p_\Theta^W \Theta(t), \tag{63}$$

[14] For sake of simplicity, also in this case we suppress Wang's constant term ϕ.

where $p_D^W = \frac{1}{r+\alpha_D}$, $p_\Pi^W = \frac{1}{(r+\alpha_D)(r+\alpha_\Pi)}$, and p^W and p_Θ^W are misspecified[15] (see Wang [35, Sez. 3, p. 254] (1993)). As a consequence, the instantaneous excess return to one share of stock satisfies

$$dQ(t) = -rp^W - (r+\alpha_\Theta)p_\Theta^W \Theta(t)dt + \frac{\sigma_{D,D}}{r+\alpha_D}dw_D(t) + \frac{(r+\alpha_D)\sigma_{D,\Pi}+\sigma_\Pi}{(r+\alpha_D)(r+\alpha_\Pi)}dw_\Pi(t) + \sigma_\Theta p_\Theta^W dw_\Theta(t). \tag{64}$$

Now, the structure of Equation (64) allows to tackle the informed investors optimization problem by exploiting the simple variable $Z_i(t) \equiv (1, \Theta(t))$ and the matrices

$$A_i \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_\Theta \end{pmatrix}, \quad Q_i^{1/2} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_\Theta \end{pmatrix},$$

$$B_i \equiv \begin{pmatrix} rp^W & (r+\alpha_\Theta)p_\Theta^W \end{pmatrix}, \quad R_i^{1/2} \equiv \begin{pmatrix} \frac{\sigma_{D,D}}{r+\alpha_D} & \frac{(r+\alpha_D)\sigma_{D,\Pi}+\sigma_\Pi}{(r+\alpha_D)(r+\alpha_\Pi)} & \sigma_\Theta p_\Theta^W \end{pmatrix}.$$

Once more, we obtain the informed investors' objective function in the form (46), where $L \equiv (\ell_{j,k})_{j,k=1}^2$ is a symmetric solution of the Riccati equation (47), with coefficients given by (48), for $j = k = 2$, and λ is a real solution of (49). The informed investors' optimal demand for the stock and their optimal consumption are given by (50) and (51). Setting $\mathring{\Psi}_i(t) \equiv (\psi_i^W, \psi_{i,\Theta}^W)\mathring{Z}_i(t)$, it then follows

$$\psi_i^W = -\frac{(r+\alpha_D)^2(r+\alpha_\Pi)^2(rp+\sigma_\Theta^2 p_\Theta^W \ell_{1,2})}{r\varphi_i((r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2(p_\Theta^W)^2 + (\sigma_\Pi+(r+\alpha_D)\sigma_{D,\Pi})^2 + (r+\alpha_\Pi)^2(\sigma_D^2-\sigma_{D,\Pi}^2))},$$

$$\psi_{i,\Theta}^W = -\frac{(r+\alpha_D)^2(r+\alpha_\Pi)^2(r+\alpha_\Theta+\sigma_\Theta^2\ell_{2,2})p_\Theta^W}{r\varphi_i((r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2(p_\Theta^W)^2 + (\sigma_\Pi+(r+\alpha_D)\sigma_{D,\Pi})^2 + (r+\alpha_\Pi)^2(\sigma_D^2-\sigma_{D,\Pi}^2))},$$

$$\psi_i^W = \frac{rp^W + \sigma_\Theta^2 p_\Theta^W \ell_{1,2}}{r\left(p^W - \varphi_i\sigma_\Theta^2(p_\Theta^W)^2\right)}, \quad \psi_{i,\Theta}^W = \frac{\left(r+\alpha_\Theta+\sigma_\Theta^2\ell_{2,2}\right)p_\Theta^W}{r\left(p^W - \varphi_i\sigma_\Theta^2(p_\Theta^W)^2\right)}. \tag{65}$$

From a technical point of view, rather than solving Equation (47) thereafter imposing the market clearing condition (see Wang [35, Sez. 4.5, p. 259] (1993)), we impose first the market clearing condition to the entries of the informed investors' optimal demand for the risky asset, obtaining

$$\ell_{1,2} = -r\left(\frac{p^W}{\sigma_\Theta^2 p_\Theta^W} + \varphi_i p_\Theta^W + \frac{\varphi_i((r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi+(\alpha_D-\alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi+(2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2 p_\Theta^W}\right) \tag{66}$$

[15] Actually, Wang considers only the particular case $\varphi_i = 1$, $\sigma_{D,\Pi} = 0$ and, following Campbell & Kyle [5] (1993), sets $p^W = -\frac{(r+\alpha_\Pi)^2\sigma_D^2+\sigma_\Pi^2}{(r+\alpha_D)^2(r+\alpha_\Pi)^2}$. However, still in Wang's approach, we derive the form of p^W in the general case.

$$\ell_{2,2} = -\left(\frac{r+\alpha_\Theta}{\sigma_\Theta^2} + r\varphi_i p_\Theta^W + \frac{r\varphi_i((r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D - \alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi}))}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2 p_\Theta^W}\right) \tag{67}$$

$$\ell_{1,2} = -r\varphi_i p_\Theta^W, \qquad \ell_{2,2} = \frac{rp^W - p_\Theta^W(r+\alpha_\Theta + r\varphi_i p_\Theta^W \sigma_\Theta^2)}{p_\Theta^W \sigma_\Theta^2}. \tag{68}$$

Then, substituting (66) and (67) in Equation (47), it follows

$$p^W = -\varphi_i \frac{(r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D - \alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})}{(r+\alpha_D)^2(r+\alpha_\Pi)^2}, \tag{69}$$

$$\ell_{1,1} = -r\varphi_i p^W, \tag{70}$$

and the following equation for p_Θ^W,

$$P_i(p_\Theta^W)Q_i(p_\Theta^W) = 0, \tag{71}$$

where

$$P_i(p_\Theta^W) = (r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2(p_\Theta^W)^2 + (r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D - \alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})$$

and

$$\begin{aligned}
Q_i(p_\Theta^W) \equiv\ & \varphi_i r^2 (r+\alpha_D)^4 (r+\alpha_\Pi)^4 \sigma_\Theta^2 (p_\Theta^W)^3 \\
& - (r+\alpha_D)^2(r+\alpha_\Pi)^2 \big((r+\alpha_\Theta)\alpha_\Theta(r+\alpha_D)^2(r+\alpha_\Pi)^2 \\
& - r^2\varphi_i^2\sigma_\Theta^2 \left((r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D-\alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})\right)\big) (p_\Theta^W)^2 \\
& + \varphi_i r^2 (r+\alpha_D)^2(r+\alpha_\Pi)^2 \left((r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D-\alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})\right) p_\Theta^W \\
& + \varphi_i^2 r^2 \left((r+\alpha_\Pi)^2\sigma_D^2 + (\sigma_\Pi + (\alpha_D-\alpha_\Pi)\sigma_{D,\Pi})(\sigma_\Pi + (2r+\alpha_D+\alpha_\Pi)\sigma_{D,\Pi})\right)^2.
\end{aligned}$$

Equation (71) provides numerical values of p_Θ^W, depending on the choice of the exogenous parameters.

To deal with the degenerate case in which all the rational investors are uninformed, we begin with observing that, in equilibrium, the absence of informed traders clearly allows the uninformed ones to infer the exact value of the current supply shock $\Theta(t)$ from the market clearing condition

$$\Psi_u(t) = 1 + \Theta(t). \tag{72}$$

Quoting Wang: "the fact that nobody knows anything enables everybody to know something". As a consequence, in terms of the variables which are observed by the investors, the putative equilibrium price takes the form

$$P(t) = p + p_D D(t) + p_{\hat{\Pi}}\hat{\Pi}(t) + p_\Theta \Theta(t). \tag{73}$$

Likewise Section 3, to discover the risky asset price which makes the informed investors' equilibrium demand for the risky asset optimal, it is still necessary to determine the equation for the dynamics of the uniformed investors estimate $\hat{\Pi}(t)$ of the unobserved $\Pi(t)$, given the observed process $D(t)$. In the stationary case, a slight modification of the arguments presented in the proof of Proposition 1 allows us to deduce that

$$d\hat{\Pi}(t) = -\alpha_\Pi \hat{\Pi}(t)dt + \frac{1}{\sigma_D}(\sigma_{D,\Pi}\sigma_\Pi + \sigma)d\tilde{w}_D(t), \tag{74}$$

where

$$\sigma = -(\alpha_\Pi \sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi) + \sqrt{(\alpha_\Pi \sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi)^2 + \sigma_{D,D}^2\sigma_\Pi^2},$$

and

$$d\tilde{w}_D(t) = \frac{1}{\sigma_D}(dD(t) - (\hat{\Pi}(t) - \alpha_D D(t))dt),$$

which in turn implies

$$dD(t) = -\alpha_D D(t)dt + \hat{\Pi}(t)dt + \sigma_D d\tilde{w}_D(t). \tag{75}$$

From (73) and (74), on account of (75), it then follows

$$\begin{aligned} dQ(t) = {} & -rp\,dt + (1-(r+\alpha_D)p_D)D(t)dt + (p_D - (r+\alpha_\Pi)p_{\hat{\Pi}})\hat{\Pi}(t)dt - (r+\alpha_\Theta)p_\Theta\Theta(t)dt. \\ & + \frac{1}{\sigma_D}\left(\sigma_D^2 p_D + (\sigma_{D,\Pi}\sigma_\Pi + \sigma)p_{\hat{\Pi}}\right) d\tilde{w}_D(t) + \sigma_\Theta p_\Theta dw_\Theta(t) \end{aligned} \tag{76}$$

Therefore, setting $Z_u(t) \equiv (1, D(t), \hat{\Pi}(t), \Theta(t))^T$, $w(t) \equiv (\tilde{w}_D(t), w_\Theta(t))^T$, and introducing the matrices

$$A_u \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\alpha_D & 1 & 0 \\ 0 & 0 & -\alpha_\Pi & 0 \\ 0 & 0 & 0 & -\alpha_\Theta \end{pmatrix}, \quad Q_u^{1/2} \equiv \begin{pmatrix} 0 & 0 \\ \sigma_D & 0 \\ \frac{1}{\sigma_D}(\sigma_{D,\Pi}\sigma_\Pi + \sigma) & 0 \\ 0 & \sigma_\Theta \end{pmatrix},$$

$$B_u = \begin{pmatrix} rp & -(1-(r+\alpha_D)p_D) & -(p_D - (r+\alpha_\Pi)p_{\hat{\Pi}}) & (r+\alpha_\Theta)p_\Theta \end{pmatrix},$$

$$R_u^{1/2} \equiv \begin{pmatrix} \frac{1}{\sigma_D}\left(\sigma_D^2 p_D + (\sigma_{D,\Pi}\sigma_\Pi + \sigma)p_{\hat{\Pi}}\right) & \sigma_\Theta p_\Theta \end{pmatrix}$$

we obtain the uninformed investors' value function in the form (31), where $G \equiv (g_{j,k})_{j,k=1}^4$ is a symmetric solution of the Riccati equation (32), with coefficients given by (33), for $j = 2$, $k = 4$, and γ is a real solution of (34). The uninformed investors' optimal demand for the stock and their optimal consumption are given by (35) and (36). Therefore, setting $\mathring{\Psi}_u(t) \equiv (\psi_u, \psi_D, \psi_{u,\hat{\Pi}}, \psi_{u,\Theta})\mathring{Z}_u(t)$, the constraint (72) implies that the coefficients of the entries of $\mathring{\Psi}_u(t)$ must satisfy the equalities

$$\psi_u = 1, \quad \psi_{u,D} = 0, \quad \psi_{u,\hat{\Pi}} = 0, \quad \psi_{u,\Theta} = 1. \tag{77}$$

Combining Equations (35) and (32), on account (77), we obtain a non-linear system yielding $p, p_D, p_{\hat{\Pi}}, p_\Theta$ and the entries $g_{j,k}$, $j,k = 1,\dots,4$. The solution of this system, requiring again a numerical approach, will be presented in Section 9.

Wang's approach turns out to be simpler and can be analytically pursued further. Indeed, Wang assumes that the putative equilibrium price has the functional form[16]

$$P(t) = p^W + p_D^W D(t) + p_\Pi^W \hat{\Pi}(t) + p_\Theta^W \Theta(t), \tag{78}$$

where $p_D^W = \frac{1}{r+\alpha_D}$, $p_\Pi^W = \frac{1}{(r+\alpha_D)(r+\alpha_\Pi)}$, and p^W and p_Θ^W are still misspecified[17] (see Wang [35, Sez. 3, p. 254] (1993)). Therefore, it is still necessary to exploit the above determined equation for the dynamics of $\hat{\Pi}(t)$, given the observed process $D(t)$. It then follows that the instantaneous excess return to one share of risky asset satisfies

$$dQ(t) = -rp^W\,dt - (r+\alpha_\Theta)p_\Theta^W \Theta(t)\,dt + \frac{(r+\alpha_\Pi)\sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi + \sigma}{(r+\alpha_D)(r+\alpha_\Pi)\sigma_D} d\tilde{w}_D(t) + p_\Theta^W \sigma_\Theta dw_\Theta(t) \tag{79}$$

The structure of Equation (79) leads to tackle the uninformed investors' optimization problem introducing the variable $Z_u(t) \equiv (1, \Theta(t))$ and the matrices

$$A_u \equiv \begin{pmatrix} 0 & 0 \\ 0 & -\alpha_\Theta \end{pmatrix}, \quad Q_u^{1/2} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \sigma_\Theta \end{pmatrix},$$

[16] For simplicity, also in this case we suppress Wang's constant term ϕ.

[17] Actually, Wang considers only the particular case $\varphi_u = 1$, $\sigma_{D,\Pi} = 0$ and sets $p^W = -\frac{((r+\alpha_\Pi)\sigma_D^2+\sigma)^2}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2}$. We derive the form of p^W in the general case.

$$B_u \equiv \begin{pmatrix} rp^W & (r+\alpha_\Theta)p^W_\Theta \end{pmatrix}, \quad R_u^{1/2} \equiv \begin{pmatrix} \frac{(r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma}{(r+\alpha_D)(r+\alpha_\Pi)\sigma_D} & \sigma_\Theta p^W_\Theta \end{pmatrix}.$$

Hence, we obtain the uninformed investors' value function again in the form (31), where $G \equiv (g_{j,k})_{j,k=1}^2$ is a symmetric solution of the Riccati equation (32), with coefficients given by (33), for $j = k = 2$, and γ is a real solution of (34). The uninformed investors' optimal demand for the stock and their optimal consumption are given by (35) and (36). Therefore, setting $\mathring{\Psi}_u(t) \equiv (\psi_u^W, \psi_{u,\Theta}^W)\mathring{Z}_{u,W}(t)$, it follows

$$\psi_u^W = -\frac{(r+\alpha_D)^2(r+\alpha_\Pi)^2(rp^W+\sigma_\Theta^2 p_\Theta^W g_{1,2})\sigma_D^2}{r\varphi_u((r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2\sigma_\Theta^2(p_\Theta^W)^2+((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2)},$$

$$\psi_{u,\Theta}^W = \frac{(r+\alpha_D)^2(r+\alpha_\Pi)^2(r+\alpha_\Theta+\sigma_\Theta^2 g_{2,2})p_\Theta^W\sigma_D^2}{r\varphi_u((r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_\Theta^2\sigma_D^2(p_\Theta^W)^2+((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2)},$$

and thanks to (72) we have

$$g_{1,2} = -r\left(\frac{p^W}{\sigma_\Theta^2 p_\Theta^W} + \varphi_u p_\Theta^W + \frac{\varphi_u((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2\sigma_\Theta^2 p_\Theta^W}\right), \tag{80}$$

$$g_{2,2} = -\left(\frac{r+\alpha_\Theta}{\sigma_\Theta^2} + r\varphi_u p_\Theta^W + \frac{r\varphi_u((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2\sigma_\Theta^2 p_\Theta^W}\right), \tag{81}$$

$$\psi_u^W = \frac{rp^W+\sigma_\Theta^2 p_\Theta^W g_{1,2}}{r\left(p^W-\varphi_u\sigma_\Theta^2(p_\Theta^W)^2\right)}. \quad \psi_{u,\Theta}^W = \frac{\left(r+\alpha_\Theta+\sigma_\Theta^2 g_{2,2}\right)p_\Theta^W}{r\left(p^W-\varphi_u\sigma_\Theta^2(p_\Theta^W)^2\right)}, \tag{82}$$

Finally, substituting (80) and (81) in Equation (32), we obtain

$$g_{1,1} = -r\varphi_u p^W, \tag{83}$$

$$p^W = -\varphi_u\frac{((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2}{(r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2}, \tag{84}$$

and the equation for p_Θ^W

$$P_u(p_\Theta^W)Q_u(p_\Theta^W) = 0, \tag{85}$$

where

$$P_u(p_\Theta^W) = (r+\alpha_D)^2(r+\alpha_\Pi)^2\sigma_D^2\sigma_\Theta^2(p_\Theta^W)^2 + ((r+\alpha_\Pi)\sigma_D^2+\sigma_{D,\Pi}\sigma_\Pi+\sigma)^2$$

and

$$
\begin{aligned}
Q(p_\Theta^W) \equiv\ & \varphi_u r^2 (r+\alpha_D)^4 (r+\alpha_\Pi)^4 \sigma_\Theta^2 \sigma_D^2 (p_\Theta^W)^3 \\
& - (r+\alpha_D)^2 (r+\alpha_\Pi)^2 \sigma_D^2 \left((r+\alpha_\Theta)(r+\alpha_D)^2 (r+\alpha_\Pi)^2 \alpha_\Theta \sigma_D^2 - r^2 \varphi_u^2 \sigma_\Theta^2 ((r+\alpha_\Pi)\sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi + \sigma)^2 \right) (p_\Theta^W)^2 \\
& + \varphi_u r^2 (r+\alpha_D)^2 (r+\alpha_\Pi)^2 \sigma_D^2 ((r+\alpha_\Pi)\sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi + \sigma)^2 p_\Theta^W \\
& + \varphi_u^2 r^2 ((r+\alpha_\Pi)\sigma_D^2 + \sigma_{D,\Pi}\sigma_\Pi + \sigma)^4.
\end{aligned}
$$

Equation (85) provides numerical values of p_Θ^W, depending on the choice of the exogenous parameters.

8 Comparing BN Equilibria with Wang's RE Equilibria

To compare our BN equilibria with Wang's ones, we first need to establish some conversion tables allowing to rewrite the price and the rational investors' optimal demand for the risky asset characterizing the former in terms of the same variables used for the latter. This is possible because the information structure of the model does not change when passing from Wang's approach to the BN one.
With regard to the price of the risky asset, note that from (16) it follows

$$\hat{\Theta}(t) = p_\Theta^{-1}(p_\Pi \Pi(t) + p_\Theta \Theta(t) - p_\Pi \hat{\Pi}(t)). \tag{86}$$

Hence, starting from the BN form of the equilibrium price (13), we can rewrite

$$P(t) = p + p_D D(t) + (p_\Pi + p_{\hat{\Pi}})\Pi(t) + (p_\Theta + p_{\hat{\Theta}})\Theta(t) + (p_{\hat{\Pi}} - p_\Pi p_\Theta^{-1} p_{\hat{\Theta}})(\hat{\Pi}(t) - \Pi(t)). \tag{87}$$

This compared with the form of Wang's RE equilibrium price (54),

$$P(t) = p^W + p_D^W D(t) + p_\Pi^W \Pi(t) + p_\Theta^W \Theta(t) + p_\Delta^W \Delta(t),$$

yields the following conversion table

$$p^W = p, \qquad p_D^W = p_D, \qquad p_\Pi^W = p_\Pi + p_{\hat{\Pi}}, \qquad p_\Theta^W = p_\Theta + p_{\hat{\Theta}}, \qquad p_\Delta^W = p_{\hat{\Pi}} - p_\Pi p_\Theta^{-1} p_{\hat{\Theta}}. \tag{88}$$

With regard to the uninformed investors' optimal inventory, on account of (13) and (16), we have

$$
\begin{aligned}
Z_u(t) =&\psi_u - \psi_{u,P}p + (\psi_{u,D} - \psi_{u,P}p_D)D(t)\\
&+ (\psi_{u,\hat{\Pi}} - \psi_{u,P}(p_\Pi + p_{\hat{\Pi}}))\hat{\Pi}(t) + (\psi_{u,\hat{\Theta}} - \psi_{u,P}(p_\Theta + p_{\hat{\Theta}}))\hat{\Theta}(t).
\end{aligned}
$$

The latter, compared with the form of Wang's RE uninformed investors' optimal inventory, yields

$$
\psi_u^W = \psi_u - \psi_{u,P}p, \qquad \psi_{i,\hat{\Theta}}^W = \psi_{u,\hat{\Theta}} - \psi_{u,P}(p_\Theta + p_{\hat{\Theta}}). \tag{89}
$$

Furthermore, the terms

$$
\psi_{u,D} - \psi_{u,P}p_D, \quad \psi_{u,\hat{\Pi}} - \psi_{u,P}(p_\Pi + p_{\hat{\Pi}})
$$

are both null or not null according to whether BN equilibrium coincides with Wang's one. Similarly, with regard to the informed investors' optimal inventory, thanks to (13) and (86), we can write

$$
\begin{aligned}
Z_i(t) = \psi_i - \psi_{i,P}p + (\psi_{i,D} - \psi_{i,P}p_D)D(t) + (\psi_{i,\Pi} - \psi_{i,P}(p_\Pi + p_{\hat{\Pi}}))\Pi(t)\\
+ (\psi_{i,\Theta} - \psi_{i,P}(p_\Theta + p_{\hat{\Theta}}))\Theta(t) - \psi_{i,P}(p_{\hat{\Pi}} - p_\Pi p_\Theta^{-1} p_{\hat{\Theta}})\Delta(t).
\end{aligned}
$$

This, compared with the form of Wang's RE informed investors' optimal inventory, yields

$$
\psi_i^W = \psi_i - \psi_{i,P}p, \qquad \psi_{i,\Theta}^W = \psi_{i,\Theta} - \psi_{i,P}(p_\Theta + p_{\hat{\Theta}}), \qquad \psi_{i,\Delta}^W = -\psi_{i,P}(p_{\hat{\Pi}} - p_\Pi p_\Theta^{-1} p_{\hat{\Theta}}). \tag{90}
$$

Also in this case the terms

$$
\psi_{i,D} - \psi_{i,P}p_D, \quad \psi_{i,\Pi} - \psi_{i,P}(p_\Pi + p_{\hat{\Pi}})
$$

are both null or not null according to whether BN equilibrium coincides with Wang's one.

Conversely, it is not difficult to check that the equilibrium price and the investor's optimal demands for the risky asset in Wang's form can be rewritten in BN form by setting

$$
p = p^W, \quad p_D = p_D^W, \quad p_\Pi = \frac{\left(p_\Pi^W - p_\Delta^W\right)\left(p_\Theta^W - p_{\hat{\Theta}}\right)}{p_\Theta^W}, \quad p_\Theta = p_\Theta^W - p_{\hat{\Theta}}, \quad p_{\hat{\Pi}} = \frac{p_\Theta^W p_\Delta^W + p_{\hat{\Theta}}(p_\Pi^W - p_\Delta^W)}{p_\Theta^W}, \quad p_{\hat{\Theta}} \in \mathbb{R}. \tag{91}
$$

$$
\psi_u = \psi_u^W + \psi_{u,P}p^W, \quad \psi_{u,D} = \psi_{u,P}p_D^W, \quad \psi_{u,\hat{\Pi}} = \psi_{u,P}p_\Pi^W, \quad \psi_{u,\hat{\Theta}} = \psi_{u,\hat{\Theta}}^W + \psi_{u,P}p_\Theta^W, \quad \psi_{u,P} \in \mathbb{R}. \tag{92}
$$

$$
\psi_i = \psi_i^W + \psi_{i,P}p^W, \quad \psi_{i,D} = \psi_{i,P}p_D^W, \quad \psi_{i,\Pi} = \psi_{i,P}p_\Pi^W, \quad \psi_{i,\Theta} = \psi_{i,\Theta}^W + \psi_{i,P}p_\Theta^W, \quad \psi_{i,P} = -\frac{\psi_{i,\Delta}^W}{p_\Delta^W}. \tag{93}
$$

Therefore, from a strictly mathematical point of view, all BN equilibria of the model could be achieved via a Walrasian auctioneer approach of Wang's type, which allows the coefficients of the putative price of the stock asset to be all misspecified. However, this general approach might obscure the BN imperfect competition interpretation.

9 Numerical Results

9.1 Preliminaries

In the sequel, due to the analytical and computational complexity concerning the search for the equilibria of the model, following Wang [35] (1993), we also restrain our analysis to the particular cases characterized by the exogenous parameter $\sigma_{D,\Pi} = 0$. In this case, setting $\sigma_{D,D} \equiv \sigma_D$, we can write

$$\begin{aligned} a^2 &\equiv \frac{((\alpha_\Pi - \alpha_\Theta)^2\sigma_D^2 + \sigma_\Pi^2)p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2}{\sigma_D^2(\sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2)}, \\ b &\equiv \frac{\alpha_\Pi \sigma_\Theta^2 p_\Theta^2 + \alpha_\Theta \sigma_\Pi^2 p_\Pi^2}{p_\Pi^2\sigma_\Pi^2 + p_\Theta^2\sigma_\Theta^2}, \\ c^2 &\equiv \frac{\sigma_\Pi^2\sigma_\Theta^2 p_\Theta^2}{p_\Pi^2\sigma_\Pi^2 + p_\Theta^2\sigma_\Theta^2}. \end{aligned} \tag{94}$$

As a consequence, we obtain

$$\begin{aligned} \sigma = {} & \frac{\sigma_D^2}{((\alpha_\Pi - \alpha_\Theta)^2\sigma_D^2 + \sigma_\Pi^2)p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2} \\ & \cdot \left(-(\alpha_\Pi \sigma_\Theta^2 p_\Theta^2 + \alpha_\Theta \sigma_\Pi^2 p_\Pi^2) + \sqrt{(\alpha_\Pi \sigma_\Theta^2 p_\Theta^2 + \alpha_\Theta \sigma_\Pi^2 p_\Pi^2)^2 + \frac{\sigma_\Pi^2 \sigma_\Theta^2}{\sigma_D^2}(((\alpha_\Pi - \alpha_\Theta)^2\sigma_D^2 + \sigma_\Pi^2)p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2)p_\Theta^2} \right), \end{aligned} \tag{95}$$

which is the same result as in Wang [35, (A.8) p. 277] (1993), on account of his shorthand $\sigma_S \equiv \sqrt{\sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2}$. Moreover,

$$h_{\hat{\Pi},D} = \frac{\sigma}{\sigma_D^2}, \quad h_{\hat{\Pi},S} = \frac{(\sigma_\Pi^2 - (\alpha_\Pi - \alpha_\Theta)\sigma)p_\Pi}{\sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2}, \quad h_{\hat{\Theta},D} = -\frac{\sigma}{\sigma_D^2}\frac{p_\Pi}{p_\Theta}, \quad h_{\hat{\Theta},S} = \frac{(\alpha_\Pi - \alpha_\Theta)p_\Pi^2\sigma + \sigma_\Theta^2 p_\Theta^2}{(\sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2)p_\Theta}, \tag{96}$$

and

$$B \circ B = \begin{pmatrix} \sigma_D^2 & 0 \\ 0 & \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 \end{pmatrix},$$

which clearly implies

$$b_{1,1} \equiv \sigma_D, \qquad b_{1,2} \equiv b_{2,1} \equiv 0, \qquad b_{2,2} \equiv \sqrt{\sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2}. \tag{97}$$

In addition, still following Wang, from now on, we choose the exogenous parameters

$$r \equiv 0.05, \quad \rho \equiv 0.20, \quad \alpha_D \equiv 1.00, \quad \alpha_\Pi \equiv 0.20, \quad \alpha_\Theta \equiv 0.40, \quad \sigma_D \equiv 1.00, \quad \sigma_\Pi \equiv 0.60. \tag{98}$$

Note that this choice implies

$$\frac{1}{r+\alpha_D} = 0.95238, \qquad \frac{1}{(r+\alpha_D)(r+\alpha_\Pi)} = 3.80952, \tag{99}$$

which are exactly the coefficients p_D^T and p_Π^T of Wang's equilibrium price. Finally, to tackle the problem of selecting the economically plausible equilibria, among all mathematically achieved candidates, we replace the uninformed [resp. informed] investors' expected utility with the principal part

$$\tilde{V}_u \equiv \frac{1}{2}(g_{1,1} + 2pg_{1,5} + p^2 g_{5,5}) + \gamma - 1 \qquad [\text{resp.} \tilde{V}_i \equiv \frac{1}{2}(\ell_{1,1} + 2p\ell_{1,5} + p^2\ell_{5,5}) + \lambda - 1]. \tag{100}$$

In fact, since in a suitable neighborhood of the origin of the Euclidean spaces of the states of the economy, $(D, \hat{\Pi}, \hat{\Theta}, P)$ and (D, Π, Θ, P), we have

$$Z_u^T G Z_u \simeq (\ell_{1,1} + 2p\ell_{1,5} + p^2\ell_{5,5}) \quad [\text{resp. } Z_i^T L Z_i \simeq \ell_{1,1} + 2p\ell_{1,5} + p^2\ell_{5,5}].$$

it follows

$$-e^{-(\rho_u t + \frac{1}{2}Z_u^T G Z_u + r\varphi_u W_u + \gamma)} \simeq \rho_u t + r\varphi_u W_u + \tilde{V}_u \quad [\text{resp. } -e^{-(\rho_i t + \frac{1}{2}Z_i^T L Z_i + r\varphi_i W_i + \lambda)} \simeq \rho_i t + r\varphi_i W_i + \tilde{V}_i].$$

Hence, the higher is the principal part of the expected utility, the higher is the expected utility itself in the considered neighborhood.

9.2 Equilibria

Given the additional exogenous parameters σ_Θ, φ_i, φ_u, and ω, our approach reveals many equilibrium candidates other than the perfect competitive equilibrium discovered by Wang. On the basis of the

Pareto efficiency criterium, we think that several of these candidates should be declared *rejectable*, since characterized by expected utilities of the two groups of rational investors whose principal parts are both lower than the corresponding ones characterizing Wang's benchmark equilibrium. By contrast, we believe that equilibrium candidates in which such principal parts of the expected utilities are both higher should be considered *feasible*. On the other hand, equilibrium candidates exist in which, with respect to Wang's benchmark, the principal part of the expected utility increases for one group of investors, while lessens for the other group. As already discussed in Introduction, we think that these candidates should still considered rejectable if the group of investors whose expected utility principal part lessens trade strategically. Nevertheless, it seems to us quite questionable the rejection of the candidates in which the group of investors whose expected utility principal part lessens trade as perfect competitors. In fact, according to whether the investors of the other group trade also as perfect competitors or strategically, we are in front of an equilibrium candidate which exhibits the same qualitative features of Wang's equilibrium or an intriguing picture of strategic investors who take profit of perfect competitors.

In the sequel, we present a small, but hopefully, representative set of the numerical results obtained via our computational procedures. Computational procedures themselves and a much larger set of results are available from the authors upon request.

In Tables 1-4 the stock price coefficients and the investors' expected utility principal part of the top three[18] equilibrium candidates of some significant degenerate cases are shown. As already

Table 1: Degenerate Cases

(a) *All Informed*, $\sigma_\theta = 2.750$, $\varphi_i = 1.000$.

	p	p_D	p_Π	p_Θ	$\tilde{V}_i$
W	-6.1315	0.9524	3.8095	-0.9781	1.4904
A	-9.0703	0.9524	-4.7619	-1.9083	-0.1766
B	-6.2496	4.0530	-0.0663	1.9780	-15.3366

(b) *All Uninformed*, $\sigma_\theta = 2.500$, $\varphi_u = 1.000$.

	p	p_D	$p_{\hat{\Pi}}$	p_Θ	$\tilde{V}_u$
W	-6.7591	0.9524	3.8095	-1.0337	1.1729
A	-10.5611	0.9524	-9.7170	-2.1420	-0.5728
B	-6.7591	0.9524	3.8095	1.7471	-12.7208

Table 2: Degenerate Cases

(a) *All Informed*, $\sigma_\theta = 3.000$, $\varphi_i = 1.000$.

	p	p_D	p_Π	p_Θ	$\tilde{V}_i$
A	-9.0703	0.9524	-4.7619	-2.3263	2.6488
W	-6.1315	0.9524	3.8095	-1.0803	2.3568
B	-6.3216	5.9422	-2.4277	0.3705	-15.3334

(b) *All Uninformed*, $\sigma_\theta = 2.750$, $\varphi_u = 1.000$.

	p	p_D	$p_{\hat{\Pi}}$	p_Θ	$\tilde{V}_u$
A	-10.5611	0.9524	-9.7170	-2.6349	2.2355
W	-6.7591	0.9524	3.8095	-1.1356	1.9433
B	-7.9780	-19.4574	29.3217	-0.1080	-15.2784

discussed in Section 7, in degenerate cases the lack of competitors forces the equilibrium demand for the stock of the only existing group of rational investors to match the stochastic supply. Hence, the stock prices characterizing the equilibrium candidates are nothing else than the prices making the rational investors' equilibrium demand optimal. Accordingly, we think it is somehow inappropriate to

[18]Ordered according to the investors' expected utility principal part

Table 3: Degenerate Cases

(a) *All Informed,* $\sigma_\theta = 1.833$, $\varphi_i = 1.500$.

	p	p_D	p_Π	p_Θ	$\tilde{V}_i$
W	-9,1973	0,9524	3,8095	-1,4669	1,6806
A	-13,6054	0,9524	-4,7619	-2,8615	0,1025
B	-9,3738	4,0418	-0,0522	2,9730	-15,1378

(b) *All Uninformed,* $\sigma_\theta = 1.666$, $\varphi_u = 1.500$.

	p	p_D	$p_{\hat{\Pi}}$	p_Θ	$\tilde{V}_u$
W	-10,1387	0,9524	3,8095	-1,5500	1,3816
A	-15,8417	0,9524	-9,7170	-3,2106	-0,2512
B	-10,1387	0,9524	3,8095	2,6158	-12,4987

Table 4: Degenerate Cases

(a) *All Informed,* $\sigma_\theta = 2.000$, $\varphi_i = 1.500$.

	p	p_D	p_Π	p_Θ	$\tilde{V}_i$
A	-13,6054	0,9524	-4,7619	-3,4894	2,9322
W	-9,1973	0,9524	3,8095	-1,6204	2,5484
B	-9,4824	5,9422	-2,4277	0,5557	-15,1305

(b) *All Uninformed,* $\sigma_\theta = 1.833$, $\varphi_u = 1.500$.

	p	p_D	$p_{\hat{\Pi}}$	p_Θ	$\tilde{V}_u$
A	-15,8417	0,9524	-9,7170	-3,9505	2,5583
W	-10,1387	0,9524	3,8095	-1,7031	2,1527
B	-11,9674	-19,4609	29,3261	-0,1572	-15,0067

refer to these candidates as perfect competitive rather than strategic. Nevertheless, while in Tables 1a-4a [resp. Tables 1b-4b] the prices characterizing Candidate W are just a linear perturbation of the risky asset fundamental value (see Campbell & Kyle [5] (1993)), those characterizing Candidate A, though correlated with the public information exactly as in W, are negatively correlated with the private information [resp. the estimate of the private information]. Thus, prices in W are informationally efficient, while prices in A are not[19]. In addition, prices in A show a much higher discount term for holding the stock and a stronger sensitivity to the stock supply shocks. Furthermore, our computational procedures suggest that the coefficients of prices in W [resp. A] are the natural limit of the coefficients of informationally efficient Wang's prices [resp. inefficient strategic prices] arising in non-degenerate cases, as the parameter ω goes to 0 or 1. Therefore, we call the equilibria of type W [resp. A] *degenerate perfect competitive* [resp. *degenerate strategic*]. The alternate of degenerate perfect competitive and strategic equilibria exhibits a pretty strong regularity, depending on the volatility of the stock supply shocks, investors' subjective risk aversion, and investors' information[20]. From the large number of cases examined in the ranges $0.250 \leq \varphi_u \leq 4.000$, $0.250 \leq \varphi_i \leq 4.500$, $0.250 \leq \sigma_\Theta \leq 12.000$, we infer that a degenerate perfect competitive equilibrium occurs assuming $\varphi_i\sigma_\Theta \leq 2.750$ or $\varphi_u\sigma_\Theta \leq 2.500$, while the condition $\varphi_i\sigma_\Theta \geq 3.000$ or $\varphi_u\sigma_\Theta \geq 2.750$ yields a degenerate strategic equilibrium. Despite degenerate cases cannot be considered fully representative, it seems to us that their analysis gives an interesting insight: the emergence of a parameter of the form

$$\mu \equiv ((1-\omega)\varphi_i + \omega k(\omega)\varphi_u)\sigma_\Theta, \tag{101}$$

[19]While prices in W can also be determined via Wang's genuine approach, prices in A, as well as in B, cannot

[20]That is $\omega = 0$ or $\omega = 1$

where $k(\omega)$ is an ω-increasing function such that $k(0) = 1$ and $k(1) \simeq 1.094$, such that, for a suitable $\bar{\mu} \in [2.750, 3.000]$, when $\mu < \bar{\mu}$ a perfect competitive equilibrium dominates and when $\mu > \bar{\mu}$ a strategic equilibrium dominates. We propose to call such a parameter *market risk perception*, to point out its dependence on the volatility of the stock supply shocks, the investors' subjective risk aversion, and the proportion between informed and uninformed investors. Such a dependence will be confirmed further ahead.

In Table 5 we show a representative selection of the equilibrium candidates achieved via our computational implementation of Wang's genuine approach[21], given $\sigma_\Theta = 2.000$, $\varphi_u = \varphi_i = 1.000$, and $\omega = 0.001$ [resp. $\omega = 0.999$]. The number n is referred to the number of times the corresponding candidate has appeared in a sample of 1000 successful trials[22]. The sum of the n's is less than 1000 due to the omission of several rejectable equilibrium candidates. The number n_1 [resp. n_2] is referred to the number of times the corresponding candidate has appeared in a sample of 1000 [resp. 5000] successful strategic deviations from Wang's perfect competitive benchmark equilibrium. To realize such deviations we have exploited our BN approach, while initializing the corresponding computational bargaining procedure with rational investors' demands obtained by random perturbations of size 0.1 [resp. 0.01] in the coefficients of the optimal perfect competitive demands. The sum of the n_1's [resp. n_2's] is less than 1000 [resp. 5000] due to the appearance of some additional rejectable deviations. The coefficients of each equilibrium candidate stock price are shown jointly with the coefficients of the corresponding rational investors' optimal demand for the stock and the principal part of the investors' expected utility. For brevity, the coefficients p_D, p_Π of the stock price and the coefficients $\psi_{u,D}$, $\psi_{i,D}$, $\psi_{u,\Pi}$, $\psi_{i,\Pi}$ of the investors' demand for the stock have been omitted[23]. Different capital letters are used to denote equilibrium candidates exhibiting different correlation patterns of the characteristic investors' optimal demands for the stock with the variables of the economy. An additional number allows to distinguish among candidates exhibiting the same pattern. A possible additional "s" indicates the Pareto sub-optimality of a candidate with respect to the one denoted by the same capital letter and number. Wang's equilibria are denoted by W. PI denotes equilibrium candidates which are Pareto inefficient with respect to W, since both the principal parts of the investors' expected utility are lower than the

[21] We recall that all equilibrium candidates achieved via Wang's genuine approach can be also achieved by our BN approach. The reason to exploit Wang's approach is to reveal the existence of multiple equilibrium candidates via this approach.

[22] Samples of 5000 successful trials have revealed no meaningful differences in the achieved equilibrium candidates.

[23] All Wang's equilibrium stock prices and rational investors' demand for the stock are characterized by $p_D = 0.95238$, $p_\Pi = 3.80952$, and $\psi_{u,D} = \psi_{i,D} = \psi_{u,\Pi} = \psi_{i,\Pi} = 0$.

corresponding ones in the benchmark W[24]. The first remarkable result is the finding of equilibrium

Table 5: Wang's approach. $\varphi_u = \varphi_i = 1.000$, $\sigma_\Theta = 2.000$. *R: rejectable, F: feasible.*

(a) $\omega = 0.001$.

	n	n_1	n_2	p	p_Θ	p_Δ	ψ_u	$\psi_{u,\hat{\Theta}}$	$\tilde{V}_u$	ψ_i	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde{V}_i$	
A1	154	21	77	-6.1309	11.5819	-0.0121	0.8878	-2.6154	55.6836	1.0001	1.0036	-0.0009	-28.8546	R
A1s	127	0	0	-6.4066	11.2778	3.6218	0.9467	-2.6531	55.3063	1.0001	1.0037	$-4.\times10^{-5}$	-41.4199	R
B1	4	0	0	-6.5132	7.3958	121.8597	1.5336	-10.7836	3.5675	0.9995	1.0118	0.1723	-8.4849	R
A3	19	0	0	-6.4023	0.9946	3.5828	0.9336	-1.0479	0.3869	1.0001	1.0020	-0.0002	-23.6083	R
C1	67	0	0	-6.4268	-0.8205	3.8916	0.9565	0.9416	0.0403	1.0000	1.0001	0.0001	-12.9940	R
A4	39	1	5	-6.1344	1.0001	0.0845	0.8512	-1.0565	-0.2688	1.0001	1.0021	-0.0039	-10.4264	R
A5	10	0	0	-6.1082	1.0600	-0.8591	0.8502	-1.1279	-0.3065	1.0001	1.0021	-0.0050	-10.2556	R
W	174	848	4744	-6.1317	-0.8030	0.0030	0.8768	0.9296	-0.5483	1.0001	1.0001	-0.0044	0.0831	F
B2	1	1	0	-6.1315	-0.8070	-0.0088	1.2868	-2.7242	-11.4878	0.9997	1.0037	0.0129	0.0871	R
PI	37	0	0	-6.5121	7.4287	123.2679	0.8724	2.5434	-9.6793	1.0001	0.9985	-0.0409	-8.4829	R

(b) $\omega = 0.999$.

	n	n_1	n_2	p	p_Θ	p_Δ	ψ_u	$\psi_{u,\hat{\Theta}}$	$\tilde{V}_u$	ψ_i	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde{V}_i$	
D1	147	2	0	-6.7585	-0.9059	3.8076	0.9998	1.0023	0.2290	1.2189	-1.2491	-2.0819	-9.6934	R
E1	3	3	0	-6.7588	-0.9051	3.8097	1.0000	1.0019	0.2276	1.0335	-0.8826	0.2402	-23.3091	R
W	128	839	4936	-6.7577	-0.9030	3.8017	0.9993	0.9995	0.2225	1.7410	1.4933	-8.6205	5.8546	F
D2	44	3	3	-6.7596	10.8892	3.8213	0.9999	1.0039	-27.5021	1.1394	-2.9285	-1.0860	59.8769	F
D2s	47	0	0	-6.7588	10.8971	3.8099	0.9999	1.0037	-27.5136	1.0557	-2.7238	-0.0379	41.7192	R
D4	3	35	16	-6.7590	1.1611	3.8183	0.9993	1.0027	-10.5786	1.6575	-1.7065	-7.5758	6.2157	F
PI	26	0	0	-6.7588	1.1511	3.8107	0.9999	0.9998	-10.5644	1.1357	1.2217	-1.0391	-10.4974	R

candidates in which at least one of the investors' optimal demands for the stock exhibits a correlation pattern with the variables of the economy different from the corresponding one characterizing Wang's perfect competitive benchmark. Namely, at least one of the two groups of investors trade strategically against the other group. Therefore, also strategic equilibrium candidates may be characterized by stock prices which are informationally efficient in the semi-strong form. However, besides Candidate PI, we think that also Candidate B2 of Table 5a [resp. D1, E1 of Table 5b] should be rejected. In fact, the uninformed [resp. informed] investors' demand for the stock is negatively correlated with their estimate of the stock supply shocks, which means a strategic trading, while the principal part of their expected utility lessens with respect to W. Similarly, Candidates B1 and C1 in Table 5a should be rejected, since the informed investors' demand for the stock is positively correlated with the uninformed investors' estimation errors of the private information, which is again a strategic trading, while the principal part of their expected utility is lower than the benchmark one. On the other hand, it seems to us that the rejection of Candidates A1, A3, A4, A5 in Table 5a [resp. D2, D4 in Table 5b] is more questionable. In fact, if compared to W these equilibrium candidates are all Pareto efficient as well. Moreover, the investors whose expected utility increases are those who trade strategically, while the investors whose expected utility lessens trade as perfect competitors. Our computational procedures suggest that this type of candidates disappear as ω goes away from 0 [resp. from 1] or at least one between

[24] All the omitted equilibrium candidates in Table 5 share this feature with Candidate PI.

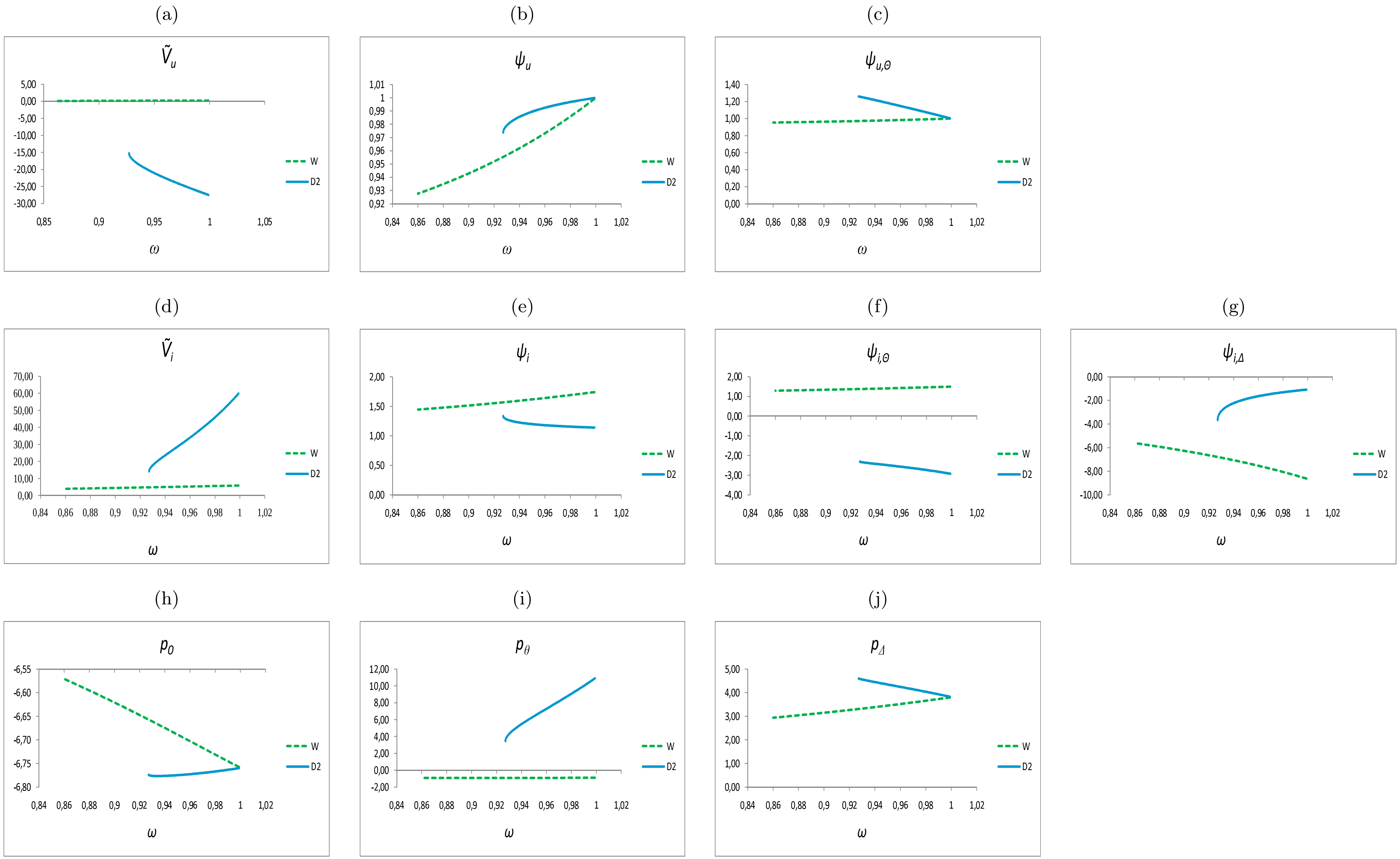
the volatility of the shocks on the stock supply and the perfect competitors' risk aversion increases. On the other hand, varying the risk aversion of the strategic investor in a large range, while keeping constant the risk aversion of majority of perfect competitors and the other parameters does not cause the equilibrium candidate to disappear. This is shown in Figures 1 -4. Hence, it seems that this type of equilibrium candidates are achievable only under the combination of a pronounced minority position of the strategic investors and a low market risk perception. In addition, in these equilibrium candidates the majority of perfect competitors are willing to accommodate a larger amount of the stock and their demand exhibits a higher positive correlation with their estimate of the stock supply shocks. Instead, the demand of the minority of strategic investors is rather strongly negatively correlated with their estimate of the stock supply shocks and the investors' private information, either they are the majority or the minority, matters less. As a balance, the corresponding stock price is positively correlated with the stock supply shocks and accounts less the estimation errors of the uninformed investors. Hence, a low market risk perception allows equilibrium candidates in which a small minority of contrarians increase their expected utility while trading against a large majority of somewhat aggressive perfect competitors. On the other hand, while it seems to us implausible that a large majority of informed perfect competitors would raise their demand for the stock up to the point of offsetting the perfectly observed shocks on the stock supply, just to end up with lessening their expected utility, an imperfect observation of the shocks might make plausible such an aggressive trading policy by the uninformed investors. In light of the above arguments, we think that Candidates A1, A3, A4, A5 of Table 5a [resp. D2, D4 of Table 5b] should be declared rejectable [resp. feasible]. Finally, we declare Candidate A1s in Table 5a [resp. D2s in Table 5b] rejectable since suboptimal to Candidate A1 [resp. D2].

In Figures 1 -4 the principal parts of the rational investors' expected utilities, the coefficients of their optimal demands, and the coefficients of the stock price[25] are plotted on varying of ω, σ_Θ, φ_u, and φ_i[26], starting from those characterizing D2 of Table 5b. It is worth noting that varying the parameter φ_i in a large range, while holding $\varphi_u = 1$ and the other parameters constants do not cause the equilibrium candidate to disappear: the aggressive trading attitude of the majority of traders is unaffected. Similar figures might be shown with reference to D4 of the same table and for equilibrium candidates A1, A3, A4, A5 of Table 5a. Some additional comments are in order. We have also tackled the case $\sigma_\Theta = 2.000$, $\varphi_u = \varphi_i = 1.000$ via our BN approach. In particular, we have generated samples of 5000 successful trials for $\omega = 0.001, 0.100, 0.009, 0.999$, and on varying of $\omega = 0.200, 0.300, \ldots, 0.800$. The first groups

[25] Except the constant coefficients $p_D = 0.95238$ and $p_\Pi = 3.80952$.

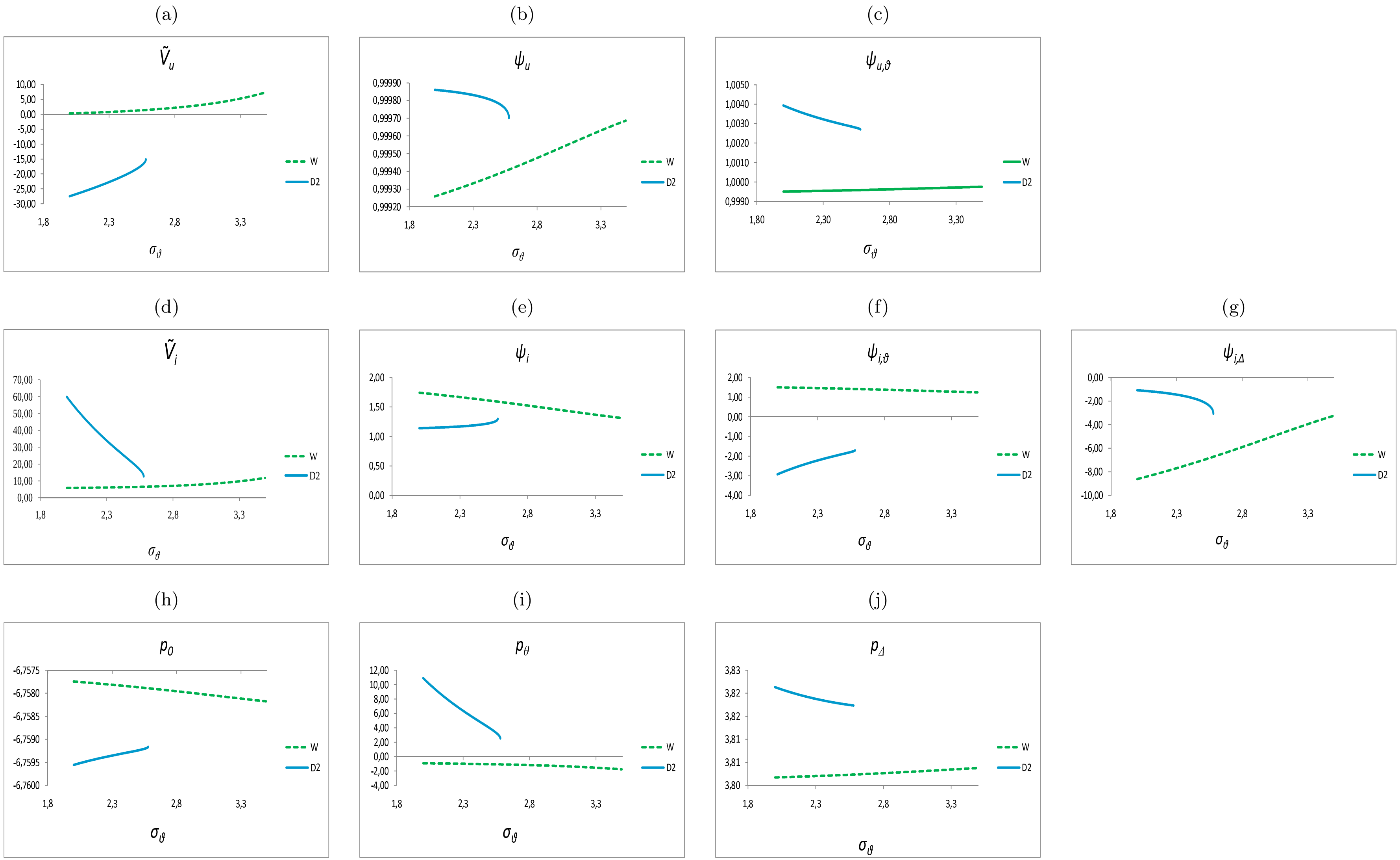
[26] The other parameters are kept constant.

Figure 1: Plot from D2 on varying of ω



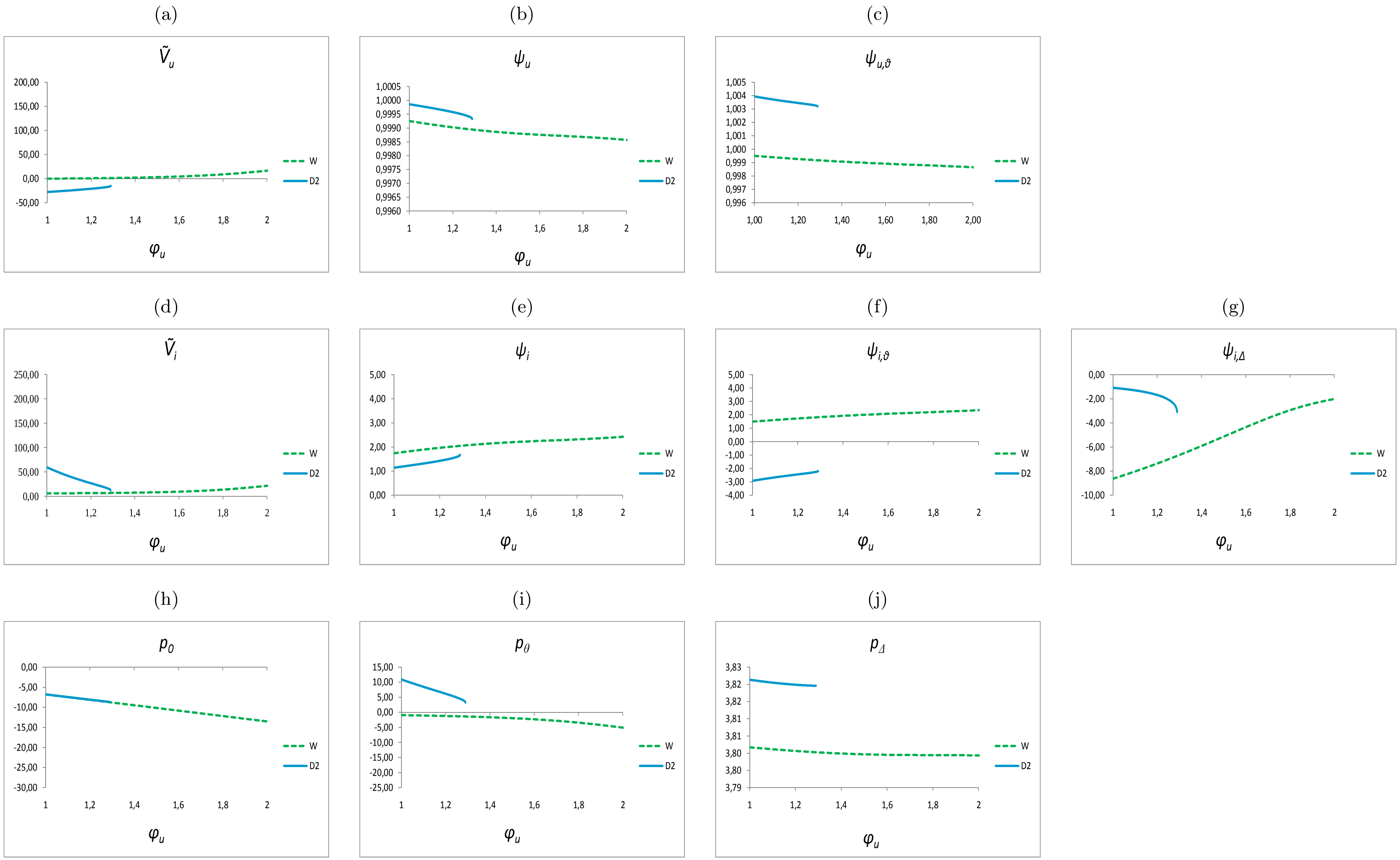
The constant coefficients $\psi_{u,D} = \psi_{u,\hat{\Pi}} = \psi_{i,D} = \psi_{i,\Pi} = 0$ and $p_D = 0.95238$, $p_\Pi = 3.80952$ have not plotted.

Figure 2: Plot from D2 on varying of σ_Θ



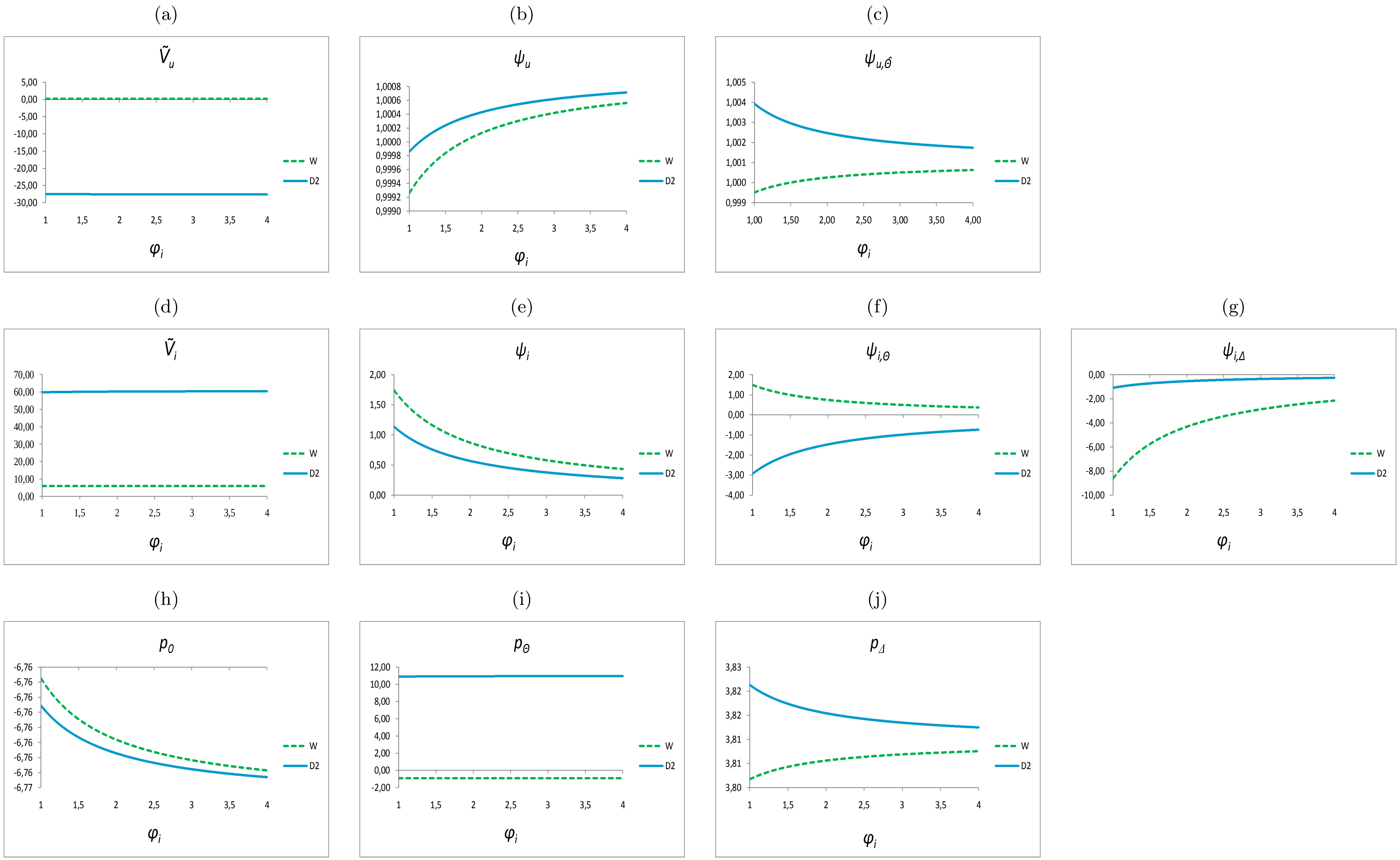
The constant coefficients $\psi_{u,D} = \psi_{u,\hat{\Pi}} = \psi_{i,D} = \psi_{i,\Pi} = 0$ and $p_D = 0.95238$, $p_\Pi = 3.80952$ have not plotted.

Figure 3: Plot from D2 on varying of φ_u



The constant coefficients $\psi_{u,D} = \psi_{u,\hat{\Pi}} = \psi_{i,D} = \psi_{i,\Pi} = 0$ and $p_D = 0.95238$, $p_\Pi = 3.80952$ have not plotted.

Figure 4: Plot from D2 on varying of φ_i



The constant coefficients $\psi_{u,D} = \psi_{u,\hat{\Pi}} = \psi_{i,D} = \psi_{i,\Pi} = 0$ and $p_D = 0.95238$, $p_\Pi = 3.80952$ have not plotted.

of samples have reveal no other feasible equilibrium candidates besides those of minority type already shown in Table 5. The second group of samples have revealed only rejectable equilibrium candidates: an equilibrium candidate of minority type has never appeared; neither has shown up an equilibrium candidate in which the informed and uninformed investors trading both strategically achieve both a higher expected utility principal part with respect to W. Clearly, the failure in revealing other feasible equilibria cannot be considered a proof of their non existence. However, the large number of trials considered suggest the suspicion that under “low” market risk perception and no minority condition, the equilibrium W should be declared the only feasible equilibrium.

In Table 6 we show all the feasible and few rejectable equilibrium candidates, which have appeared in a sample of 1000 successful trials obtained via our BN approach, given $\varphi_u = \varphi_i = 1.000$, $\sigma_\Theta = 3.000$, and various values of ω. The same notices and conventions as in the former Tables apply. In particular, we distinguish among the presented candidates by retaining the classification that we have used in the full samples. In Table 6 Candidates A1, F1, K1, S1 are examples of candidates which should be rejected on the ground that rational investors are unlikely to trade strategically to end up with lessening their perfect competitive expected utility[27]. On the other hand, both Candidates B1, E1, G2 and H2 Pareto efficiently dominate the benchmark W, while they are characterized by rational investors’ demands for the risky asset exhibiting a non-null correlation with the private information and its estimate. Even more, in Candidates E1 and G2 the investors’ demands exhibit even a non-null correlation with the public information. Hence, in these equilibrium candidates both the groups of rational investors trade strategically and achieve a higher expected utility than W. Differently than W, the informed investors exploit their private information explicitly, not only to take profit of the uninformed investors’ estimation errors. In B1 and E1, where the informed investors are the large majority, their demand for the risky asset is negatively correlated with the private information. Hence, the informed investors are willing to use the component $\psi_{i,\Pi}$ of their demand to transmit false information to the market: they transmit sell [resp. buy] signals when their information would suggest them to buy [resp. sell]. In turn, the uninformed investors use their estimate of the private information not only to detect the stock supply shocks, but also to react to the signal transmitted by the informed investors: they do not take the bait in the informed investors’ demand and they buy or sell as suggested by their estimate of the private information, by this taking profit of the contrarian component $\psi_{i,\Pi}$. However, the informed investors exploit the uninformed estimation errors to better calibrate their demand: as they observe

[27] All the Candidates which have not shown in this and the following tables share the same structure of the principal part of the expected utility with Candidates A1 and K1 or F1 and S1.

Table 6: BN approach. $\varphi_u = \varphi_i = 1.000$, $\sigma_\Theta = 2.000$. *R: rejectable, F: feasible.*

(a) $\omega = 0.001$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
A1	1	0		-9,5147	-2,8942	-1,6752	-2,6173	0,0027	4,3152	5,9526	3,3306	3,2666	45,8046	0,9967	-0,0060	-0,0033	0,9977	-0,0012	-16,6983
B1	10	5		-9,0686	0,9524	-4,7542	-2,3375	0,0049	3,4721	0,0000	7,8893	2,6192	31,1754	0,9975	0,0000	-0,0079	0,9984	-0,0026	2,6704
E1	9	0		-9,0689	0,9486	-4,7498	-2,3352	0,0106	2,9779	-2,5859	9,9463	2,1716	8,8308	0,9980	0,0026	-0,0100	0,9988	-0,0055	2,6617
E1s	1	0		-9,5183	-2,9288	-1,6422	-2,6087	0,0214	2,8184	-3,3295	11,4394	2,2435	8,0435	0,9982	0,0033	-0,0115	0,9988	-0,0100	-16,7135
W	261	730		-6,1318	0,9524	3,8095	-1,0805	0,0027	0,8186	0,0000	0,0000	0,8836	1,0602	1,0002	0,0000	0,0000	1,0001	-0,0031	2,3599
F1	6	0		-9,0698	0,9524	-4,7583	-2,3460	0,0083	1,3336	0,0000	1,8675	-1,2122	-22,4584	0,9997	0,0000	-0,0019	1,0022	-0,0043	2,7443

(b) $\omega = 0.100$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
B1	10			-8,8179	0,9524	-3,7656	-3,0205	0,4995	2,5151	0,0000	4,8798	2,3243	31,2977	0,8317	0,0000	-0,5422	0,8529	-0,1775	3,0890
E1	2			-8,8716	0,4727	-3,2367	-2,9193	1,1953	2,2897	-2,1363	7,1610	2,0130	8,0210	0,8567	0,2374	-0,7957	0,8874	-0,4561	2,7813
PC	1			-6,4500	0,9524	3,8095	-1,1640	3,7746	0,9526	0,0000	0,0000	0,9609	2,4925	1,0053	0,0000	0,0000	1,0043	-0,0032	-10,4113
W	659			-6,1627	0,9524	3,8095	-1,1030	0,2718	0,8239	0,0000	0,0000	0,8855	1,2313	1,0196	0,0000	0,0000	1,0127	-0,3156	2,6892

(c) $\omega = 0.200$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
PC	5			-6,4828	0,9524	3,8095	-1,1760	3,7772	0,9576	0,0000	0,0000	0,9652	2,5516	1,0106	0,0000	0,0000	1,0087	-0,0066	-10,3494
W	775			-6,1988	0,9524	3,8095	-1,1264	0,5538	0,8309	0,0000	0,0000	0,8887	1,4171	1,0423	0,0000	0,0000	1,0278	-0,6421	3,0589

(d) $\omega = 0.500$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
PC	9			-6,5833	0,9524	3,8095	-1,2129	3,7869	0,9729	0,0000	0,0000	0,9782	2,7360	1,0271	0,0000	0,0000	1,0218	-0,0182	-10,1562
H2s	3			-9,6705	0,9524	-6,5386	-2,9147	-3,5793	0,8008	0,0000	-0,8496	0,8319	2,6088	1,1992	0,0000	0,8496	1,1681	1,6942	9,0144
W	803			-6,3437	0,9524	3,8095	-1,1998	1,4961	0,8646	0,0000	0,0000	0,9072	2,0502	1,1354	0,0000	0,0000	1,0928	-1,7493	4,4362

(e) $\omega = 0.600$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
H2s	2			-9,8254	0,9524	-7,0487	-3,0218	-4,5576	0,8313	0,0000	-0,7238	0,8587	3,4829	1,2530	0,0000	1,0857	1,2119	2,1476	10,5297
PC	3			-6,6176	0,9524	3,8095	-1,2256	3,7908	0,9782	0,0000	0,0000	0,9825	2,8000	1,0327	0,0000	0,0000	1,0262	-0,0225	-10,0892
W	823			-6,4069	0,9524	3,8095	-1,2239	1,8585	0,8815	0,0000	0,0000	0,9177	2,2815	1,1778	0,0000	0,0000	1,1235	-2,1943	5,0010

(f) $\omega = 0.999$.

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
K1	1	0	0	-10,5606	0,9524	-9,7171	-3,3490	-9,7172	1,0000	0,0000	0,0000	1,0017	6,9437	1,0472	0,0000	-0,0270	-0,6968	-0,0529	-34,4445
G2	59	1	0	-10,5547	0,9510	-9,6964	-3,3469	-9,7032	0,9964	0,0006	-0,0127	0,9976	6,8662	4,5466	-0,6033	12,7369	3,4006	10,7077	33,9399
H2	32	46	3	-10,5546	0,9524	-9,6973	-3,3462	-9,7016	0,9965	0,0000	-0,0119	0,9976	6,8640	4,5062	0,0000	11,9207	3,3848	10,6301	55,1263
H2s	3	0	0	-10,5591	0,9524	-9,7089	-3,3274	-9,7007	0,9994	0,0000	-0,0025	0,9995	6,8231	1,5703	0,0000	2,5332	1,4713	4,9673	17,3917
W	122	734	4986	-6,7580	0,9524	3,8095	-1,2784	3,8030	0,9995	0,0000	0,0000	0,9997	3,0686	1,4622	0,0000	0,0000	1,3383	-5,1282	7,8387
S1	16	14	1	-6,3030	8,5820	-5,7275	0,3705	-5,7248	1,0000	0,0111	-0,0139	1,0013	-15,3836	0,9799	-11,1035	13,8793	-0,3249	6,4865	18,9626
PC	41	3	0	-6,7588	0,9524	3,8095	-1,2781	3,8095	0,9999	0,0000	0,0000	1,0000	3,0686	1,0560	0,0000	0,0000	1,0440	-0,0416	-9,8078

that the uninformed investors are overestimating [resp. underestimating] their private information, thereby formulating an excessive [resp. defective] demand for the risky asset, the informed investors weaken [resp. strengthen] their own demand. In G2 and H2 where the informed investors are the small minority, their demand for the risky asset is strongly positively correlated with the private information. The informed investors are not concerned to use the component $\psi_{i,\Pi}$ of their demand to transmit fair signals to the market: they transmit buy [resp. sell] signals as suggested by their private information. However, the uninformed investors use again their estimates of the private information to formulate a strategic response: they buy [resp. sell] as their estimates would suggest to sell [buy]. In addition, also in these cases, the informed investors exploit the uninformed investors' estimation errors to better calibrate their demand. As a matter of fact, these strategic tradings increase the expected utility of both the groups of rational investors. It seems to us worth noting that the early fully strategic Pareto dominant equilibrium candidates occur under exactly the same exogenous parameters used by Wang. However, the dominance of these candidates on the benchmark W is only local in ω. In fact, as ω goes away from 0 [resp. 1], Candidates B1 and E1 [resp. G2 and H2] cease to dominate W. As long as the dominance occurs, the risky asset price is quite strongly negatively correlated with the private information, and it exhibits a stronger discount term for holding the stock and a much higher sensitivity to the stock supply shocks. Moreover, while in B1 and E1 the estimation errors of the small minority of the uninformed investors play little role, in G2 and H2 the risky asset price turns out to be quite strongly negatively correlated with the uninformed investors' estimation errors, as a result of the strategic demand of the large majority of uninformed investors. This can be seen in Figure 5 [resp. 6]. Another relevant feature of Table 6 is the systematic occurrence of Candidate PC[28], which is a perfect competitive equilibrium candidate other than W, Pareto suboptimal to W only when ω is close to 1. In PC, also obtained by our exploitation of genuine Wang's approach, the informed investors restrain the amount of the risky asset they trade on the basis of their private information. As a consequence, their utility lessens while the utility of uninformed investor increases (see Figure 7). It may be also interesting that Candidate H2s, which is suboptimal to H2 in Table 6f, becomes optimal in Tables 6e and 6d. Finally, it is worth noting that in the samples characterized by $\omega = 0.300, 0.400$, not shown here, no other equilibrium candidates than W and PC have appeared, and that samples characterized by $\omega = 0.900, 0.800, 0.700$ have revealed no other equilibrium candidates than those shown in Tables 6f and 6e. While the local dominance of Candidates G2, H2, and H2s on W may be justified on the basis of a strong perception of the market

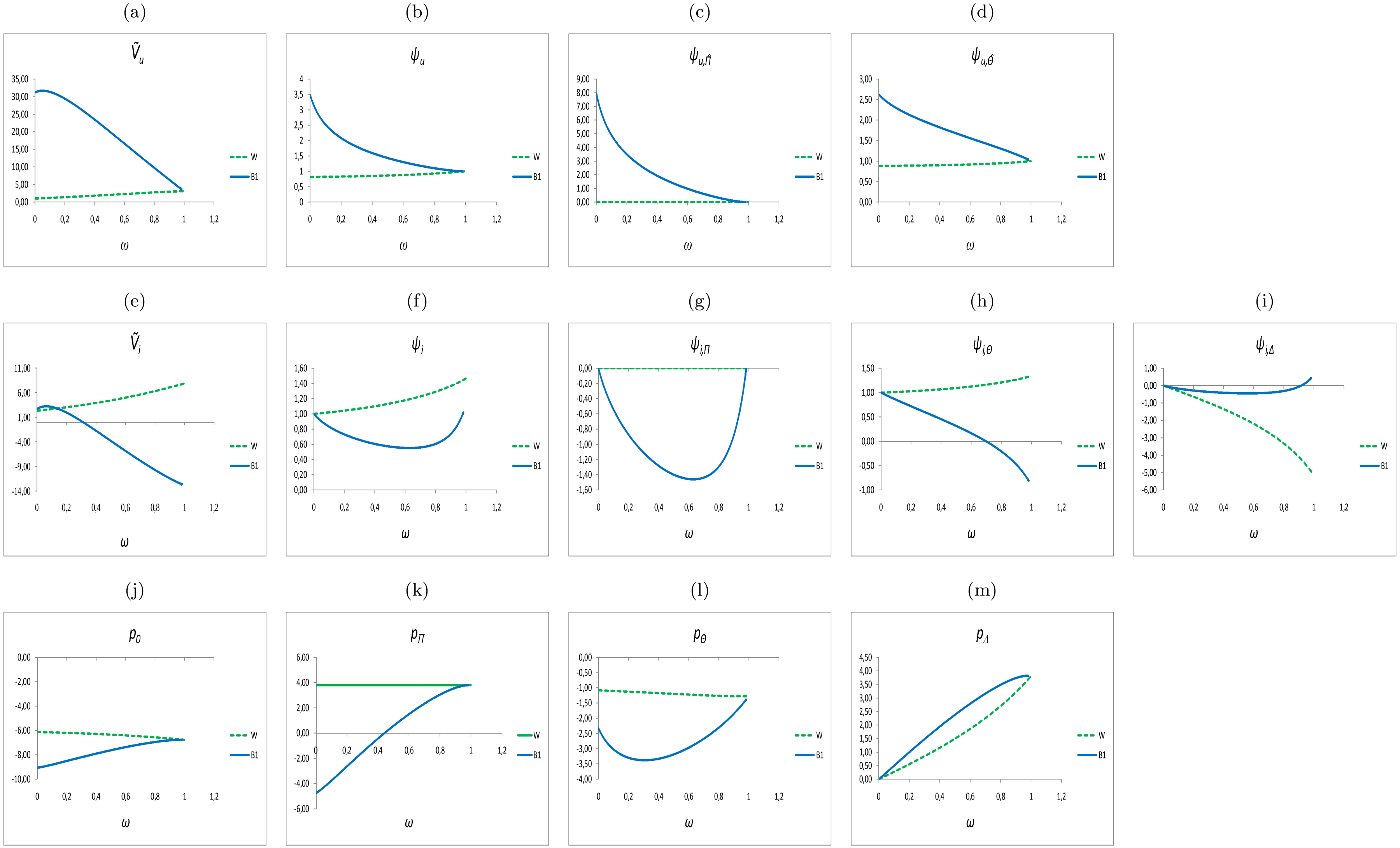
[28]Though not achieved in the sample of Table 6a PC still exists.

risk, due to the presence of a large number of uninformed investors, the local dominance of Candidates B1 and E1 requires some additional consideration. Likewise Candidates A1, A3, A4, A5 of Table 5a, a small minority of uninformed investors may increases their utility by a strategic trading against a large majority of informed investors. However, differently than Candidates A1, A3, A4, A5, a higher perception of market risk leads informed investors to formulate themselves a strategic response. This slightly increases their own expected utility. Still remains the issue of the plausibility of a market model in which a small minority of uninformed investors trades against a large majority of informed investors. Finally, with regard to the perfect competitive equilibrium PC, in general it is rather natural that on the lessening of the quantity of risky asset traded on the basis of the private information the expected utility of the informed investors also lessens, while the expected utility of the uninformed investors increases. The exception when the informed investors are the very small minority in the market might be explained by the possibility that making a very small use of their private information the informed investors transmit a very small piece of information to the market, which ends up with damaging also the uninformed investors. In Tables 7a and 7b] [resp. 7c and 7d] we show a selection of the equilibrium candidates derived via the computational implementation of our BN approach, given $\varphi_u = \varphi_i = 1.000$, $\sigma_\Theta = 3.500$ [resp. $\varphi_u = \varphi_i = 1.500$, $\sigma_\Theta = 3.000$], $\omega = 0.001$ and $\omega = 0.999$. With respect to Table 6 the only new relevant features are: the appearance of the strategic Candidate D1 [resp. D1] which dominates W for all ω; the re-emergence of an alternative perfect competitive Candidate PC which is dominated by W for no ω. More detailed accounts of these findings are shown in Figures 8, 9. Our analysis of many other additional cases has confirmed the main features discussed above.

10 Conclusions

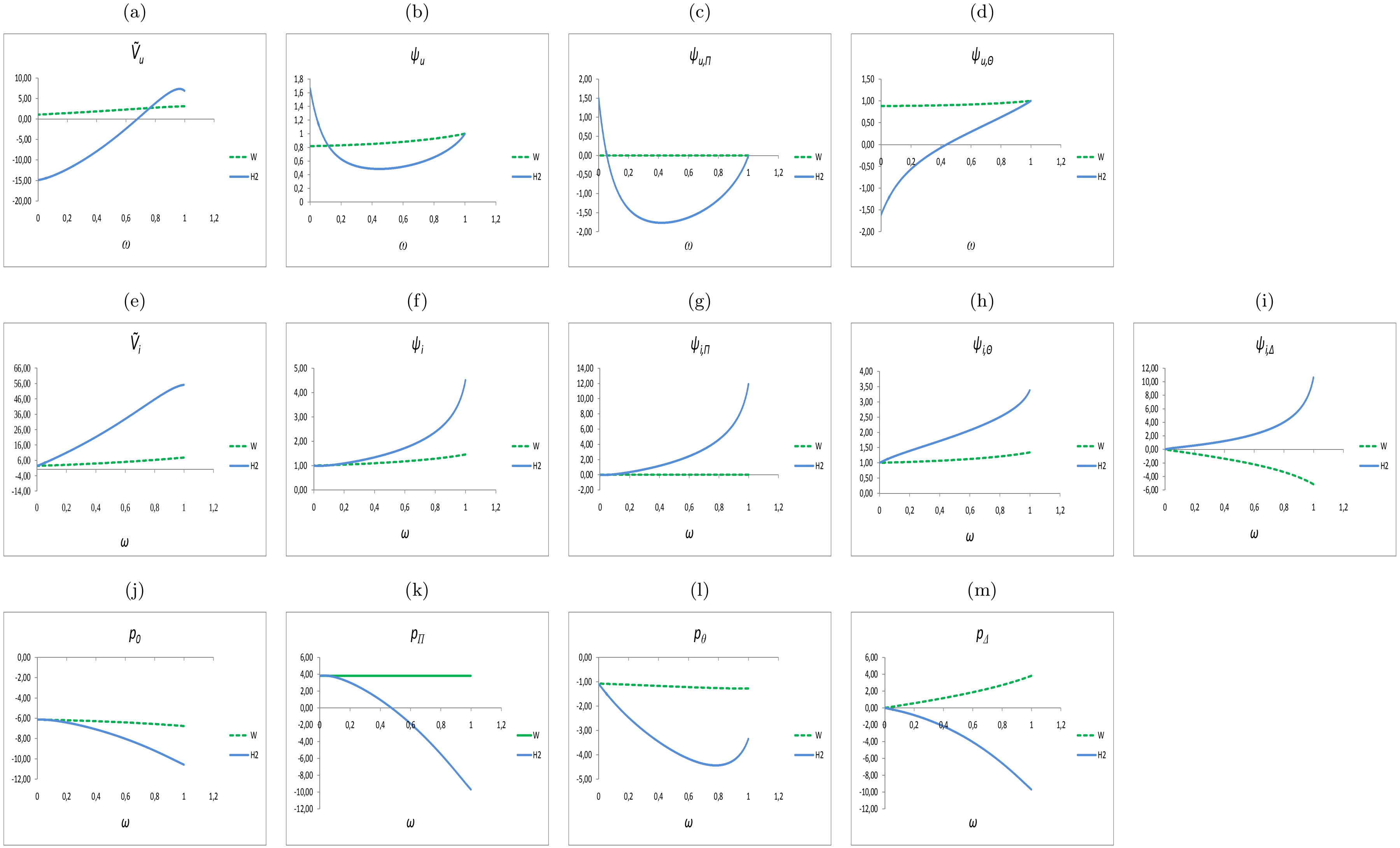
In this paper we have studied the celebrate Wang's model both in a BNE perspective. Our analysis reveals new results that might constitute a non trivial contribution to the theory of economic equilibrium with incomplete financial markets under asymmetric information. First, reproducing the original Wang's REE procedure, we have discovered the existence of Pareto efficient equilibria additional to the one revealed by Wang in his paper. Second, pursuing the BNE approach, computationally supported by a bargain-style computational procedure, we have replied all Wang's RE equilibria and we have also discovered new purely BN equilibria with strategic flavour. We found out that "market risk perception"(as we called it) matters in Pareto ranking the equilibrium candidates. In fact, first in degenerate cases, later in the general ones, we showed how risky assets supply, investor's risk aversion and the parameter

Figure 5: Plot from B1 on varying of ω



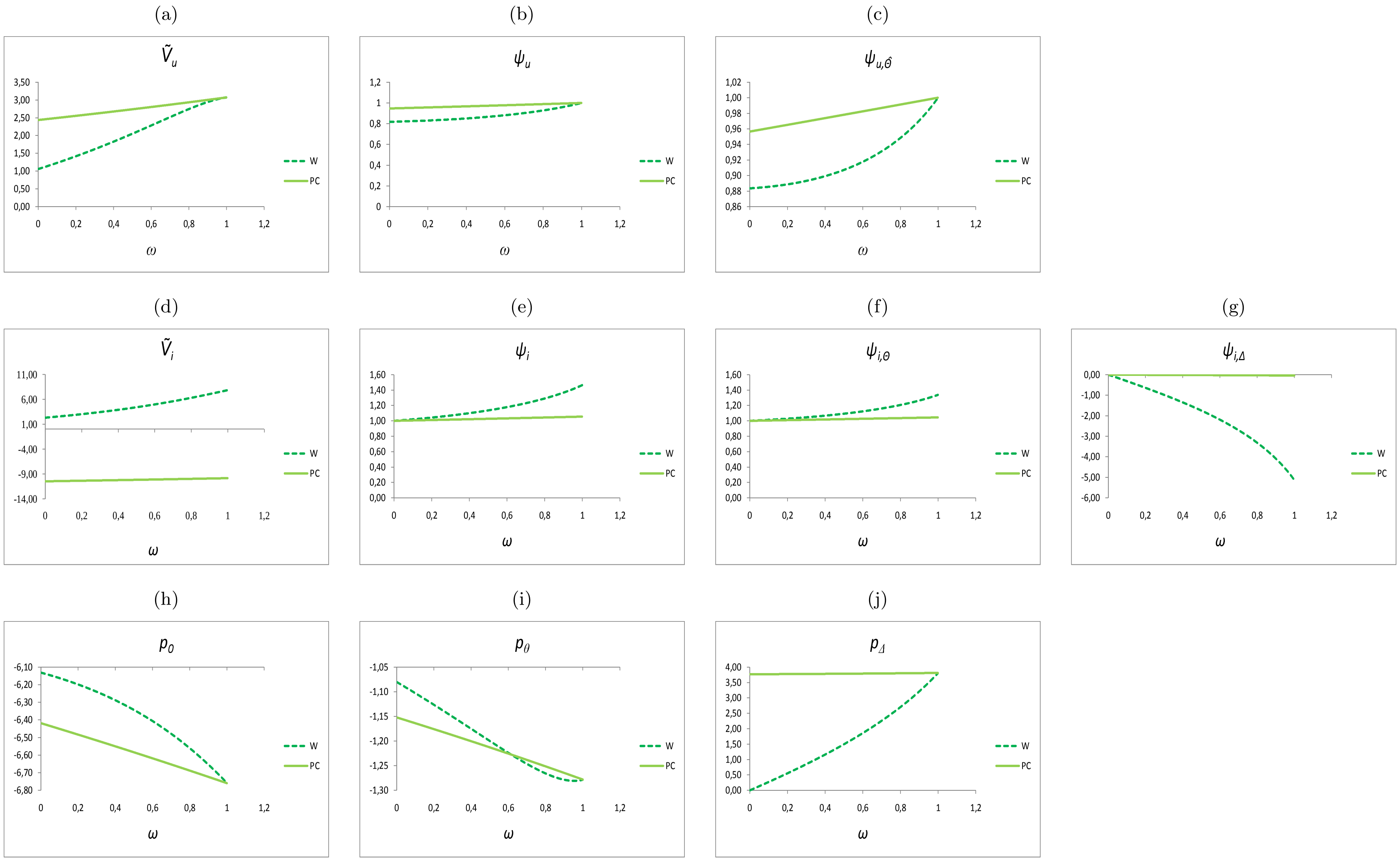
The constant coefficients $\psi_{u,D} = \psi_{i,D} = 0$ and $p_D = 0.95238$ have not plotted.

Figure 6: Plot from H2 on varying of ω



The constant coefficients $\psi_{u,D} = \psi_{i,D} = 0$ and $p_D = 0.95238$ have not plotted.

Figure 7: Plot from PC on varying of ω



The constant coefficients $\psi_{u,D} = \psi_{u,\hat{\Pi}} = \psi_{i,D} = \psi_{i,\Pi} = 0$ and $p_D = 0.95238$, $p_\Pi = 3.80952$ have not plotted.

Table 7: BN approach.

(a) $\varphi_u = \varphi_i = 1.000, \quad \sigma_\Theta = 3.500, \quad \omega = 0.001.$

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
B1	10			-9,0675	0,9524	-4,7511	-3,5266	0,0048	2,2949	0,0000	4,2731	2,0177	43,3959	0,9987	0,0000	-0,0043	0,9990	-0,0016	13,0913
C1	5			-9,0681	0,9457	-4,7435	-3,5257	0,0141	1,9620	-2,5414	6,8196	1,6911	19,1583	0,9990	0,0025	-0,0068	0,9993	-0,0045	13,0876
D1	6			-9,0714	0,9524	-4,7651	-3,5249	-0,0064	0,7470	0,0000	-1,0217	0,7787	7,4701	1,0003	0,0000	0,0010	1,0002	0,0021	13,1152
W	422			-6,1319	0,9524	3,8095	-1,4215	0,0027	0,8216	0,0000	0,0000	0,8756	3,7540	1,0002	0,0000	0,0000	1,0001	-0,0023	5,5244

(b) $\varphi_u = \varphi_i = 1.000, \quad \sigma_\Theta = 3.500, \quad \omega = 0.999.$

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
D0	1			-10,5605	0,9524	-9,7169	-4,8877	-9,7168	0,9999	0,0000	0,0000	1,0000	21,2847	1,0565	0,0000	0,0186	1,0386	0,0365	9,3182
L1	3			-10,5600	0,9479	-9,7080	-4,8873	-9,7064	0,9998	0,0012	-0,0022	0,9998	21,2793	1,1777	-1,1566	2,2295	1,1558	2,5575	5,3757
D1	2			-10,5592	0,9524	-9,7101	-4,8869	-9,7033	0,9997	0,0000	-0,0014	0,9997	21,2729	1,3476	0,0000	1,4436	1,3021	2,8309	32,3420
L2	5			-10,5563	0,9358	-9,6777	-4,8879	-9,6777	0,9988	0,0042	-0,0102	0,9990	21,2571	2,1703	-4,2409	10,1755	1,9808	10,1649	21,9185
L3	63			-10,5537	0,9522	-9,6940	-4,8908	-9,6995	0,9981	0,0000	-0,0064	0,9985	21,2535	2,9229	-0,0340	6,3855	2,5142	5,2648	54,5548
D2	37			-10,5537	0,9524	-9,6942	-4,8908	-9,6994	0,9981	0,0000	-0,0063	0,9985	21,2534	2,9203	0,0000	6,3388	2,5127	5,2611	75,0649
D2s	9			-10,5541	0,9524	-9,6943	-4,8898	-9,6946	0,9982	0,0000	-0,0060	0,9986	21,2522	2,7520	0,0000	6,0263	2,3991	5,9573	58,9648
PC	21			-6,7588	0,9524	3,8095	-1,7653	3,8094	0,9999	0,0000	0,0000	1,0000	7,3996	1,0574	0,0000	0,0000	1,0393	-0,0586	-5,3044
W	121			-6,7582	0,9524	3,8095	-1,7653	3,8038	0,9997	0,0000	0,0000	0,9998	7,3980	1,3122	0,0000	0,0000	1,2429	-3,2498	11,8204

(c) $\varphi_u = \varphi_i = 1.500, \quad \sigma_\Theta = 3.000, \quad \omega = 0.001.$

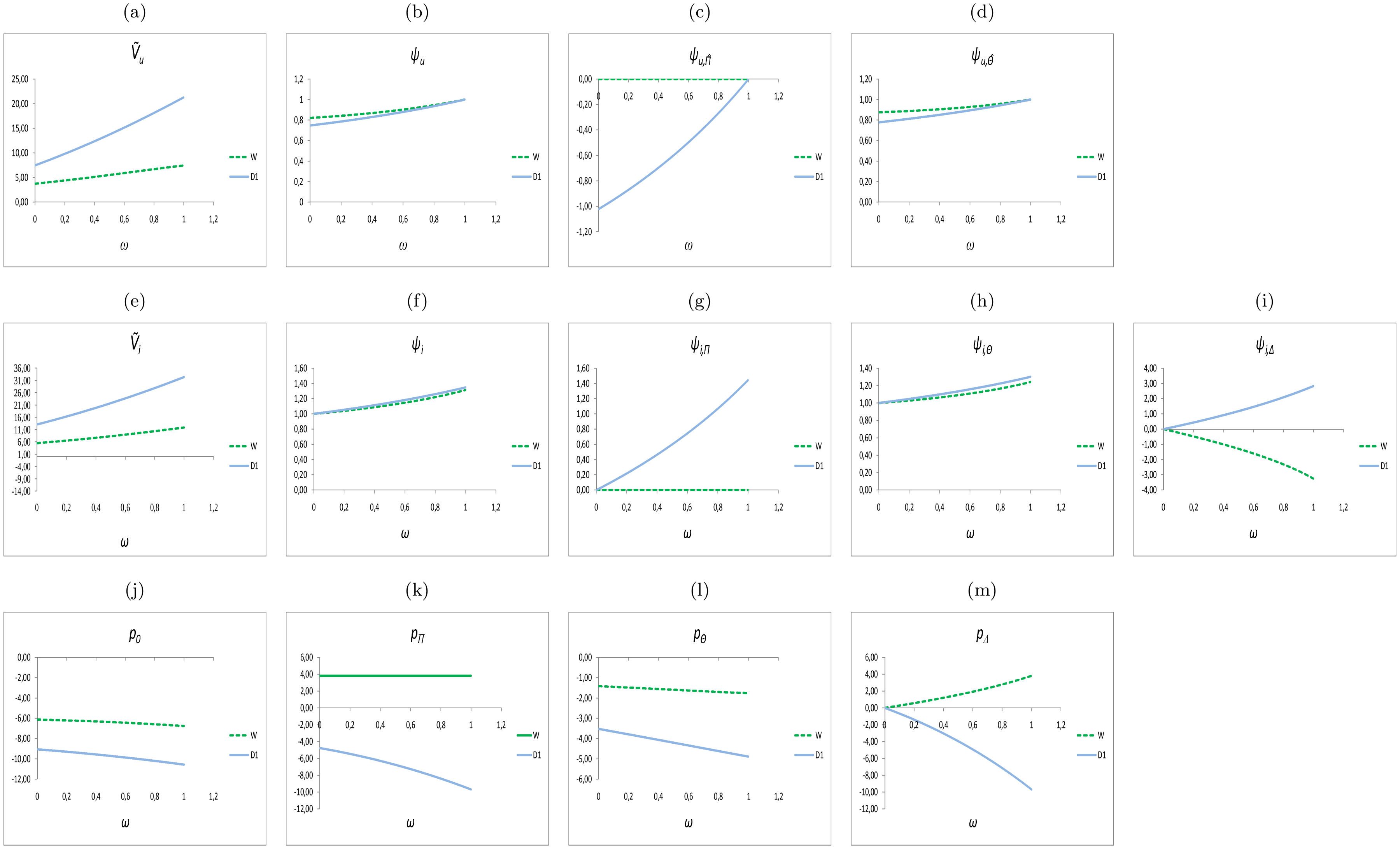
	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
A1	3	0		-14,2739	-2,9121	-1,6581	-9,0117	0,0046	1,8039	0,8114	0,8569	1,7416	104,7759	0,9992	-0,0008	-0,0009	0,9993	-0,0005	31,6125
B1	8	4		-13,6008	0,9524	-4,7503	-8,3456	0,0049	1,6730	0,0000	1,5490	1,6159	89,0703	0,9993	0,0000	-0,0016	0,9994	-0,0006	47,6246
C1	2	0		-13,6022	0,9419	-4,7383	-8,3464	0,0184	1,4628	-1,4248	3,1352	1,4034	59,5667	0,9995	0,0014	-0,0031	0,9996	-0,0023	47,6366
D1	2	1		-13,6073	0,9524	-4,7657	-8,3501	-0,0075	0,8078	0,0000	-0,4817	0,8252	37,1685	1,0002	0,0000	0,0005	1,0002	0,0010	47,7043
W	403	632		-9,1980	0,9524	3,8095	-4,0750	0,0030	0,8649	0,0000	0,0000	0,8925	20,0709	1,0001	0,0000	0,0000	1,0001	-0,0008	23,4881

(d) $\varphi_u = \varphi_i = 1.500, \quad \sigma_\Theta = 3.000, \quad \omega = 0.999.$

	n	n_1	n_2	p	p_D	p_Π	p_Θ	p_Δ	ψ_u	$\psi_{u,D}$	$\psi_{u,\hat\Pi}$	$\psi_{u,\hat\Theta}$	$\tilde V_u$	ψ_i	$\psi_{i,D}$	$\psi_{i,\Pi}$	$\psi_{i,\Theta}$	$\psi_{i,\Delta}$	$\tilde V_i$
N1	2	0		-15,8400	0,9480	-9,7081	-10,5622	-9,7064	0,9999	0,0005	-0,0009	0,9999	62,6347	1,1378	-0,4582	0,9186	1,1228	1,0827	50,9763
D1	2	4		-15,8390	0,9524	-9,7108	-10,5613	-9,7047	0,9998	0,0000	-0,0006	0,9998	62,6224	1,2345	0,0000	0,5934	1,2129	1,1636	79,1352
D2	3	0		-15,8313	0,9524	-9,6943	-10,5570	-9,6946	0,9989	0,0000	-0,0024	0,9991	62,5379	2,0538	0,0000	2,3667	1,9414	2,3389	123,9548
O1	48	2		-15,8303	0,9532	-9,6947	-10,5569	-9,6995	0,9988	-0,0001	-0,0024	0,9990	62,5294	2,1600	0,0862	2,3716	2,0295	1,9177	122,6877
D3	25	49		-15,8303	0,9524	-9,6937	-10,5569	-9,6996	0,9988	0,0000	-0,0025	0,9990	62,5289	2,1694	0,0000	2,4887	2,0369	1,9250	143,5175
N2	1	0		-17,4621	23,5765	-46,6497	-12,1751	-46,6496	0,9998	0,0025	-0,0040	0,9998	53,0386	1,2317	-2,4524	4,0141	1,2139	4,0243	79,9078
N3	2	0		-17,4498	23,5623	-46,6020	-12,1692	-46,6095	0,9986	0,0043	-0,0094	0,9988	52,9053	2,3888	-4,3436	9,3477	2,2319	8,7359	181,3185
PC	26	6		-10,1381	0,9524	3,8095	-4,9626	3,8094	0,9999	0,0000	0,0000	1,0000	29,4022	1,0563	0,0000	0,0000	1,0389	-0,0299	17,5884
W	140	661		-10,1375	0,9524	3,8095	-4,9621	3,8048	0,9998	0,0000	0,0000	0,9999	29,3958	1,1669	0,0000	0,0000	1,1395	-0,9537	34,8972

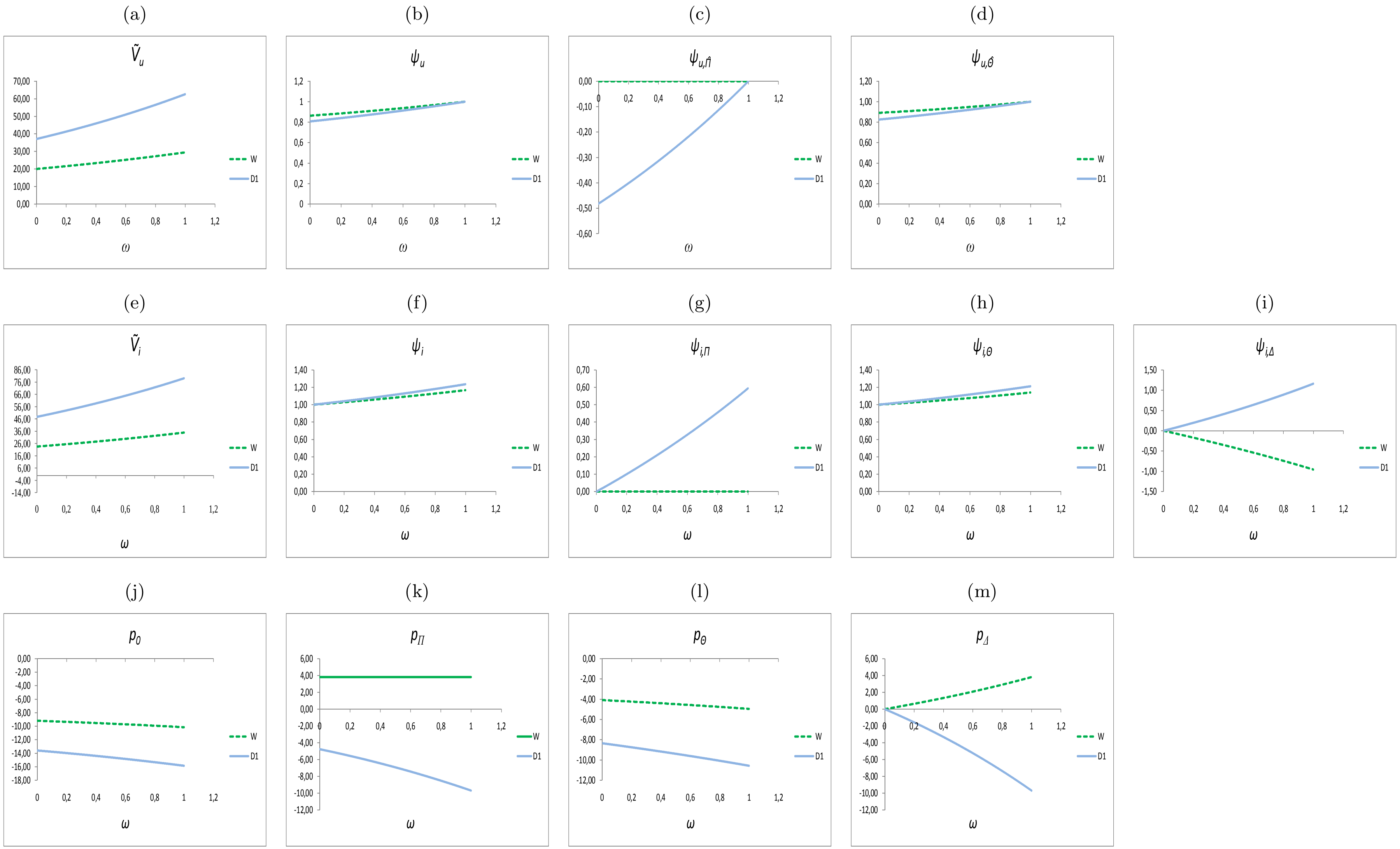
The existence of an equilibrium candidate which Pareto efficiently dominates W could be shown also in case $\varphi_u = \varphi_i = 1.500, \sigma_\Theta = 2.500$. However, the achievement of this candidate via our BN procedures seems to be little frequent.

Figure 8: Plot from D1 of Table (7a), (7b) on varying of ω



The constant coefficients $\psi_{u,D} = \psi_{i,D} = 0$ and $p_D = 0.95238$ have not plotted.

Figure 9: Plot from D1 of Table (7c), (7d) on varying of ω



The constant coefficients $\psi_{u,D} = \psi_{i,D} = 0$ and $p_D = 0.95238$ have not plotted.

of informationally asymmetry ω (or the proportion of different investors) take an important role in inducing investors to behave rationally or strategically. In fact, while under "low" risky asset supply volatility and investors' risk aversion both Wang's approach and ours lead to the same equilibria, under "high" risky asset supply volatility or investors' risk aversion, equilibrium candidates occur in which the investors of the two groups trading both strategically achieve both a higher utility. The economical interpretation seems to us intriguing: as rational investors' perception of market risk is low, then they trade as perfect competitors and consequently informationally efficient equilibria are achieved, but as rational investors' perception of market risk is high, they prefer to cheat their competitors trading strategically consequently leading to informationally inefficient equilibria.

Appendix

Proof of Proposition 1 The uninformed investors' filtering problem can be managed more easily by introducing a suitable matrix notation. Setting $X(t) \equiv (\Pi(t), \Theta(t))^T$ for the vector of the unobservable processes and $Y(t) \equiv (D(t), S(t))^T$ for the vector of the signals, we can write

$$\begin{aligned} dX(t) &= A_{X,X}X(t)dt + A_{X,Y}Y(t)dt + Q^{1/2}_{X,X}dw_X(t) + Q^{1/2}_{X,Y}dw_Y(t), \\ dY(t) &= A_{Y,X}X(t)dt + A_{Y,Y}Y(t)dt + Q^{1/2}_{Y,X}dw_X(t) + Q^{1/2}_{Y,Y}dw_Y(t), \end{aligned}$$

where

$$A_{X,X} \equiv \begin{pmatrix} -\alpha_\Pi & 0 \\ 0 & -\alpha_\Theta \end{pmatrix}, \quad A_{X,Y} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^{1/2}_{X,X} \equiv \begin{pmatrix} \sigma_\Pi & 0 \\ 0 & \sigma_\Theta \end{pmatrix}, \quad Q^{1/2}_{X,Y} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

$$A_{Y,X} \equiv \begin{pmatrix} 1 & 0 \\ -\alpha_\Pi p_\Pi & -\alpha_\Theta p_\Theta \end{pmatrix}, \quad A_{Y,Y} \equiv \begin{pmatrix} -\alpha_D & 0 \\ 0 & 0 \end{pmatrix}, \quad Q^{1/2}_{Y,X} \equiv \begin{pmatrix} \sigma_{D,\Pi} & 0 \\ \sigma_\Pi p_\Pi & \sigma_\Theta p_\Theta \end{pmatrix}, \quad Q^{1/2}_{Y,Y} \equiv \begin{pmatrix} \sigma_{D,D} \\ 0 \end{pmatrix},$$

and

$$w_X(t) \equiv (w_\Pi(t), w_\Theta(t))^T, \qquad w_Y(t) \equiv w_D(t).$$

Therefore, in terms of Liptser & Shiryayev's notations, we have

$$b \circ b \equiv Q_{X,X}^{1/2} \left(Q_{X,X}^{1/2}\right)^T + Q_{X,Y}^{1/2} \left(Q_{X,Y}^{1/2}\right)^T = \begin{pmatrix} \sigma_\Pi^2 & 0 \\ 0 & \sigma_\Theta^2 \end{pmatrix},$$

$$b \circ B \equiv Q_{X,X}^{1/2} \left(Q_{Y,X}^{1/2}\right)^T + Q_{X,Y}^{1/2} \left(Q_{Y,Y}^{1/2}\right)^T = \begin{pmatrix} \sigma_{D,\Pi}\sigma_\Pi & \sigma_\Pi^2 p_\Pi \\ 0 & \sigma_\Theta^2 p_\Theta \end{pmatrix},$$

$$B \circ B \equiv Q_{Y,X}^{1/2} \left(Q_{Y,X}^{1/2}\right)^T + Q_{Y,Y}^{1/2} \left(Q_{Y,Y}^{1/2}\right)^T = \begin{pmatrix} \sigma_D^2 & \sigma_{D,\Pi}\sigma_\Pi p_\Pi \\ \sigma_{D,\Pi}\sigma_\Pi p_\Pi & \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 \end{pmatrix}$$

Now, under the non-degeneracy assumption $\det(B \circ B) = \sigma_{D,D}^2 \sigma_\Pi^2 p_\Pi^2 + \sigma_D^2 \sigma_\Theta^2 p_\Theta^2 \neq 0$, there exists

$$(B \circ B)^{-1} = \frac{1}{\sigma_{D,D}^2 \sigma_\Pi^2 p_\Pi^2 + \sigma_D^2 \sigma_\Theta^2 p_\Theta^2} \begin{pmatrix} \sigma_\Pi^2 p_\Pi^2 + \sigma_\Theta^2 p_\Theta^2 & -\sigma_{D,\Pi}\sigma_\Pi p_\Pi \\ -\sigma_{D,\Pi}\sigma_\Pi p_\Pi & \sigma_D^2 \end{pmatrix}.$$

Hence, setting

$$\Sigma(t) \equiv \begin{pmatrix} \sigma_{1,1}(t) & \sigma_{1,2}(t) \\ \sigma_{1,2}(t) & \sigma_{2,2}(t) \end{pmatrix} \equiv \begin{pmatrix} \mathbf{E}[(\Pi(t) - \hat{\Pi}(t))^2] & \mathbf{E}[(\Pi(t) - \hat{\Pi}(t))(\Theta(t) - \hat{\Theta}(t))] \\ \mathbf{E}[(\Pi(t) - \hat{\Pi}(t))(\Theta(t) - \hat{\Theta}(t))] & \mathbf{E}[(\Theta(t) - \hat{\Theta}(t))^2] \end{pmatrix},$$

we are in a position to apply Liptser & Shiryayev [27, Vol. I, Thm 10.3, p. 392] and write the dynamics for $\hat{X}(t) \equiv \mathbf{E}[\, X(t) \,|\mathfrak{F}_u(t)]$, obtaining

$$\begin{aligned} d\hat{X}(t) &= A_{X,X}\hat{X}(t)dt + A_{X,Y}Y(t)dt \\ &\quad + \left(b \circ B + \Sigma(t)A_{Y,X}^T\right)(B \circ B)^{-1}\left(dY(t) - A_{Y,X}\hat{X}(t)dt - A_{Y,Y}Y(t)dt\right) \end{aligned} \tag{102}$$

and

$$\begin{aligned} \dot{\Sigma}(t) &= A_{X,X}\Sigma(t) + \Sigma(t)A_{X,X}^T + b \circ b \\ &\quad - \left(b \circ B + \Sigma(t)A_{Y,X}^T\right)(B \circ B)^{-1}\left(b \circ B + \Sigma(t)A_{Y,X}^T\right)^T. \end{aligned} \tag{103}$$

On the other hand, since the process $S(t) \equiv p_\Pi \Pi(t) + p_\Theta \Theta(t)$ is observed, we have

$$p_\Theta(\Theta(t) - \hat{\Theta}(t)) = -p_\Pi(\Pi(t) - \hat{\Pi}(t)).$$

Assuming $p_\Theta \neq 0$, it then follows

$$\sigma_{2,2}(t) = \frac{p_\Pi^2}{p_\Theta^2}\mathbf{E}[(\Pi(t)-\hat{\Pi}(t))^2] \equiv \frac{p_\Pi^2}{p_\Theta^2}\sigma_{1,1}(t), \quad \text{and} \quad \sigma_{1,2}(t) = -\frac{p_\Pi}{p_\Theta}\mathbf{E}[(\Pi(t)-\hat{\Pi}(t))^2] \equiv -\frac{p_\Pi}{p_\Theta}\sigma_{1,1}(t).$$

Therefore, setting $\sigma_{1,1}(t) \equiv \sigma(t)$, we are led to seek $\Sigma(t)$ in the form

$$\begin{pmatrix} \sigma(t) & -\frac{p_\Pi}{p_\Theta}\sigma(t) \\ -\frac{p_\Pi}{p_\Theta}\sigma(t) & \frac{p_\Pi^2}{p_\Theta^2}\sigma(t) \end{pmatrix}.$$

It then follows,

$$A_{X,X}\Sigma(t) + \Sigma(t)A_{X,X}^T + b\circ b = \begin{pmatrix} \sigma_\Pi^2 - 2\alpha_\Pi\sigma(t) & (\alpha_\Pi+\alpha_\Theta)\frac{p_\Pi}{p_\Theta}\sigma(t) \\ (\alpha_\Pi+\alpha_\Theta)\frac{p_\Pi}{p_\Theta}\sigma(t) & \sigma_\Theta^2 - 2\alpha_\Theta\frac{p_\Pi^2}{p_\Theta^2}\sigma(t) \end{pmatrix}$$

and

$$\left(b\circ B + \Sigma(t)A_{Y,X}^T\right)(B\circ B)^{-1}\left(b\circ B + \Sigma(t)A_{Y,X}^T\right)^T = \frac{1}{\sigma_{D,D}^2\sigma_\Pi^2 p_\Pi^2 + \sigma_D^2\sigma_\Theta^2 p_\Theta^2}\begin{pmatrix} \tilde{\sigma}_{1,1}(t) & \tilde{\sigma}_{1,2}(t) \\ \tilde{\sigma}_{2,1}(t) & \tilde{\sigma}_{2,2}(t) \end{pmatrix},$$

where

$$\tilde{\sigma}_{1,1}(t) = \left(\sigma_{D,D}^2\left((\alpha_\Pi-\alpha_\Theta)\sigma(t)-\sigma_\Pi^2\right)^2 + \left(\sigma_\Pi + (\alpha_\Pi-\alpha_\Theta)\sigma_{D,\Pi}\right)^2\sigma^2(t)\right)p_\Pi^2 + \sigma_\Theta^2\left(\sigma_{D,\Pi}\sigma_\Pi + \sigma(t)\right)^2 p_\Theta^2$$

$$\begin{aligned}\tilde{\sigma}_{1,2}(t) = &\left((\alpha_\Pi-\alpha_\Theta)\sigma_{D,D}^2\sigma_\Pi^2 - \left((\alpha_\Pi-\alpha_\Theta)^2\sigma_{D,D}^2 + ((\alpha_\Pi-\alpha_\Theta)\sigma_{D,\Pi}+\sigma_\Pi)^2\right)\sigma(t)\right)\frac{p_\Pi^3}{p_\Theta}\sigma(t) \\ &- \sigma_\Theta^2\left((\sigma_{D,\Pi}\sigma_\Pi+\sigma(t))^2 + (\alpha_\Pi-\alpha_\Theta)\sigma_D^2\sigma(t) - \sigma_D^2\sigma_\Pi^2\right)p_\Pi p_\Theta\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{2,2}(t) = &\,\sigma_D^2\sigma_\Theta^4 p_\Theta^2 + 2\left((\alpha_\Pi-\alpha_\Theta)\sigma_D^2 + \sigma_\Pi\sigma_{D,\Pi}\right)\sigma_\Theta^2 p_\Pi^2\sigma(t) \\ &+ \left(\sigma_\Theta^2 p_\Pi^2 + \left((\alpha_\Pi-\alpha_\Theta)^2\sigma_{D,D}^2 + ((\alpha_\Pi-\alpha_\Theta)\sigma_{D,\Pi}+\sigma_\Pi)^2\right)\frac{p_\Pi^4}{p_\Theta^2}\right)\sigma^2(t)\end{aligned}$$

Therefore, in terms of the shorthand (), Equation (103) yields the single ordinary differential equation for $\sigma(t)$

$$\dot{\sigma}(t) = -a^2\sigma^2(t) - 2b\sigma(t) + c^2, \tag{104}$$

whose general integral is given by (22). Moreover, (104) admits also a stationary positive solution $\sigma(t) \equiv \sigma$ characterized by

$$a^2\sigma^2 + 2b\sigma - c^2 = 0.$$

This completes the proof. $\square$

Proof of Proposition 2 Thanks to the Verification Theorem (see [12, Th 3.1, p.163; Th 5.1, p.172], see also [11, Th. 4.1, p. 159]), to show that (31) is the uninformed investor's objective function (28) we need to prove that

(i) the function (31) is a solution of the Bellman equation

$$\partial_t V(t, Z_u, W_u) + \max_{\Psi_u, c_u} \left\{ \mathcal{G}_u V(t, Z_u, W_u) - e^{-(\rho_u t + \varphi_u c_u)} \right\} = 0, \tag{105}$$

where $\mathcal{G}_u$ is the infinitesimal generator of the diffusion process $(Z_u(t), W_u(t))$;

(ii) the control $(\mathring{\Psi}_u(t), \mathring{c}_u(t))$ satisfies

$$(\mathring{\Psi}_u(t), \mathring{c}_u(t)) \in \arg\max\{\mathcal{G}_u V(t, \mathring{Z}_u(t), \mathring{W}_u(t)) - e^{-(\rho_u t + \varphi_u \mathring{c}_u(t))}\} = 0,$$

where $(\mathring{Z}_u(t), \mathring{W}_u(t))$ is a solution of (29), (30) corresponding to the choice of $(\mathring{\Psi}_u(t), \mathring{c}_u(t))$;

(ii) the transversality condition

$$\lim_{T\to+\infty} \mathbf{E}_{t,Z_u,W_u} \left[V(t+T, \mathring{Z}_u(t+T), \mathring{W}_u(t+T)) \right] = 0, \tag{106}$$

where $(\mathring{Z}_u(t), \mathring{W}_u(t))$ is a solution of (29), (30) given $(\mathring{\Psi}_u(t), \mathring{c}_u(t))$, holds true.

Suppressing the index u in the remainder of the proof, to show that $V(t, Z, W) = -e^{-(\rho t + \frac{1}{2} Z^T G Z + \varphi r W + \gamma)}$ is a solution of (105), we start to determine the operator $\mathcal{G}$. A straightforward computation yields

$$\mathcal{G} \equiv \frac{1}{2} \sum_{j,k=1}^{5} Q_{j,k} \partial^2_{Z_j,Z_k} + \Psi \sum_{j=1}^{5} R^{1/2} \left(Q^{1/2} \right)^T_j \partial^2_{W,Z_j} + \frac{1}{2} \Psi^2 R \partial^2_{W,W} \tag{107}$$
$$+ \sum_{j=1}^{5} (AZ)_j \, \partial_{Z_j} + (rW - c - \Psi B Z) \, \partial_W.$$

On the other hand, using V as a shorthand for $V(t, Z, W)$, we have

$$
\begin{aligned}
\partial_{Z_j} V &= -(Z^T G)_j \, V, \\
\partial_W V &= -r\varphi \, V, \\
\partial^2_{Z_j, Z_k} V &= \left(GZZ^T G - G\right)_{j,k} \, V, \\
\partial^2_{Z_j, W} V &= r\varphi (Z^T G)_j \, V, \\
\partial^2_{W,W} V &= r^2 \varphi^2 \, V.
\end{aligned}
$$

Therefore, we can write

$$
\begin{aligned}
\mathcal{G}V = & \frac{1}{2} \left(\sum_{j,k=1}^{5} Q_{j,k} \left(GZZ^T G - G\right)_{j,k} \right) V + \frac{1}{2} r^2 \varphi^2 R \Psi^2 \, V \\
& + r\varphi \left(\sum_{j=1}^{5} \left(R^{1/2} \left(Q^{1/2}\right)^T \right)_j (Z^T G)_j \right) \Psi V \\
& - \left(\sum_{j=1}^{5} (AZ)_j (Z^T G)_j \right) V - r\varphi \left(rW - c - BZ\Psi \right) \, V.
\end{aligned} \tag{108}
$$

Now, thanks to the properties of the trace functional, we have

$$
\begin{aligned}
\sum_{j,k=1}^{5} Q_{j,k} \left(GZZ^T G - G\right)_{j,k} \, V &= \mathbf{tr} \left(\left(Q^{1/2}\right)^T \left(GZZ^T G - G\right) Q^{1/2} \right) \\
&= Z^T GQGZ - \mathbf{tr} \left(\left(Q^{1/2}\right)^T GQ^{1/2} \right).
\end{aligned} \tag{109}
$$

Moreover,

$$
\sum_{j=1}^{5} \left(R^{1/2} \left(Q^{1/2}\right)^T \right)_j (Z^T G)_j = R^{1/2} \left(Q^{1/2}\right)^T GZ, \tag{110}
$$

and

$$
\sum_{j=1}^{5} (AZ)_j (Z^T G)_j = Z^T GAZ. \tag{111}
$$

Hence, combining (108) with (109)-(111), it follows

$$\begin{aligned}\mathcal{G}V = &\frac{1}{2}Z^T GQGZ - \frac{1}{2}\mathbf{tr}\left(\left(Q^{1/2}\right)^T GQ_u^{1/2}\right)V \\ &+ r\varphi R^{1/2}\left(Q^{1/2}\right)^T GZ\,\Psi V + \frac{1}{2}r^2\varphi^2 R\,\Psi^2 V \\ &- Z^T GAZ\,V - r\varphi\left(rW - c - BZ\Psi\right)V.\end{aligned} \tag{112}$$

The latter, on account of

$$\partial_t V = -\rho V, \quad \text{and} \quad Z^T GAZ = \frac{1}{2}(Z^T A^T GZ + Z^T GAZ),$$

allows us to rewrite Equation (105) in the form

$$\begin{aligned}&\left(-\rho + \frac{1}{2}Z^T GQGZ - \frac{1}{2}(Z^T A^T GZ + Z^T GAZ) - \frac{1}{2}\mathbf{tr}\left(\left(Q^{1/2}\right)^T GQ_u^{1/2}\right) - \varphi r^2 W\right)V \\ &+ \max_{\Psi,c}\{r\varphi((R^{1/2}\left(Q^{1/2}\right)^T G + B)Z\Psi + \frac{1}{2}r\varphi R\Psi^2)V + \varphi rcV - e^{-(\rho t+\varphi c)}\} \\ &= 0.\end{aligned} \tag{113}$$

Therefore, setting

$$J \equiv J(t, Z, W, \Psi) \equiv ((R^{1/2}\left(Q^{1/2}\right)^T G + B)Z\Psi + \frac{1}{2}r\varphi R\Psi^2)V,$$

and

$$K \equiv K(t, Z, W, c) \equiv r\varphi cV - e^{-(\rho t+\varphi c)},$$

we can write

$$\begin{aligned}&\max_{\Psi,c}\left\{r\varphi((R^{1/2}\left(Q^{1/2}\right)^T G + B)Z\Psi + \frac{1}{2}r\varphi R\Psi^2)V + \varphi rcV - e^{-(\rho t+\varphi c)}\right\} \\ &= r\varphi\max_{\Psi}\{J(t, Z, W, \Psi)\} + \max_{c}\{K(t, Z, W, c)\}.\end{aligned}$$

Now, we have

$$\frac{dJ}{d\Psi} = ((R^{1/2}\left(Q^{1/2}\right)^T G + B)Z + r\varphi\Psi R)V, \quad \text{and} \quad \frac{d^2 J}{d\Psi^2} = r\varphi RV.$$

Hence, maximizing J [resp. K] with respect to Ψ [resp. c], the first order condition yields the desired (35) [resp. (36)]. Moreover, the second order condition $r\varphi RV \leq 0$ [resp. $-e^{-\varphi c} \leq 0$] guarantees that Ψ [resp.c] is optimal for J [resp. K]. As a consequence,

$$\max_{\Psi}\{J(t,Z,W,\Psi)\} = -\frac{1}{2r\varphi R}\left(Z^T\left(GQ^{1/2}(R^{1/2})^T + B^T\right)\left(R^{1/2}\left(Q^{1/2}\right)^T G + B\right)Z\right)V$$

and

$$\max_{c}\{K(t,Z,W,c)\} = r\left(\frac{1}{2}Z^T GZ + r\varphi W + \gamma - \log(r) + 1\right)V.$$

What shown above implies that the Bellman equation (113) takes the form

$$\begin{aligned}
&\left(-\rho + \frac{1}{2}Z^T GQGZ - \frac{1}{2}(Z^T A^T GZ + Z^T GAZ) - \frac{1}{2}\mathbf{tr}\left(\left(Q^{1/2}\right)^T GQ_u^{1/2}\right) - \varphi r^2 W\right)V \\
&- \frac{1}{2R}\left(Z^T(GQ^{1/2}(R^{1/2})^T + B^T\right)\left(R^{1/2}\left(Q^{1/2}\right)^T G + B\right)ZV \\
&+ r\left(\frac{1}{2}Z^T GZ + r\varphi W + \gamma - \log(r) + 1\right)V \\
&= 0,
\end{aligned}$$

that is

$$\begin{aligned}
&\frac{1}{2}Z^T\left(GQG - \frac{1}{R}\left(G^T Q^{1/2}(R^{1/2})^T + B^T\right)\left(R^{1/2}\left(Q^{1/2}\right)^T G + B\right) - A^T G - GA + rG\right)ZV \qquad (114)\\
&+ \left(r\gamma + r(1-\log(r)) - \rho - \frac{1}{2}\mathbf{tr}\left(\left(Q^{1/2}\right)^T GQ^{1/2}\right)\right)V \\
&= 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&GQG - \frac{1}{R}\left(GQ^{1/2}(R^{1/2})^T + B^T\right)\left(R^{1/2}\left(Q^{1/2}\right)^T G + B\right) - A^T G - GA + rG \qquad (115)\\
&= \frac{1}{R}\left(GQ^{1/2}\left(RI_2 - \left(R^{1/2}\right)^T R^{1/2}\right)\left(Q^{1/2}\right)^T G\right) \\
&- \frac{1}{R}\left(G\left(Q^{1/2}\left(R^{1/2}\right)^T B + R\left(A - \frac{1}{2}rI_5\right)\right) + \left(B^T R^{1/2}\left(Q^{1/2}\right)^T + R\left(A^T - \frac{1}{2}rI_5\right)G\right) + B^T B\right).
\end{aligned}$$

Therefore, combining (114) with (115), it follows that $V(t,Z,W)$ is a solution of the Bellman equation (105), provided that the matrix G and the parameter γ are chosen to fulfill (32) and (34), respectively.

We are left with proving that the transversality condition (106) holds true. To this goal, we apply Itô's formula to the identity

$$
\begin{aligned}
&V(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t)) - V(t, \mathring{Z}(t), \mathring{W}(t)) \\
&\quad = \int_t^{t+\Delta t} dV(s, \mathring{Z}(s), \mathring{W}(s))
\end{aligned}
$$

which allows to write

$$
\begin{aligned}
&V(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t)) - V(t, \mathring{Z}(t), \mathring{W}(t)) \\
&\quad = \int_t^{t+\Delta t} (\partial_s V(s, \mathring{Z}(s), \mathring{W}(s)) + \mathcal{G}V(s, \mathring{Z}(s), \mathring{W}(s)))\, ds \\
&\quad\quad + \int_t^{t+\Delta t} Q_{Z,W}^{1/2} \nabla_{Z,W} V(s, \mathring{Z}(s), \mathring{W}(s))\, d\tilde{w}(s),
\end{aligned}
\tag{116}
$$

where $Q_{Z,W}^{1/2}$ stands for the diffusion matrix of the process $(\mathring{Z}(s), \mathring{W}(s))$ and $\nabla_{Z,W}$ denotes the gradient operator in the state space of $(\mathring{Z}(s), \mathring{W}(s))$. On the other hand, since $V(t, Z, W)$ is a solution of the Bellman equation (105) and $(\mathring{Z}(s), \mathring{W}(s))$ corresponds to an optimal control, we have

$$
\begin{aligned}
&\int_t^{t+\Delta t} (\partial_s V(s, \mathring{Z}(s), \mathring{W}(s)) + \mathcal{G}V(s, \mathring{Z}(s), \mathring{W}(s)))\, ds \\
&\quad = \int_t^{t+\Delta t} e^{-(\rho s + \psi \mathring{c}(s))}\, ds.
\end{aligned}
$$

By virtue of the latter, applying the expectation operator on both the sides of (116), we obtain

$$
\begin{aligned}
&\frac{\mathbf{E}_{t,Z,W}\left[V(t+\Delta t, \mathring{Z}(t+\Delta t), \mathring{W}(t+\Delta t))\right] - \mathbf{E}_{t,Z,W}\left[V(t, \mathring{Z}(t), \mathring{W}(t))\right]}{\Delta t} \\
&= \frac{1}{\Delta t}\mathbf{E}_{t,Z,W}\left[\int_t^{t+\Delta t} e^{-(\rho s + \psi \mathring{c}(s))}\, ds\right],
\end{aligned}
$$

and, passing to the limit as $\Delta t \to 0$, it follows

$$
\frac{d\mathbf{E}_{t,Z,W}\left[V(t, \mathring{Z}(t), \mathring{W}(t))\right]}{dt} = \mathbf{E}_{t,Z,W}\left[e^{-(\rho t + \psi \mathring{c}(t))}\right].
$$

Now, by virtue of c-first order condition,

$$
e^{-(\rho t + \psi \mathring{c}(t))} = -\rho V(t, \mathring{Z}(t), \mathring{W}(t)).
$$

Hence, $\mathbf{E}_{t,Z,W}\left[V(t,\mathring{Z}(t),\mathring{W}(t))\right]$ satisfies the differential equation

$$\frac{d\mathbf{E}_{t,Z,W}\left[V(t,\mathring{Z}(t),\mathring{W}(t))\right]}{dt}=-\rho\mathbf{E}_{t,Z,W}\left[V(t,\mathring{Z}(t),\mathring{W}(t))\right],$$

and the desired transversality condition follows. $\square$

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