Regularity and Pareto Improving of financial equilibria with endogenous borrowing restrictions

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Abstract

We consider a pure exchange, general equilibrium model, with two periods and a finite number of states, goods, numeraire assets, and households. Participation on the asset market is restricted in a household specific manner, imposing upper bounds on the amounts of borrowing which can be obtained using assets. Those bounds are assumed to depend on asset and good prices - hence the reference to endogenous borrowing restrictions in the title. After having established existence of equilibria, we show that for an open and dense subset of the set of the economies, equilibria are finite and regular. We then analyze optimality properties. We first show generic suboptimality. Afterwards, we restrict our attention to the significant set of economies in which a sufficiently high number of participation constraints is "strictly" binding, i.e., utilities of involved households could be increased relaxing those constraints. We prove that, generically in that set, associated equilibria are Pareto improvable through a local change of those constraints.

Keywords: General equilibrium; Restricted participation; Financial markets; Numeraire assets; Suboptimality; Pareto Improving.

JEL classification: D50, D53, D61.

1 Introduction

We present a general equilibrium model with uncertainty and restricted, personalized participation on the asset markets. Each household chooses a consumption vector and an asset demand as in a standard model with incomplete or potentially complete markets, with the additional constraint that asset demand is *restricted* to belong to a household specific portfolio set.

Some papers on the topic are available in the literature. Siconolfi (1988) and, in a multiperiod framework, Angeloni and Cornet (2006) show existence of equilibria assuming the portfolio set is a closed, convex subset of a Euclidean space containing zero for each household and a neighborhood of zero for some of them. Balasko, Cass and Siconolfi (1990) analyze, in a model with nominal assets, the case in which the portfolio constraint set is a linear subspace and show that if enough households are subject to similar constraints, then restricted participation leads to a degree of real indeterminacy comparable to the one of the incomplete markets case. Polemarchakis and Siconolfi (1997) prove existence for a case of restricted participation with real assets, i.e., when payoffs are denominated in multiple commodities. Cass, Siconolfi and Villanacci (2001) study the case in which portfolio sets are described by smooth restriction functions, and show generic regularity of equilibria.[1] In a model with numeraire assets and outside money with restrictions for both type of assets, Carosi (2001) proves generic inefficiency of equilibria and effectiveness of monetary policy. Martins Da-Rocha and Triki (2005) present an original proof of existence. Won and Hahn (2007) discuss the presence of redundant assets. Hens, Herings and Predtetchinskii (2006) consider a model with one good per spot and give conditions for the existence of arbitrage possibilities when households cannot exchange at all some assets. In the same framework, Herings and Schmedders (2006) study the case of transaction costs proportional to the units (or values) of traded assets and present homotopy arguments to study equilibria. Basak, Cass, Licari and Pavlova (2008) describe how participation constraints may generate indeterminacy of equilibria, presenting an economy with two goods and two households for which adding a constraint may generate additional inefficient (and sunspot) equilibria. Aouani and Cornet (2009) consider a two period model with households having nonordered preferences and show existence of equilibria for the case of linear constraints. Carosi, Gori and Villanacci (2009) present a case in which the portfolio set is price dependent. After having shown existence, they prove that if constraints are homogeneous of degree zero with respect to spot prices, equilibria are typically finite and regular, and present a robust example of indeterminacy in the nonhomogeneous case.[2]

In our model, each household has to satisfy a specific constraint on the asset portfolio which depends on asset and good prices, an assumption which makes the financial restrictions endogenous. That feature is shared with the model in Carosi, Gori and Villanacci (2009). Restriction functions in the present paper are however different from the ones used there: our specification of the restriction functions does not require that, for each asset, there is a household whose participation does not depend on the demand of that asset. In particular, we assume that each household cannot sell more than a given quantity, depending on prices, of each asset. In other words, we impose upper bounds on the amounts of borrowing which can be obtained using assets. Examples of that restriction are the presence of no short sale constraints of the form $-z_h^a \leq 0$, where z_h^a is the demand of asset a by household h, or constraints on borrowing like $-z_h^a \leq \sigma$, with $\sigma > 0$. Both above restrictions are of the exogenous type, but they can be easily thought as endogenous. In fact σ could be assumed to be fixed today and in a near future, but it would surely change in a relatively different economic situation (described in terms of prices and fundamentals). Another related example of endogenous borrowing restriction is the case of the presence of an upper bound equal to a proportion of the individual wealth, a short cut to incorporate the well known moral hazard problems.

We can now present the main results of the paper. We consider a pure exchange, general equilibrium model, with two periods and a finite number of states, goods, assets, and households. Assets give the right to receive a certain quantity of the numeraire commodity. Participation on the asset market is restricted in a household specific manner. Economies are described by endowments of commodities, utility functions, asset yield matrices, and borrowing functions. After having established existence of equilibria, we show that for an open and dense subset of the set of the economies, equilibria are finite and regular. We then analyze optimality properties. We first show generic suboptimality. Afterwards, we restrict our attention to the significant set of economies in which a sufficiently high number of participation constraints is "strictly" binding, i.e., utilities of involved households could be increased relaxing those constraints. We believe that even casual observation of financial restrictions engineered and introduced in financial markets shows that constraints are created with the purpose of being strictly binding for some households. We prove that, generically in the set of economies with strictly binding constraints, associated equilibria are Pareto improvable through a local change of those constraints.

The paper is organized in a standard manner. Section 2 presents the set-up of the model and the definition of equilibrium. Section 3 lists the main results (existence, generic regularity, generic suboptimality and possibility of Pareto improvements), whose proofs are contained in the Appendix.

[1] In an unpublished note, Lisboa (1995) discusses indeterminacy and optimality properties in the model introduced in Cass, Siconolfi and Villanacci (2001).

[2] Many finance papers focus on how constraints on asset markets affect equilibrium consumption and prices: among them see the textbook on corporate finance by Tirole (2006) and the literature quoted there.

2 Set-up of the model

Our restricted participation model builds up on the very standard two-period, pure exchange economy with uncertainty and financial markets. We consider a commodity market in which $C \geq 2$ types of different commodities, denoted by $c \in \mathcal{C} = \{1, 2, \ldots, C\}$, are traded both today and tomorrow. We assume that tomorrow only one among $S \geq 1$ possible states of the world, denoted by $s \in \{1, \ldots, S\}$, will occur. We denote today by $s = 0$ and we define $\mathcal{S} = \{0, 1, \ldots, S\}$. Asset markets open in the first period, and there are $A \geq 1$ assets traded, denoted by $a \in \mathcal{A} = \{1, 2, \ldots, A\}$. We assume $A \leq S$, see (7) below[3]. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. The time structure of the model is as follows: today, households exchange commodities and assets, and consumption takes place. Then, tomorrow, uncertainty is resolved, households honor their financial obligations, and they again exchange and then consume commodities.

Let[4] $x_h^c(s) \in \mathbb{R}_{++}$ be the consumption of commodity c in state s by household h, and let $e_h^c(s) \in \mathbb{R}_{++}$ be the endowment of commodity c in state s owned by household h. We define

$$\begin{aligned} x_h(s) &= (x_h^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad x_h = (x_h(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad x = (x_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}, \\ e_h(s) &= (e_h^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad e_h = (e_h(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad e = (e_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}, \end{aligned}$$

where $G = C(S + 1)$.

Household h's preferences are represented by a utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$. We assume, for every $h \in \mathcal{H}$,

$$u_h \in C^2(\mathbb{R}^G_{++}); \tag{1}$$

$$\forall\, x_h \in \mathbb{R}^G_{++},\ Du_h(x_h) \gg 0\,; \tag{2}$$

$$\forall\, v \in \mathbb{R}^G \setminus \{0\}\,,\ x_h \in \mathbb{R}^G_{++},\ Du_h(x_h)v = 0 \Rightarrow\ v\, D^2u_h(x_h)\, v < 0\,; \tag{3}$$

$$\forall\, \underline{x}_h \in \mathbb{R}^G_{++},\ \left\{x_h \in \mathbb{R}^G_{++} : u_h(x_h) \geq u_h(\underline{x}_h)\right\} \text{ is closed in the topology of } \mathbb{R}^G. \tag{4}$$

Let $\mathcal{U}$ be the set of vectors $u = (u_h)_{h\in\mathcal{H}}$ of such utility functions. In what follows we are sometimes going to deal with the next stronger version of Assumptions (1) and (3):

$$u_h \in C^3(\mathbb{R}^G_{++}); \tag{5}$$

$$\forall\, v \in \mathbb{R}^G \setminus \{0\}\,,\ x_h \in \mathbb{R}^G_{++},\ v\, D^2u_h(x_h)\, v < 0\,. \tag{6}$$

With innocuous abuse of notation, we still denote by $\mathcal{U}$ the set of utility functions satisfying Assumptions (5) and (6), as well.

We denote by $p^c(s) \in \mathbb{R}_{++}$ the price of commodity c at spot s, by $q^a \in \mathbb{R}$ the price of asset a and by $z_h^a \in \mathbb{R}$ the quantity of asset a held by household h. Moreover we define

$$\begin{aligned} p(s) &= (p^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad p = (p(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad q = (q^a)_{a\in\mathcal{A}} \in \mathbb{R}^A, \\ z_h &= (z_h^a)_{a\in\mathcal{A}} \in \mathbb{R}^A, \qquad\quad z = (z_h)_{h\in\mathcal{H}} \in \mathbb{R}^{AH}. \end{aligned}$$

Furthermore, we denote by $y^a(s) \in \mathbb{R}$ the yield in state s of asset a in units of the numeraire commodity, we choose to be commodity C, $y\,(s) = (y^a(s))_{a\in\mathcal{A}} \in \mathbb{R}^A$ is the vector of asset yields in state s and

$$Y = \begin{bmatrix} y^1(1) & \cdots & y^a(1) & \cdots & y^A(1) \\ \vdots & & \vdots & & \vdots \\ y^1(s) & \ldots & y^a(s) & \ldots & y^A(s) \\ \vdots & & \vdots & & \vdots \\ y^1(S) & \ldots & y^a(S) & \ldots & y^A(S) \end{bmatrix}$$

[3]Such assumption is done with loss of generality, but it is commonly made and technically convenient.

[4]For every positive integer N, we define the binary relations $\gg$, $\geq$ and $>$ over $\mathbb{R}^N$ as follows: given $v = (v_1, \ldots, v_N)$ and $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$, we write

$$\begin{array}{lll} v \gg w & \text{if} & v_i > w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v \geq w & \text{if} & v_i \geq w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v > w & \text{if} & v \geq w \text{ and } v \neq w. \end{array}$$

We define also the sets $\mathbb{R}^N_+ = \{v \in \mathbb{R}^N : v \geq 0\}$ and $\mathbb{R}^N_{++} = \{v \in \mathbb{R}^N : v \gg 0\}$.

is the corresponding yield matrix. It greatly simplifies our analysis (but it is not without loss of generality) to assume that

$$\operatorname{rank} Y = A \leq S. \tag{7}$$

Let $\mathbb{M}(S, A)$ be the space of the $S \times A$ matrices with real elements and let

$$\mathcal{Y} = \{Y \in \mathbb{M}(S, A) : Y \text{ satisfies } (7)\}.$$

Consistently with our restricted participation framework, we assume that each household has only partial access, in a personalized manner, to the asset market. In particular, we suppose that each household cannot sell more than a fixed quantity, depending on commodity and asset prices, of each asset. We assume that, for every $h \in \mathcal{H}$, there is a function

$$\sigma_h : \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^A, \quad (p, q) \mapsto (\sigma^a_h(p, q))_{a\in\mathcal{A}}$$

such that, for every $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $\sigma^a_h(p, q)$ represents the largest quantity of asset a that household h can sell at prices (p, q), i.e., the maximum amount she can borrow using asset a. In other words, we add the following constraints to household h's maximization problem:

$$\forall\, a \in \mathcal{A},\ -z^a_h \leq \sigma^a_h(p, q).$$

We call each function σ_h borrowing function. We assume that,

$$\forall\, h \in \mathcal{H},\ \sigma_h \in C^2(\mathbb{R}^G_{++} \times \mathbb{R}^A, \mathbb{R}^A); \tag{8}$$

$$\forall\, h \in \mathcal{H},\ a \in \mathcal{A},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A,\ \sigma^a_h(p, q) \geq 0; \tag{9}$$

$$\forall\, a \in \mathcal{A},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A,\ \sum_{h=1}^{H} \sigma^a_h(p, q) > 0; \tag{10}$$

$$\forall\, h \in \mathcal{H},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A,\ \sigma_h\left(p(0), \ldots, p(S), q\right) = \sigma_h\Big(\frac{p(0)}{p^C(0)}, \ldots, \frac{p(S)}{p^C(S)}, \frac{q}{p^C(0)}\Big). \tag{11}$$

Let us denote by Σ the set of vectors $\sigma = (\sigma_h)_{h\in\mathcal{H}}$ of such functions. Assumption (8) allows to use differential techniques. Assumption (9) permits no participation on the financial markets. Assumption (10) insures that each asset is nontrivially exchanged, as $\sum_{h=1}^{H} \sigma^a_h(p, q) = 0$ would imply that asset a is not traded in equilibrium. Assumption (11) says that participation constraints are not affected by nominal, i.e., price, changes. In what follows we are sometimes going to deal with the next stronger version of Assumptions (9) and (10):

$$\forall\, h \in \mathcal{H},\ a \in \mathcal{A},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A,\ \sigma^a_h(p, q) > 0\,. \tag{12}$$

With innocuous abuse of notation, we still denote by Σ the set of borrowing functions satisfying Assumption (12), as well.

We define the set of economies as $\mathcal{E} = \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y} \times \Sigma$ with generic element $E = (e, u, Y, \sigma)$, and for given $(p, q, E) \in \mathbb{R}^G_{++} \times \mathbb{R}^A \times \mathcal{E}$, we assume household $h \in \mathcal{H}$ has to solve the following maximization problem:

$$\max_{(x_h, z_h)} u_h(x_h) \quad \text{s.t.}$$

$$\begin{cases} p(0)x_h(0) + qz_h = p(0)e_h(0) \\ p(s)x_h(s) = p(s)e_h(s) + p^C(s)y(s)z_h, \quad s \in \{1, \ldots, S\} \\ z_h + \sigma_h(p, q) \geq 0. \end{cases} \tag{13}$$

Note that, because of (11), (13) is invariant with respect to price normalization spot by spot. Then, without loss of generality, we can assume that commodity C is the unit of measure of the exchanges.

We are now ready to give the definition of equilibrium we use in our framework.

Definition 1. *Let us consider $E \in \mathcal{E}$. We say that $\theta = \left((x_h, z_h)_{h\in\mathcal{H}}, p, q\right) \in \mathbb{R}^{GH}_{++} \times \mathbb{R}^{AH} \times \mathbb{R}^{G}_{++} \times \mathbb{R}^{A} = \Theta$ is an equilibrium for E if, for every $h \in \mathcal{H}$, (x_h, z_h) solves Problem* (13) *at (p, q, E), (x, z) satisfies market clearing conditions, that is,*

$$\sum_{h=1}^{H} (x_h - e_h) = 0 \quad \text{and} \quad \sum_{h=1}^{H} z_h = 0,$$

and, for every $s \in \mathcal{S}$, $p^C(s) = 1$. We denote by $\Theta(E) \subseteq \Theta$ and $X(E) \subseteq \mathbb{R}^{GH}_{++}$ the set of equilibria and the set of equilibrium allocations for economy $E \in \mathcal{E}$, respectively.

We are also interested in considering the intervention of a planner and in studying its effects on the possibility of Pareto improving equilibria. We assume the planner has the power to (locally) change households' borrowing functions by a "proportionality" factor t^a_h. Therefore, defined

$$t = (t^a_h)_{a\in\mathcal{A},h\in\mathcal{H}} \in (-1, +\infty)^{AH} = \mathcal{T},$$

the presence of the planner transforms each economy $E = (e, u, Y, \sigma) \in \mathcal{E}$ into an economy $E(t) = (e, u, Y, \sigma(t)) \in \mathcal{E}$, where $\sigma(t) = (\sigma^a_h(t))_{a\in\mathcal{A},h\in\mathcal{H}}$ and, for every $a \in \mathcal{A}, h \in \mathcal{H}$,

$$\sigma^a_h(t) = (1 + t^a_h)\sigma^a_h.$$

3 Main results

In this section, we present our main results. Proofs are deferred to the Appendix. We stress that Theorems 2, 3, 4 below can be proved both assuming (5), (6) and (12), as well as without assuming (5), (6) and (12) in the definition of economy. On the contrary, in order to prove Theorem 5 we need to assume (5), (6) and (12). In what follows we prove Theorems 2, 3, 4 without assuming (5), (6) and (12). The proof assuming those further conditions is exactly the same.
An indispensable preliminary result in every general equilibrium model is existence of equilibria. The following existence result can be immediately deduced by Theorems 2 and 3 in Gori, Pireddu and Villanacci (2010).

Theorem 2. *For every $E \in \mathcal{E}$, $\Theta(E) \neq \varnothing$.*

Consider now the Hausdorff topological vector space

$$\mathscr{V} = \mathbb{R}^{GH}_{++} \times \left[C^3(\mathbb{R}^{G}_{++})\right]^H \times \mathbb{R}^{AS} \times \left[C^2(\mathbb{R}^{G}_{++} \times \mathbb{R}^{A}, \mathbb{R}^{A})\right]^H, \tag{14}$$

endowed with the product topology of the natural topologies on each of the spaces in the Cartesian product. In what follows, we endow $\mathcal{E} \subseteq \mathscr{V}$ with the topology induced by $\mathscr{V}$.
Given S finite set, we denote by $|S|$ its cardinality. Next theorem shows that typically in the set of economies, equilibria are finite in number and depend smoothly upon elements in that set.

Theorem 3. *There exists an open and dense set $\mathcal{D} \subseteq \mathcal{E}$ such that, for every $E^* \in \mathcal{D}$,*

$$\Theta(E^*) = \{\theta^{i*}\}_{i=1}^{k}, \tag{15}$$

where k is a positive integer, and there exist an open neighborhood $V(E^*) \subseteq \mathcal{E}$ of E^* and, for every $i \in \{1, \ldots, k\}$, an open neighborhood $O(\theta^{i*}) \subseteq \Theta$ of θ^{i*} and $g_i : V(E^*) \to O(\theta^{i*})$ such that [5]*:*

$$g_i \in C^1,\ g_i(E^*) = \theta^{i*} \text{ and } O(\theta^{i*}) \cap O(\theta^{j*}) = \varnothing,\ \text{for } i \neq j; \tag{16}$$

$$\left\{(E, \theta) \in V(E^*) \times O(\theta^{i*}) : \theta \in \Theta(E)\right\} = \operatorname{graph}(g_i); \tag{17}$$

$$\{(E, \theta) \in V(E^*) \times \Theta : \theta \in \Theta(E)\} = \bigcup_{i=1}^{k} \operatorname{graph}(g_i). \tag{18}$$

[5]Notice that the maps g_i are C^1 on a topological vector space. The precise definition of this concept can be found in the Appendix, Section A.2.

The theorem below shows the easily conjectured typical inefficiency of equilibria.

Theorem 4. *If $A < S$, there exists an open and dense set $\mathcal{D}^{\triangle} \subseteq \mathcal{E}$ such that, for every $E \in \mathcal{D}^{\triangle}$, each $x \in X(E)$ is not Pareto Optimal.*

A consequence of the proof of Theorem 3 is that equilibria associated with economies in the set $\mathcal{D}$ are such that if a borrowing constraint holds with equality, i.e., it is binding, utility of the involved household could be increased changing that constraint. The presence of that kind of constraints is a necessary condition for a successful intervention of a planner acting on participation restrictions. Theorem 5 shows in fact that, if in correspondence to a certain equilibrium there is a number of binding constraints at least equal to the number of households, then it is possible to Pareto improve upon the considered equilibrium.[6] Let $\Lambda : \Theta \times \mathcal{E} \to \mathbb{N}$ be defined as

$$\Lambda(\theta, E) = |\left\{(a, h) \in \mathcal{A} \times \mathcal{H} : z_h^a + \sigma_h^a(p, q) = 0\right\}|. \tag{19}$$

Theorem 5. *There exists an open and dense set $\mathcal{D}^{\Diamond} \subseteq \mathcal{E}$ such that, if*

$$(\theta^*, E^*) \in \left\{(\theta, E) \in \Theta \times \mathcal{D}^{\Diamond} : \theta \in \Theta(E), \Lambda(\theta, E) \geq H\right\},$$

then, for every open neighborhood $\mathcal{V}(0) \subseteq \mathcal{T}$ of 0, there exist $t \in \mathcal{V}(0)$ and $\theta \in \Theta\left(E^*(t)\right)$ such that $(u_h(x_h))_{h \in \mathcal{H}} \gg (u_h(x_h^*))_{h \in \mathcal{H}}$.

A Appendix

A.1 Preliminary notation and results

Define the vectors

$$\begin{array}{ll} x_h^{\setminus}(s) = (x_h^c(s))_{c \in \{1,\ldots,C-1\}} \in \mathbb{R}_{++}^{C-1}, & x_h^{\setminus} = \left(x_h^{\setminus}(s)\right)_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{G-(S+1)}, \\ e_h^{\setminus}(s) = (e_h^c(s))_{c \in \{1,\ldots,C-1\}} \in \mathbb{R}_{++}^{C-1}, & e_h^{\setminus} = \left(e_h^{\setminus}(s)\right)_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{G-(S+1)}. \end{array}$$

Because of the validity of $S+1$ Walras' laws in our model, the significant market clearing conditions in Definition 1 are in fact

$$\sum_{h=1}^{H} \left(x_h^{\setminus} - e_h^{\setminus}\right) = 0 \quad \text{and} \quad \sum_{h=1}^{H} z_h = 0.$$

Since we are going to study equilibria in terms of first order conditions associated with households' maximization problems and (significant) market clearing conditions, define

$$\Xi = \mathbb{R}_{++}^{GH} \times \mathbb{R}^{(S+1)H} \times \mathbb{R}^{AH} \times \mathbb{R}^{AH} \times \mathbb{R}_{++}^{G} \times \mathbb{R}^{A}$$

with generic element

$$\xi = \left((x_h, \lambda_h, z_h, \mu_h)_{h \in \mathcal{H}}, p, q\right) = (x, \lambda, z, \mu, p, q),$$

and the function[7]

$$\mathcal{F} : \Xi \times \mathcal{E} \times \mathcal{T} \to \mathbb{R}^{\dim(\Xi)},$$

[6] In more technical terms, the proof of Theorem 3 shows that each borrowing constraint for each household is not in a so-called "border line case", i.e., it is not the case that both the constraint and the associated multiplier are equal to zero. Therefore, if a borrowing constraint holds with equality, the associated multiplier is positive and the constraint is said to be "strictly binding". In other words, all binding constraints are strictly binding. For every economy E belonging to the set $\mathcal{D}$ introduced in Theorem 3 and for every $\theta \in \Theta(E)$, the integer $\Lambda(\theta, E)$ defined in (19) denotes the cardinality of the set of strictly binding constraints associated with the pair (θ, E).

[7] $\dim(\Xi)$ denotes the dimension of the manifold (open set) Ξ.

$$
\mathcal{F}(\xi,E,t)=\left[\begin{array}{ll}
(20.1) & D_{x_h(s)}u_h(x_h)-\lambda_h(s)p(s)\\[6pt]
(20.2) & \begin{array}{l}-p(0)(x_h(0)-e_h(0))-qz_h\\ -p(s)(x_h(s)-e_h(s))+p^C(s)y(s)z_h, \quad s\in\{1,\ldots,S\}\end{array}\\[6pt]
(20.3) & -\lambda_h(0)q^a+\sum_{s=1}^{S}\lambda_h(s)p^C(s)y^a(s)+\mu_h^a\\[6pt]
(20.4) & \min\left\{\mu_h^a, z_h^a+(1+t_h^a)\sigma_h^a(p,q)\right\}\\[6pt]
(20.5) & \sum_{h=1}^{H}\left(x_h^{\backslash}-e_h^{\backslash}\right)\\[6pt]
(20.6) & \sum_{h=1}^{H} z_h\\[6pt]
(20.7) & p^C(s)-1
\end{array}\right] \tag{20}
$$

Given now $(E,t)\in\mathcal{E}\times\mathcal{T}$, it is immediate to prove that if $\theta=\left((x_h,z_h)_{h\in\mathcal{H}},p,q\right)\in\Theta(E(t))$, then there exists a unique $(\lambda_h,\mu_h)_{h\in\mathcal{H}}\in\mathbb{R}^{(S+1)H}\times\mathbb{R}^{AH}$ such that $\xi=\left((x_h,\lambda_h,z_h,\mu_h)_{h\in\mathcal{H}},p,q\right)\in\Xi$ solves system $\mathcal{F}(\xi,E,t)=0$. Sometimes we will call such ξ the extended equilibrium associated with θ. Vice versa, if $\xi=\left((x_h,\lambda_h,z_h,\mu_h)_{h\in\mathcal{H}},p,q\right)\in\Xi$ solves system $\mathcal{F}(\xi,E,t)=0$, then $\left((x_h,z_h)_{h\in\mathcal{H}},p,q\right)\in\Theta(E(t))$.
Let us also introduce the function $\mathcal{F}_0:\Xi\times\mathcal{E}\to\mathbb{R}^{\dim(\Xi)}$, $\mathcal{F}_0(\xi,E)=\mathcal{F}(\xi,E,0)$. The following is a consequence of Theorem 2:

Theorem 6. *For every $E\in\mathcal{E}$, there exists $\xi\in\Xi$ such that $\mathcal{F}_0(\xi,E)=0$.*

As a corollary it follows that, since $\mathcal{F}(\xi,E,t)=\mathcal{F}_0(\xi,E(t))$, for every $(E,t)\in\mathcal{E}\times\mathcal{T}$, system $\mathcal{F}(\xi,E,t)=0$ has always solutions in the variable ξ.

Later we are going to use the following result (see Villanacci et al. (2002)).

Theorem 7. *Let m,p,n and α be nonnegative integers, and let M,Ω and N be C^α manifolds of dimensions m,p and n, respectively. Let $F:M\times\Omega\to N$ be a C^α function. Assume $\alpha>\max\{m-n,0\}$. If y is a regular value for F, then there exists a full measure subset Ω^* of Ω such that for any $\omega\in\Omega^*$, y is a regular value for $F_\omega:M\to N$, $x\mapsto F_\omega(x)=F(x,\omega)$.*

Let $\mathscr{V}$ be a topological Hausdorff vector space, $V\subseteq\mathscr{V}$ be an open set and $f:V\to\mathbb{R}^n$ be a function. We say that $f\in C^0(V,\mathbb{R}^n)$ if f is continuous, while $f\in C^1(V,\mathbb{R}^n)$ if it is continuous, there exists the limit

$$
df(v,w)=\lim_{\varepsilon\to 0}\frac{f(v+\varepsilon w)-f(v)}{\varepsilon},\quad \forall v\in V, w\in\mathscr{V},
$$

and the function $df:V\times\mathscr{V}\to\mathbb{R}^n$ is continuous.
Given now *any* (not necessarily open) set $X\subseteq\mathscr{V}$, and $f:X\to\mathbb{R}^n$, we say $f\in C^0(X,\mathbb{R}^n)$ if f is continuous with respect to the topology induced by $\mathscr{V}$ on X, while, as in the finite dimensional setting, $f\in C^1(X,\mathbb{R}^n)$ if for every $v_0\in X$ there exists an open neighborhood of v_0 in $\mathscr{V}$, say $V(v_0)$, and a function $\overline{f}:V(v_0)\to\mathbb{R}^n$ such that $\overline{f}\in C^1(V(v_0),\mathbb{R}^n)$ and, for every $v\in V(v_0)\cap X$, $f(x)=\overline{f}(x)$.
Those definitions allow to state the following implicit function theorem which is a simplified version of Theorem 2.3 in Glöckner (2006).

Theorem 8. *Let us consider $f:O\times V\to\mathbb{R}^n$, where O is an open subset of $\mathbb{R}^n$ and V is an open subset of a topological Hausdorff vector space $\mathscr{V}$. Assume $f\in C^1(O\times V,\mathbb{R}^n)$ and let $(x_0,v_0)\in O\times V$ such that $f(x_0,v_0)=0$ and $D_xf(x_0,v_0)$ is invertible*[8]*. Then there exist $O(x_0)\subseteq O$ open neighborhood of x_0, $V(v_0)\subseteq V$ open neighborhood of v_0 and $g:V(v_0)\to O(x_0)$ such that*

1. *$g\in C^1(V(v_0),O(x_0))$,*
2. *$g(v_0)=x_0$,*
3. *$\{(x,v)\in O(x_0)\times V(v_0):f(x,v)=0\}=\{(x,v)\in O(x_0)\times V(v_0):x=g(v)\}$.*

[8]Note that if $f\in C^1(O\times V,\mathbb{R}^n)$ then, for every $v\in V$, $f(\cdot,v):O\to\mathbb{R}^n$, $x\mapsto f(x,v)$, belongs to $C^1(O,\mathbb{R}^n)$ and thus, for every $(x,v)\in O\times V$, $D_xf(x,v)$ is well defined.

A.2 Generic regularity of equilibria

In what follows we will need the next lemma. We recall that a function $f : A \to B$, with A and B topological spaces, is proper if, for every $K \subseteq B$ compact set, $f^{-1}(K) \subseteq A$ is compact, too. We also recall that any proper and continuous function is closed, i.e., it maps closed sets onto closed sets.

Lemma 9. *$\mathcal{F}$ is continuous on $\Xi \times \mathcal{E} \times \mathcal{T}$ and*

$$\pi : \mathcal{F}^{-1}(0) \to \mathcal{E}, \quad (\xi, E, t) \mapsto \pi(\xi, E, t) = E$$

is proper. In particular, $\mathcal{F}_0$ is continuous on $\Xi \times \mathcal{E}$ and

$$\pi_0 : \mathcal{F}_0^{-1}(0) \to \mathcal{E}, \quad (\xi, E) \mapsto \pi_0(\xi, E) = E$$

is proper.

Proof. The continuity of $\mathcal{F}$ is immediate. In order to show that π is proper, we have to prove that each sequence $(\xi^{[n]}, E^{[n]}, t^{[n]})_{n\in\mathbb{N}}$ in $\mathcal{F}^{-1}(0)$, such that $(E^{[n]}, t^{[n]})$ converges in $\mathcal{E}$, admits a converging subsequence in $\mathcal{F}^{-1}(0)$. Let us assume that

$$(E^{[n]}, t^{[n]}) = (e^{[n]}, u^{[n]}, Y^{[n]}, \sigma^{[n]}, t^{[n]}) \to (\overline{E}, \overline{t}) = (\overline{e}, \overline{u}, \overline{Y}, \overline{\sigma}, \overline{t}) \in \mathcal{E} \times \mathcal{T}.$$

Then it suffices to show that, up to a subsequence, $(\xi^{[n]})_{n\in\mathbb{N}}$ converges to a certain $\overline{\xi} \in \Xi$: indeed the condition $\mathcal{F}(\overline{\xi}, \overline{E}, \overline{t}) = 0$ follows by the continuity of $\mathcal{F}$.

As we are going to use a diagonal argument, every time we say that a sequence converges we mean it has a converging subsequence.

Let us start with the convergence of $x^{[n]}$. For a fixed $h \in \mathcal{H}$, we know that, for every $n \in \mathbb{N}$, $\left(x_h^{[n]}, z_h^{[n]}\right)$ is solution to the problem

$$\max_{(x_h, z_h)} u_h^{[n]}(x_h) \quad \text{s.t.}$$

$$\begin{cases} p^{[n]}(0)x_h(0) + q^{[n]}z_h = p^{[n]}(0)e_h^{[n]}(0) \\ p^{[n]}(s)x_h(s) = p^{[n]}(s)e_h^{[n]}(s) + p^{C,[n]}(s)y^{[n]}(s)z_h, \quad s \in \{1, \dots, S\} \\ z_h^a + (1 + t_h^{a,[n]})\sigma_h^{a,[n]}(p^{[n]}, q^{[n]}) \geq 0, \end{cases}$$

and then, since $(e_h^{[n]}, 0)$ belongs to the constraint set, it has to be $u_h^{[n]}(x_h^{[n]}) \geq u_h^{[n]}(e_h^{[n]})$. As $(e_h^{[n]})_{n\in\mathbb{N}}$ converges to $\overline{e}_h \in \mathbb{R}^G_{++}$, it holds that the compact set $S_h = \{e_h^{[n]}\}_{n=1}^{\infty} \cup \{\overline{e}_h\}$ is a subset of $\mathbb{R}^G_{++}$ and we have

$$u_h^{[n]}(x_h^{[n]}) \geq u_h^{[n]}(e_h^{[n]}) \geq \min_{x_h \in S_h} u_h^{[n]}(x_h) \geq \min_{x_h \in S_h} \overline{u}_h(x_h) - \varepsilon_n,$$

for a suitable sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_{++}$ such that $\varepsilon_n \to 0$ if $n \to \infty$, by the definition of the topology on $C^2(\mathbb{R}^G_{++})$. Indeed we can define, for every $n \in \mathbb{N}$,

$$\varepsilon_n = \max_{w \in S_h} \left| u_h^{[n]}(w) - \overline{u}_h(w) \right|.$$

Let $x_h^* \in S_h$ be such that $\min_{x_h \in S_h} \overline{u}_h(x_h) = \overline{u}_h(x_h^*)$, and let $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^G$ and $\delta > 0$ be small enough such that $x_h^* - 2\delta\mathbf{1} \in \mathbb{R}^G_{++}$. Obviously, since by (2), $\overline{u}_h(x_h^*) > \overline{u}_h(x_h^* - \delta\mathbf{1})$, there exists n_1 such that $n \geq n_1$ implies $\overline{u}_h(x_h^*) - \varepsilon_n \geq \overline{u}_h(x_h^* - \delta\mathbf{1})$ and thus, for every $n \geq n_1$,

$$u_h^{[n]}(x_h^{[n]}) \geq \overline{u}_h(x_h^* - \delta\mathbf{1}). \tag{21}$$

Of course, because of the validity of $S + 1$ Walras' laws in our model, we can also assume that, for every $n \geq n_1$,

$$0 \ll x_h^{[n]} \leq \sum_{h=1}^{H} e_h^{[n]} \leq \sum_{h=1}^{H} \overline{e}_h + \mathbf{1}.$$

Our purpose now is to prove that for infinite values of n it is $\overline{u}_h(x_h^{[n]}) \geq \overline{u}_h(x_h^* - 2\delta\mathbf{1})$.
Let $\widehat{x}_h \in \left[0, \sum_{h=1}^H e_h + \mathbf{1}\right]$ be a cluster point of $(x_h^{[n]})_{n\geq n_1}$. Then we can assume $x_h^{[n]} \to \widehat{x}_h$. Consider any $\widetilde{x}_h \in \mathbb{R}^G_{++}$ such that $\overline{u}_h(\widetilde{x}_h) = \overline{u}_h(x_h^* - 2\delta\mathbf{1})$. If we take n large enough, by (21), it is $u_h^{[n]}(x_h^{[n]}) - u_h^{[n]}(\widetilde{x}_h) \geq 0$. Then, for n sufficiently large,

$$0 \leq u_h^{[n]}(x_h^{[n]}) - u_h^{[n]}(\widetilde{x}_h) \leq D_{x_h} u_h^{[n]}(\widetilde{x}_h)(x_h^{[n]} - \widetilde{x}_h)$$

$$= \left(D_{x_h} u_h^{[n]}(\widetilde{x}_h) - D_{x_h}\overline{u}_h(\widetilde{x}_h)\right)(x_h^{[n]} - \widetilde{x}_h) + D_{x_h}\overline{u}_h(\widetilde{x}_h)(x_h^{[n]} - \widetilde{x}_h).$$

Taking the limit as $n \to \infty$ in the previous inequality, we achieve

$$D_{x_h}\overline{u}_h(\widetilde{x}_h)(\widehat{x}_h - \widetilde{x}_h) \geq 0.$$

Then

$$\widehat{x}_h \subseteq \bigcap_{\widetilde{x}_h \in \{y\in\mathbb{R}^G_{++} : \overline{u}_h(y) = \overline{u}_h(x_h^* - 2\delta\mathbf{1})\}} \left\{y \in \mathbb{R}^G : D_{x_h}\overline{u}_h(\widetilde{x}_h)(y - \widetilde{x}_h) \geq 0\right\}. \tag{22}$$

Since the right hand side of (22) is exactly $\{y \in \mathbb{R}^G : \overline{u}_h(y) \geq \overline{u}_h(x_h^* - 2\delta\mathbf{1})\}$, which is a subset of $\mathbb{R}^G_{++}$ by (4), then $\widehat{x}_h \in \mathbb{R}^G_{++}$ and the proof is complete.
As regards the convergence of $\lambda^{[n]}$, from (20.1), (20.7) and (2) we find that, for every $h \in \mathcal{H}$ and $s \in \mathcal{S}$,

$$\lambda_h^{[n]}(s) = D_{x_h^C(s)} u_h^{[n]}(x_h^{[n]}) \to D_{x_h^C(s)}\overline{u}_h(\overline{x}_h) = \overline{\lambda}_h(s) \in \mathbb{R}_{++},$$

since $D_{x_h^C(s)} u_h^{[n]} \to D_{x_h^C(s)}\overline{u}_h$ uniformly on compact subsets of $\mathbb{R}^G_{++}$. Then, from (20.1) and (2), it follows that, for every $s \in \mathcal{S}$,

$$p^{[n]}(s) = \frac{D_{x_h(s)} u_h^{[n]}(x_h^{[n]})}{\lambda_h^{[n]}(s)} \to \frac{D_{x_h(s)}\overline{u}_h(\overline{x}_h)}{\overline{\lambda}_h(s)} = \overline{p}(s) \in \mathbb{R}^C_{++}$$

and thus $(p^{[n]})_{n\in\mathbb{N}}$ converges to an element of $\mathbb{R}^G_{++}$. Fix now $a \in \mathcal{A}$ and consider the sequence $(q^{a,[n]})_{n\in\mathbb{N}}$. We claim that it converges if there exist $h \in \mathcal{H}$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$,

$$z_h^{a,[n_k]} + (1 + t_h^{a,[n_k]})\sigma_h^{a,[n_k]}(p^{[n_k]}, q^{[n_k]}) > 0.$$

Indeed, if this is true, from (20.3) and (20.4) we get

$$q^{a,[n_k]} = \frac{1}{\lambda_h^{[n_k]}(0)} \sum_{s=1}^S \lambda_h^{[n_k]}(s) p^{C,[n_k]}(s) y^{a,[n_k]}(s) \to \frac{1}{\overline{\lambda}_h(0)} \sum_{s=1}^S \overline{\lambda}_h(s)\overline{p}^C(s)\overline{y}^a(s) = \overline{q}^a.$$

In order to show the claim, assume by contradiction that, for every $h \in \mathcal{H}$, there exists $\nu_h \in \mathbb{N}$ such that, for every $n \geq \nu_h$,

$$z_h^{a,[n]} + (1 + t_h^{a,[n]})\sigma_h^{a,[n]}(p^{[n]}, q^{[n]}) = 0.$$

Summing up on $h \in \mathcal{H}$, and using (20.6), we find

$$0 = \sum_{h=1}^H z_h^{a,[n]} + \sum_{h=1}^H (1 + t_h^{a,[n]})\sigma_h^{a,[n]}(p^{[n]}, q^{[n]}) = \sum_{h=1}^H (1 + t_h^{a,[n]})\sigma_h^{a,[n]}(p^{[n]}, q^{[n]}).$$

From (10), the right hand side of the above equality has to be positive. Then the contradiction is found and the convergence of $(q^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^A$ follows. Fix now $h \in \mathcal{H}$ and $a \in \mathcal{A}$ and consider the sequence $(z_h^{a,[n]})_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, we have

$$-(1 + t_h^{a,[n]})\sigma_h^{a,[n]}(p^{[n]}, q^{[n]}) \leq z_h^{a,[n]} = -\sum_{h'\in\mathcal{H}, h'\neq h} z_{h'}^{a,[n]} \leq \sum_{h'\in\mathcal{H}, h'\neq h} (1 + t_{h'}^{a,[n]})\sigma_{h'}^{a,[n]}(p^{[n]}, q^{[n]}).$$

Since $p^{[n]} \to \overline{p}$, $q^{[n]} \to \overline{q}$, $\sigma_h^{a,[n]} \to \overline{\sigma}_h^a$ uniformly on compact subsets of $\mathbb{R}^G_{++} \times \mathbb{R}^A$ and $t^{[n]} \to \overline{t}$, there exist $\underline{K}_h^a$, $\overline{K}_h^a \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$,

$$\underline{K}_h^a \leq z_h^{a,[n]} \leq \overline{K}_h^a.$$

Thus $z_h^{a,[n]} \to \overline{z}_h^a$ and the convergence of $(z^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^{AH}$ is proved.
Finally, fixed $h \in \mathcal{H}$ and $a \in \mathcal{A}$, let us study the convergence of the sequence $(\mu_h^{a,[n]})_{n\in\mathbb{N}}$. If $\overline{z}_h^a + (1 + \overline{t}_h^a)\overline{\sigma}_h^a(\overline{p}, \overline{q}) > 0$, then by (20.4) it follows that $\mu_h^{a,[n]} = 0$ if n is large enough and the convergence is proved. If instead $\overline{z}_h^a + (1 + \overline{t}_h^a)\overline{\sigma}_h^a(\overline{p}, \overline{q}) = 0$, then from (20.3) we obtain

$$\mu_h^{a,[n]} = \lambda_h^{[n]}(0)q^{a,[n]} - \sum_{s=1}^{S} \lambda_h^{[n]}(s)p^{C,[n]}(s)y^{a,[n]}(s) \to \overline{\lambda}_h(0)\overline{q}^a - \sum_{s=1}^{S} \overline{\lambda}_h(s)\overline{p}^C(s)\overline{y}^a(s) = \overline{\mu}_h^a,$$

and the convergence of $(\mu^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^{AH}$ follows. The proof is complete. $\square$

Lemma 10. *The set*

$$\mathcal{D}_1 = \{E \in \mathcal{E} : \mathcal{F}_0(\xi, E) = 0 \ \Rightarrow\ \forall h \in \mathcal{H}, \forall a \in \mathcal{A}, \max\{\mu_h^a, z_h^a + \sigma_h^a(p, q)\} > 0\}$$

is open and dense in $\mathcal{E}$.

Proof. We have

$$\mathcal{D}_1 = \mathcal{E} \setminus \pi_0\Big(\bigcup_{a\in\mathcal{A}, h\in\mathcal{H}} \Delta^{a,h}\Big)$$

where, for every $a \in \mathcal{A}$ and $h \in \mathcal{H}$,

$$\Delta^{a,h} = \{(\xi, E) : \mathcal{F}_0(\xi, E) = 0 \ \text{ and } \ \max\{\mu_h^a, z_h^a + \sigma_h^a(p, q)\} = 0\}.$$

Immediately we have that

$$\bigcup_{a\in\mathcal{A}, h\in\mathcal{H}} \Delta^{a,h}$$

is closed in $\mathcal{F}_0^{-1}(0)$. Since π_0 is proper,

$$\pi_0\Big(\bigcup_{a\in\mathcal{A}, h\in\mathcal{H}} \Delta^{a,h}\Big)$$

is closed in $\mathcal{E}$ and then $\mathcal{D}_1$ is open in $\mathcal{E}$.

Fix now $(u, Y, \sigma) \in \mathcal{U} \times \mathcal{Y} \times \Sigma$ and define

$$\widetilde{\mathcal{F}} : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim(\Xi)}, \quad (\xi, e) \mapsto \widetilde{\mathcal{F}}(\xi, e) = \mathcal{F}_0(\xi, e, u, Y, \sigma).$$

In order to prove the density of $\mathcal{D}_1$ in $\mathcal{E}$, we note that it is sufficient to show that, for every $(u, Y, \sigma) \in \mathcal{U} \times \mathcal{Y} \times \Sigma$, the set

$$\mathcal{O}_1\,(u, Y, \sigma) = \big\{e \in \mathbb{R}^{GH}_{++} : \widetilde{\mathcal{F}}(\xi, e) = 0 \ \Rightarrow\ \forall h \in \mathcal{H}, \forall a \in \mathcal{A}, \max\{\mu_h^a, z_h^a + \sigma_h^a(p, q)\} > 0\big\}$$

is dense in $\mathbb{R}^{GH}_{++}$.

Let us call $\mathfrak{Q}_h$ the family of all possible *tri-partitions*[9] $\mathbf{Q}_h = \big\{\mathcal{Q}_h^{(1)}, \mathcal{Q}_h^{(2)}, \mathcal{Q}_h^{(3)}\big\}$ of the set $\mathcal{A}$ and let $Q_h^{(i)} = |\mathcal{Q}_h^{(i)}|$, for $i \in \{1, 2, 3\}$. Define then $\mathfrak{Q} = \bigtimes_{h\in\mathcal{H}} \mathfrak{Q}_h$, with generic element $\mathbf{Q} = (\mathbf{Q}_h)_{h\in\mathcal{H}}$, and

$$\mathfrak{Q}^* = \{\mathbf{Q} \in \mathfrak{Q} : \exists h \in \mathcal{H} \text{ such that } \mathcal{Q}_h^{(3)} \neq \varnothing\}.$$

Fixed $\mathbf{Q} \in \mathfrak{Q}^*$, define $k\,(\mathbf{Q}) = \sum_{h\in\mathcal{H}} Q_h^{(3)} > 0$ and

$$\widetilde{\mathcal{F}}^{\mathbf{Q}} : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim(\Xi)+k(\mathbf{Q})},$$

[9] In this context, the term partition is used in a loose manner, as its elements are allowed to be the empty set.

$$
\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi,e)=\left[\begin{array}{ll}
(23.1) & D_{x_h(s)}u_h(x_h)-\lambda_h(s)p(s)\\
(23.2) & \begin{array}{l} -p(0)(x_h(0)-e_h(0))-qz_h \\ -p(s)(x_h(s)-e_h(s))+p^C(s)y(s)z_h, \quad s\in\{1,\ldots,S\}\end{array}\\
(23.3) & -\lambda_h(0)q^a+\sum\limits_{s=1}^{S}\lambda_h(s)p^C(s)y^a(s)+\mu_h^a\\
(23.4) & \mu_h^a, \quad h\in\mathcal{H},\, a\in\mathcal{Q}_h^{(1)}\cup\mathcal{Q}_h^{(3)}\\
(23.5) & z_h^a+\sigma_h^a(p,q), \quad h\in\mathcal{H},\, a\in\mathcal{Q}_h^{(2)}\cup\mathcal{Q}_h^{(3)}\\
(23.6) & \sum\limits_{h=1}^{H}\left(x_h^{\setminus}-e_h^{\setminus}\right)\\
(23.7) & \sum\limits_{h=1}^{H}z_h\\
(23.8) & p^C(s)-1
\end{array}\right] \tag{23}
$$

We are going to show that 0 is a regular value for $\widetilde{\mathcal{F}}^{\mathbf{Q}}$. If that is the case, from Theorem 7, there exists $\mathcal{O}_1^{\mathbf{Q}}(u,Y,\sigma)$ full measure subset of $\mathbb{R}^{GH}_{++}$ such that, for every $e\in\mathcal{O}_1^{\mathbf{Q}}(u,Y,\sigma)$, 0 is a regular value for $\widetilde{\mathcal{F}}^{\mathbf{Q}}(\cdot,e)$, that is, there are no solutions to $\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi,e)=0$. It is immediate to prove that

$$
\mathcal{O}_1(u,Y,\sigma)\supseteq\bigcap_{\mathbf{Q}\in\mathfrak{Q}^*}\mathcal{O}_1^{\mathbf{Q}}(u,Y,\sigma)
$$

and thus $\mathcal{O}_1(u,Y,\sigma)$ has full measure in $\mathbb{R}^{GH}_{++}$ and then it is dense therein.

Let us finally show that, for every $\mathbf{Q}\in\mathfrak{Q}^*$, 0 is a regular value for $\widetilde{\mathcal{F}}^{\mathbf{Q}}$. A suitable submatrix of the Jacobian matrix of the function $\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi,e)$, that we still denote by $D\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi,e)$, is presented in the table below. The components of the function are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the corresponding partial Jacobian is displayed.

The rank computation for $D\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi,e)$ is performed through row and column operations. The starred matrices in the same super-row have the property that some of (possibly all) their columns are used to obtain a full row rank matrix that we employ to "clean up" its super-row, since all the other elements of its super-column are null. The symbol $\circledcirc$ indicates instead a possibly nonzero matrix whose values are insignificant for our argument.

A suitable order in which the appropriate elementary super-column operations have to be performed is the one indicated in the last column of the table.

The above methodology and notation will be used in several rank computations throughout the paper.

	x_1	λ_1	z_1	$\mu_1^{(1,3)}$	$\mu_1^{(2)}$	$e_1^{\setminus}$	$e_1^{\diamond}$	$\dots$	x_H	λ_H	z_H	$\mu_H^{(13)}$	$\mu_H^{(2)}$	$e_H^{\setminus}$	$e_H^{\diamond}$	p^C	
$(23.1)_1$	D^2u_1*	$-\Phi(p)^T$														$\circledcirc$	3
$(23.2)_1$	$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$			$\Phi^{\setminus}(p)$	$I*$									$\circledcirc$	1
$(23.3)_1$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T*$		$\circledcirc$	$\circledcirc$											$\circledcirc$	4
$(23.4)_1$				$I*$													5
$(23.5)_1$			$I(\mathcal{A}, \mathcal{Q}_1^{(2)} \cup \mathcal{Q}_1^{(3)})^T$													$\circledcirc$	
$\vdots$								$\ddots$								$\vdots$	$\vdots$
$(23.1)_H$									D^2u_H*	$-\Phi(p)^T$						$\circledcirc$	3
$(23.2)_H$									$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$			$\Phi^{\setminus}(p)$	$I*$	$\circledcirc$	1
$(23.3)_H$										$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T*$		$\circledcirc$	$\circledcirc$			$\circledcirc$	4
$(23.4)_H$												$I*$					5
$(23.5)_H$											$I(\mathcal{A}, \mathcal{Q}_H^{(2)} \cup \mathcal{Q}_H^{(3)})^T$					$\circledcirc$	
(23.6)	L					$-I*$		$\dots$	L					$-I$			2
(23.7)			I					$\dots$			I						
(23.8)																$I*$	6

(24)

In the previous table, for $h \in \mathcal{H}$, we have set

$$\mu_h^{(i,j)} = (\mu_h^a)_{a \in \mathcal{Q}_h^{(i)} \cup \mathcal{Q}_h^{(j)}}\ ,\ i,j \in \{1,2,3\}.$$

Moreover I is an identity matrix of appropriate dimension, $e_h^{\diamond} = \left(e_h^C(s)\right)_{s\in\mathcal{S}} \in \mathbb{R}_{++}^{S+1}$,

$$\Phi(p) = \begin{bmatrix} p^1(0) & \cdots & p^{C-1}(0) & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & p^1(S) & \cdots & p^{C-1}(S) & 1 \end{bmatrix}_{(S+1)\times G}$$

$$\Phi^{\backslash}(p) = \begin{bmatrix} p^1(0) & \dots & p^{C-1}(0) & & & & \\ & & & \ddots & & & \\ & & & & p^1(S) & \dots & p^{C-1}(S) \end{bmatrix}_{(S+1)\times(G-(S+1))}$$

$$L = \begin{bmatrix} I_{C-1} & 0 & & & \\ & & \ddots & & \\ & & & I_{C-1} & 0 \end{bmatrix}_{G-(S+1)\times G}$$

where I_{C-1} is the identity matrix of dimension $C-1$. Given two finite sets S and T, with $S \supseteq T$, we have denoted by

$$I(S,T) = (\delta_{st})_{s\in S,\, t\in T} \in \mathbb{R}^{|S|} \times \mathbb{R}^{|T|}$$

the matrix with generic element $\delta_{st} = 1$, if $s = t$, and $\delta_{st} = 0$, else. Notice that rank $I(S,T) = |T|$.
The following is the table remaining after the first six steps of our procedure:

	z_1	$\dots$	z_H
$(23.5)_1$	$I(\mathcal{A}, \mathcal{Q}_1^{(2)} \cup \mathcal{Q}_1^{(3)})^T$		
$\vdots$		$\ddots$	
$(23.5)_H$			$I(\mathcal{A}, \mathcal{Q}_H^{(2)} \cup \mathcal{Q}_H^{(3)})^T$
(23.7)	I	$\dots$	I

Observe now that

$$\forall a \in \mathcal{A},\ \exists h(a) \in \mathcal{H} \text{ such that } z_{h(a)}^a + \sigma_{h(a)}^a(p,q) > 0. \tag{25}$$

Indeed, otherwise we would find that there exists $a \in \mathcal{A}$ such that, for every $h \in \mathcal{H}$, $z_h^a + \sigma_h^a(p,q) = 0$. Hence $0 = \sum_h z_h^a = -\sum_h \sigma_h^a(p,q) < 0$, a contradiction. Then, considering for any $a \in \mathcal{A}$ the column in correspondence to $z_{h(a)}^a$, we notice that with all such columns it is possible to get an identity matrix on the last super-row, that does not interfere with the nonnull columns of the matrices in super-rows $(23.5)_1, \dots, (23.5)_H$. Since, for every $h \in \mathcal{H}$, $I(\mathcal{A}, \mathcal{Q}_h^{(2)} \cup \mathcal{Q}_h^{(3)})^T$ has full row rank, the fullness of rank for $D\widetilde{\mathcal{F}}^{\mathbf{Q}}(\xi, e)$ thus follows. $\square$

Let us now introduce the following objects by using a generality which will be useful later. Call $\mathfrak{P}_h$ the family of all possible bi-partitions[10] $\mathbf{P}_h = \left\{\mathcal{P}_h^{(1)}, \mathcal{P}_h^{(2)}\right\}$ of the set $\mathcal{A}$ and let $P_h^{(i)} = |\mathcal{P}_h^{(i)}|$, for $i \in \{1,2\}$. Define $\mathfrak{P} = \bigtimes_{h\in\mathcal{H}} \mathfrak{P}_h$, with generic element $\mathbf{P} = (\mathbf{P}_h)_{h\in\mathcal{H}}$. Fixed $\mathbf{P} \in \mathfrak{P}$, consider

$$\mathcal{F}^{\mathbf{P}} : \Xi \times \mathscr{V} \times \mathcal{T} \to \mathbb{R}^{\dim(\Xi)},$$

[10] Also in this case the term partition in used in a loose form, as some of its elements are allowed to be the empty set.

$$
\mathcal{F}^{\mathbf{P}}(\xi, E, t) = \begin{bmatrix}
(26.1) & D_{x_h(s)} u_h(x_h) - \lambda_h(s) p(s) \\
(26.2) & \begin{array}{l} -p(0)(x_h(0) - e_h(0)) - q z_h \\ -p(s)(x_h(s) - e_h(s)) + p^C(s) y(s) z_h, \quad s \in \{1, \ldots, S\} \end{array} \\
(26.3) & -\lambda_h(0) q^a + \sum_{s=1}^{S} \lambda_h(s) p^C(s) y^a(s) + \mu_h^a \\
(26.4) & \mu_h^a, \quad h \in \mathcal{H},\, a \in \mathcal{P}_h^{(1)} \\
(26.5) & z_h^a + (1 + t_h^a) \sigma_h^a(p, q), \quad h \in \mathcal{H},\, a \in \mathcal{P}_h^{(2)} \\
(26.6) & \sum_{h=1}^{H} \left(x_h^{\setminus} - e_h^{\setminus} \right) \\
(26.7) & \sum_{h=1}^{H} z_h \\
(26.8) & p^C(s) - 1
\end{bmatrix} \tag{26}
$$

where $\mathscr{V}$ has been defined in (14).

Lemma 11. *For every* $\mathbf{P} \in \mathfrak{P}$, $\mathcal{F}^{\mathbf{P}} \in C^1(\Xi \times \mathscr{V} \times \mathcal{T}, \mathbb{R}^{\dim(\Xi)})$.

Proof. Of course $\mathcal{F}^{\mathbf{P}}$ is continuous. We have to show that

$$
d\mathcal{F}^{\mathbf{P}} : (\Xi \times \mathscr{V} \times \mathcal{T}) \times (\mathbb{R}^{\dim(\Xi)} \times \mathscr{V} \times \mathcal{T}) \to \mathbb{R}^{\dim(\Xi)}
$$

is well defined and continuous. Consider then

$$
(\xi, E, t) \in \Xi \times \mathscr{V} \times \mathcal{T}, \quad (\xi^*, E^*, t^*) \in \mathbb{R}^{\dim(\Xi)} \times \mathscr{V} \times \mathcal{T}.
$$

It suffices to show that

$$
\lim_{\varepsilon \to 0} \frac{\mathcal{F}^{\mathbf{P}}(\xi + \varepsilon \xi^*, E + \varepsilon E^*, t + \varepsilon t^*) - \mathcal{F}^{\mathbf{P}}(\xi, E, t)}{\varepsilon}
$$

exists and it is continuous and that can be easily done. $\square$

Lemma 12. *The set*

$$
\mathcal{D}_2 = \left\{ E \in \mathcal{E} : \forall \mathbf{P} \in \mathfrak{P},\ \mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0 \ \Rightarrow\ \det D_\xi \mathcal{F}^{\mathbf{P}}(\xi, E, 0) \neq 0 \right\}
$$

is open and dense in $\mathcal{E}$.

Proof. Openness follows from continuity of the considered functions. In order to show density, it is sufficient to prove that, for every $(u, Y, \sigma) \in \mathcal{U} \times \mathcal{Y} \times \Sigma$, the set

$$
\mathcal{O}_2(u, Y, \sigma) = \left\{ e \in \mathbb{R}^{GH}_{++} : (e, u, Y, \sigma) \in \mathcal{D}_2 \right\}
$$

is dense in $\mathbb{R}^{GH}_{++}$. Fix (u, Y, σ) and $\mathbf{P} \in \mathfrak{P}$, and define

$$
\widetilde{\mathcal{F}}^{\mathbf{P}} : \Xi \times \mathbb{R}^{GH}_{++} \to \mathbb{R}^{\dim(\Xi)},
$$

$$\widetilde{\mathcal{F}}^{\mathbf{P}}(\xi,e)=\left[\begin{array}{ll}(27.1) & D_{x_h(s)}u_h(x_h)-\lambda_h(s)p(s)\\ (27.2) & \begin{array}{l}-p(0)(x_h(0)-e_h(0))-qz_h\\ -p(s)(x_h(s)-e_h(s))+p^C(s)y(s)z_h, \quad s\in\{1,\ldots,S\}\end{array}\\ (27.3) & -\lambda_h(0)q^a+\sum\limits_{s=1}^{S}\lambda_h(s)p^C(s)y^a(s)+\mu_h^a\\ (27.4) & \mu_h^a, \quad h\in\mathcal{H},\ a\in\mathcal{P}_h^{(1)}\\ (27.5) & z_h^a+\sigma_h^a(p,q), \quad h\in\mathcal{H},\ a\in\mathcal{P}_h^{(2)}\\ (27.6) & \sum\limits_{h=1}^{H}\left(x_h^{\setminus}-e_h^{\setminus}\right)\\ (27.7) & \sum\limits_{h=1}^{H}z_h\\ (27.8) & p^C(s)-1\end{array}\right] \tag{27}$$

We are going to prove that, for every $\mathbf{P}\in\mathfrak{P}$, 0 is a regular value for $\widetilde{\mathcal{F}}^{\mathbf{P}}$, so that

$$\mathcal{O}^{\mathbf{P}}(u,Y,\sigma)=\left\{e\in\mathbb{R}^{GH}_{++} : 0 \text{ is a regular value for } \widetilde{\mathcal{F}}^{\mathbf{P}}(\cdot,e)\right\}$$

is a full measure subset of $\mathbb{R}^{GH}_{++}$. Since

$$\mathcal{O}_2(u,Y,\sigma)\supseteq\bigcap_{\mathbf{P}\in\mathfrak{P}}\mathcal{O}^{\mathbf{P}}(u,Y,\sigma),$$

we get the desired result. The rank computation for a suitable submatrix of the Jacobian matrix of the function $\widetilde{\mathcal{F}}^{\mathbf{P}}(\xi,e)$ is performed through row and column operations, similarly to the proof of Lemma 10, and it is presented in the table below.

	x_1	λ_1	z_1	$\mu_1^{(1)}$	$\mu_1^{(2)}$	$e_1^{\setminus}$	$e_1^{\diamond}$	$\dots$	x_H	λ_H	z_H	$\mu_H^{(1)}$	$\mu_H^{(2)}$	$e_H^{\setminus}$	$e_H^{\diamond}$	p^C	
$(27.1)_1$	D^2u_1*	$-\Phi(p)^T$														$\circledcirc$	3
$(27.2)_1$	$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$			$\Phi^{\setminus}(p)$	$I*$									$\circledcirc$	1
$(27.3)_1$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T*$		$\circledcirc$	$\circledcirc$											$\circledcirc$	4
$(27.4)_1$				$I*$													5
$(27.5)_1$			$I(\mathcal{A},\mathcal{P}_1^{(2)})^T$													$\circledcirc$	
$\dots$								$\ddots$								$\dots$	$\dots$
$(27.1)_H$									D^2u_H*	$-\Phi(p)^T$						$\circledcirc$	3
$(27.2)_H$									$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$			$\Phi^{\setminus}(p)$	$I*$	$\circledcirc$	1
$(27.3)_H$										$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T*$		$\circledcirc$	$\circledcirc$			$\circledcirc$	4
$(27.4)_H$												$I*$					5
$(27.5)_H$											$I(\mathcal{A},\mathcal{P}_H^{(2)})^T$					$\circledcirc$	
(27.6)	L					$-I*$		$\dots$	L					$-I$			2
(27.7)			I					$\dots$			I						
(27.8)																$I*$	6

The symbols have the same meaning as in (24).
The remaining steps are exactly as in the proof of Lemma 10, with the only difference that, for every $h \in \mathcal{H}$, the matrix $I(\mathcal{A}, \mathcal{P}_h^{(2)})^T$ has now replaced $I(\mathcal{A}, \mathcal{Q}_h^{(2)} \cup \mathcal{Q}_h^{(3)})^T$. Condition (25) however still holds and thus we can conclude in the same manner. $\square$

Proof of Theorem 3. Define $\mathcal{D} = \mathcal{D}_1 \cap \mathcal{D}_2$. Of course, $\mathcal{D}$ is open and dense in $\mathcal{E}$ and $\mathcal{D}$ is the set of economies $E \in \mathcal{E}$ such that, for every $\xi \in \Xi$ with $\mathcal{F}_0(\xi, E) = 0$, the following conditions hold:

$$\forall h \in \mathcal{H}, \forall a \in \mathcal{A} \text{ either } z_h^a + \sigma_h^a(p,q) > 0 \text{ or } \mu_h^a > 0,$$

$$\mathcal{F}_0 \text{ is } C^1 \text{ in a neighborhood of } (\xi, E),$$
$$\det D_\xi \mathcal{F}_0(\xi, E) \neq 0.$$

We are then left with showing that for any $E^* \in \mathcal{D}$ all conditions in Theorem 3 are satisfied.
From Theorem 2 and Lemma 9, we get

$$\{\xi \in \Xi : \mathcal{F}_0(\xi, E^*) = 0\} = \{\xi^{i*}\}_{i=1}^k, \tag{28}$$

where k is a positive integer. Then, from Theorem 8, there exist an open neighborhood $V(E^*) \subseteq \mathcal{E}$ of E^* and, for every $i \in \{1, \ldots, k\}$, an open neighborhood $O(\xi^{i*}) \subseteq \Xi$ of ξ^{i*} and $\varphi_i : V(E^*) \to O(\xi^{i*})$ such that:

$$\varphi_i \in C^1, \varphi_i(E^*) = \xi^{i*}, \text{ and } O(\xi^{i*}) \cap O(\xi^{j*}) = \varnothing, \text{ for } i \neq j\,, \tag{29}$$

$$\left\{(\xi, E) \in O(\xi^{i*}) \times V(E^*) : \mathcal{F}_0(\xi, E) = 0\right\} = \left\{(\xi, E) \in O(\xi^{i*}) \times V(E^*) : \xi = \varphi_i(E)\right\}. \tag{30}$$

Moreover, again from Lemma 9,

$$\{(\xi, E) \in \Xi \times V(E^*) : \mathcal{F}_0(\xi, E) = 0\} = \bigcup_{i=1}^k \{(\xi, E) \in \Xi \times V(E^*) : \xi = \varphi_i(E)\}. \tag{31}$$

Of course, (28), (29), (30) and (31) imply (15), (16), (17) and (18), respectively.

□

A.3 Generic Pareto Suboptimality of equilibria

Proof of Theorem 4. The existence of the set $\mathcal{D}^\triangle$ follows by a transversality argument. More precisely, let us fix $\mathbf{P} \in \mathfrak{P}$. We recall that, given $E \in \mathcal{E}$, if $\xi = (x, \lambda, z, \mu, p, q) \in \Xi$ is such that $\mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0$, then x is a Pareto Optimal allocation if and only if for every $s, s' \in \mathcal{S}$ and $h, h' \in \mathcal{H}$, it holds that

$$\frac{\lambda_h(s)}{\lambda_h(s')} = \frac{\lambda_{h'}(s)}{\lambda_{h'}(s')}\,.$$

Thus we are going to show that generically this cannot happen when $A < S$. More precisely, we prove that the set

$$\mathcal{D}^{\triangle,\mathbf{P}} = \left\{E \in \mathcal{D} : \mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0 \Rightarrow \exists s, s' \in \mathcal{S} \text{ and } h, h' \in \mathcal{H} \text{ with } \frac{\lambda_h(s)}{\lambda_h(s')} \neq \frac{\lambda_{h'}(s)}{\lambda_{h'}(s')}\right\}$$

is open and dense in $\mathcal{D}$ and thus in $\mathcal{E}$, so that the desired open and dense set $\mathcal{D}^\triangle$ can then be defined as

$$\mathcal{D}^\triangle = \bigcap_{\mathbf{P} \in \mathfrak{P}} \mathcal{D}^{\triangle,\mathbf{P}}.$$

Openness of $\mathcal{D}^{\triangle,\mathbf{P}}$ immediately follows, while, in order to show density, it is sufficient to prove that, for every $(u, Y, \sigma) \in \mathcal{U} \times \mathcal{Y} \times \Sigma$, the set

$$\mathcal{O}^{\triangle,\mathbf{P}}(u, Y, \sigma) = \left\{e \in \mathbb{R}^{GH}_{++} : (e, u, Y, \sigma) \in \mathcal{D}^{\triangle,\mathbf{P}}\right\}$$

is dense in $\mathbb{R}^{GH}_{++}$. Fix then $(u, Y, \sigma) \in \mathcal{U} \times \mathcal{Y} \times \Sigma$. We observe that, since $\operatorname{rank} \mathrm{Y} = \mathrm{A} < \mathrm{S}$, without loss of generality, we can assume that the matrix obtained by Y deleting the first row, say Y_1, still has rank equal to A. Consider then the subset of $\mathcal{O}^{\triangle,\mathbf{P}}(u, Y, \sigma)$ given by

$$\mathcal{O}_1^{\triangle,\mathbf{P}}(u, Y, \sigma) = \left\{e \in \mathbb{R}^{GH}_{++} : \mathcal{F}^{\mathbf{P}}(\xi, e, u, Y, \sigma, 0) = 0 \Rightarrow \lambda_H(1) - \frac{\lambda_1(1)}{\lambda_1(0)}\lambda_H(0) \neq 0\right\}.$$

We will show that such set is dense in $\mathbb{R}^{GH}_{++}$ by using Theorem 7. The proof that $D_{(\xi,e)} \begin{bmatrix} \mathcal{F}^{\mathbf{P}}(\xi, e, u, Y, \sigma, 0) \\ \lambda_H(1) - \frac{\lambda_1(1)}{\lambda_1(0)}\lambda_H(0) \end{bmatrix}$ has full row rank is performed through row and column operations, similarly to the proof of Lemma 10, and it is presented in the table below.

	x_1	λ_1	z_1	$\mu_1^{(1)}$	$\mu_1^{(2)}$	$e_1^{\setminus}$	$e_1^{\diamond}$	$\dots$	x_H	λ_H	z_H	$\mu_H^{(1)}$	$\mu_H^{(2)}$	$e_H^{\setminus}$	$e_H^{\diamond}$	p^C	
$(26.1)_1$	D^2u_1*	$-\Phi(p)^T$														$\circledcirc$	3
$(26.2)_1$	$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$			$\Phi^{\setminus}(p)$	$I*$									$\circledcirc$	1
$(26.3)_1$		$\begin{bmatrix}-q\\Y\end{bmatrix}^T*$		$\circledcirc$	$\circledcirc$											$\circledcirc$	4
$(26.4)_1$				$I*$													5
$(26.5)_1$			$I(\mathcal{A},\mathcal{P}_1^{(2)})^T$													$\circledcirc$	
$\vdots$								$\ddots$								$\vdots$	$\vdots$
$(26.1)_H$									D^2u_H*	$-\Phi(p)^T$						$\circledcirc$	3
$(26.2)_H$									$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$			$\Phi^{\setminus}(p)$	$I*$	$\circledcirc$	1
$(26.3)_H$										$\begin{bmatrix}-q\\Y\end{bmatrix}^T*$		$\circledcirc$	$\circledcirc$			$\circledcirc$	4
$(26.4)_H$												$I*$					5
$(26.5)_H$											$I(\mathcal{A},\mathcal{P}_H^{(2)})^T$					$\circledcirc$	
(26.6)	L					$-I*$		$\dots$	L					$-I$			2
(26.7)			I					$\dots$			I						
(26.8)																$I*$	6
$\lambda_H(1)-\frac{\lambda_1(1)}{\lambda_1(0)}\lambda_H(0)$		$\circledcirc$								$\left(-\frac{\lambda_1(1)}{\lambda_1(0)},1,0,\dots,0\right)*$							7

The symbols have the same meaning as in (24).
Notice that in Steps 4 and 7 we use the fact that $\operatorname{rank} \mathrm{Y}_1 = \mathrm{A} < \mathrm{S}$. Indeed, there exist A columns of Y_1^T which allow to clean up $(33.3)_H$ and, in particular, they permit to cancel $-q^T$ and the remaining columns of Y^T. Moreover the columns of Y_1^T do not interfere with the vector $\big(-\frac{\lambda_1(1)}{\lambda_1(0)}, 1, 0, \ldots, 0\big)$ in the last row. Then, in Step 7, we can exploit the positive term $-\frac{\lambda_1(1)}{\lambda_1(0)}$ to clean up the corresponding row.
The remaining steps are exactly as in the proof of Lemma 10, with the only difference that, for every $h \in \mathcal{H}$, the matrix $I(\mathcal{A}, \mathcal{P}_h^{(2)})^T$ has now replaced $I(\mathcal{A}, \mathcal{Q}_h^{(2)} \cup \mathcal{Q}_h^{(3)})^T$. Condition (25) however still holds and thus we can conclude in the same manner. □

A.4 Generic Pareto Improving

Proof of Theorem 5. We stress that along the present proof we need the stronger Assumptions (5), (6) and (12). Let us define $\widetilde{\Lambda}(\xi, E)$ in analogy with (19). More precisely[11], let

$$\mathbf{P} : \Xi \times \mathcal{E} \to (\mathscr{P}(\mathcal{A}))^{2H}, \quad \mathbf{P}(\xi, E) = \big(\mathcal{P}_h^{(1)}(\xi, E), \mathcal{P}_h^{(2)}(\xi, E)\big)_{h \in \mathcal{H}},$$

with

$$\mathcal{P}_h^{(1)}(\xi, E) = \{a \in \mathcal{A} : \mu_h^a = 0\} \quad \text{and} \quad \mathcal{P}_h^{(2)}(\xi, E) = \{a \in \mathcal{A} : z_h^a + \sigma_h^a(p, q) = 0\}.$$

Define then

$$\widetilde{\Lambda} : \Xi \times \mathcal{E} \to \mathbb{N}, \quad \widetilde{\Lambda}(\xi, E) = \sum_{h \in \mathcal{H}} P_h^{(2)}(\xi, E),$$

where, for every $(\xi, E) \in \Xi \times \mathcal{E}$, $P_h^{(2)}(\xi, E) = |\mathcal{P}_h^{(2)}(\xi, E)|$.
Notice that, given $E \in \mathcal{E}$ and $\theta \in \Theta(E)$, if $\xi \in \Xi$ is the extended equilibrium associated with θ, then $\widetilde{\Lambda}(\xi, E) = \Lambda(\theta, E)$. Moreover, if $(\xi, E) \in \Xi \times \mathcal{D}$ and $\mathcal{F}_0(\xi, E) = 0$, then

$$\mathcal{P}_h^{(1)}(\xi, E) \cup \mathcal{P}_h^{(2)}(\xi, E) = \mathcal{A}, \quad \mathcal{P}_h^{(1)}(\xi, E) \cap \mathcal{P}_h^{(2)}(\xi, E) = \varnothing.$$

Let us introduce now

$$\Gamma : \Xi \times \mathcal{E} \times \mathcal{T} \to \mathbb{R}^H, \quad \Gamma(\xi, E, t) = (u_1(x_1), \ldots, u_H(x_H)),$$

$$\mathfrak{P}_H = \Big\{\mathbf{P} \in \mathfrak{P} : \sum_{h \in \mathcal{H}} P_h^{(2)} \geq H\Big\},$$

$$\mathcal{D}^{\diamond} = \left\{ E \in \mathcal{D} : \forall \mathbf{P} \in \mathfrak{P}_H, \ \begin{cases} \mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0 \\ \mathcal{F}_0(\xi, E) = 0 \end{cases} \Rightarrow \ \operatorname{rank} D_{(\xi,t)} \begin{bmatrix} \mathcal{F}^{\mathbf{P}} \\ \Gamma \end{bmatrix} (\xi, E, 0) = \dim(\Xi) + H \right\}.$$

Define also

$$\mathcal{W}_{PI} = \left\{ (\xi, E) \in \Xi \times \mathcal{E} : \begin{array}{l} \mathcal{F}_0(\xi, E) = 0 \text{ and, for every open neighborhood } \mathcal{V}(0) \subseteq \mathcal{T} \text{ of } 0, \\ \exists t^* \in \mathcal{V}(0), \xi^* \in \Xi, \mathcal{F}(\xi^*, E, t^*) = 0, \ (u_h(x_h^*))_{h \in \mathcal{H}} \gg (u_h(x_h))_{h \in \mathcal{H}} \end{array} \right\}$$

and

$$\mathcal{W} = \Big\{(\xi, E) \in \Xi \times \mathcal{D}^{\diamond} : \mathcal{F}_0(\xi, E) = 0, \widetilde{\Lambda}(\xi, E) \geq H\Big\}.$$

It is immediate to prove that if $\mathcal{W} \subseteq \mathcal{W}_{PI}$, then the conclusions of Theorem 5 holds. In fact, if

$$(\theta^*, E^*) \in \big\{(\theta, E) \in \Theta \times \mathcal{D}^{\diamond} : \theta \in \Theta(E), \Lambda(\theta, E) \geq H\big\}$$

and ξ^* is the extended equilibrium associated with θ^*, then $(\xi^*, E^*) \in \mathcal{W} \subseteq \mathcal{W}_{PI}$, so that (θ^*, E^*) fulfills the conclusions of Theorem 5.
In order to check that $\mathcal{W} \subseteq \mathcal{W}_{PI}$, let $(\xi^*, E^*) \in \mathcal{W}$. By definition of $\mathcal{W}$, $\widetilde{\Lambda}(\xi^*, E^*) \geq H$ and then $\mathbf{P}(\xi^*, E^*) \in \mathfrak{P}_H$. Moreover, $\mathcal{F}^{\mathbf{P}(\xi^*, E^*)}(\xi^*, E^*, 0) = 0$ and, since $E^* \in \mathcal{D}_2$,

$$\det D_\xi \mathcal{F}^{\mathbf{P}(\xi^*, E^*)}(\xi^*, E^*, 0) \neq 0.$$

[11]Given a set S, we denote by $\mathscr{P}(S)$ its power set.

Hence, by the implicit function theorem, there exists a neighborhood $V(0) \subseteq \mathcal{T}$ of 0 and a C^1 function

$$\xi : V(0) \to \Xi, \quad t \mapsto \xi(t),$$

such that, for $t \in V(0)$, $\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi(t), E^*, t) = 0$ and $\xi(0) = \xi^*$. Then, in order to prove that $(\xi^*, E^*) \in \mathcal{W}_{PI}$, it is sufficient to show that $\Gamma(\xi(t))$ is essentially surjective[12] at 0. This is true if $D_t\Gamma(\xi(t))|_{t=0}$ is surjective and, since by the implicit function theorem,

$$D_t\Gamma(\xi(t))|_{t=0} = D_\xi\Gamma(\xi^*)D_t\xi^* = D_\xi\Gamma(\xi^*)[-D_\xi\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0)]^{-1}D_t\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0),$$

then by the Gaussian elimination in block form (cf. Villanacci et al. (2002), Chapter 1, Lemma 8), we find that $D_\xi\Gamma(\xi^*)[-D_\xi\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0)]^{-1}D_t\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0)$ has full rank if and only if

$$\begin{bmatrix} D_\xi\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0) & D_t\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0) \\ D_\xi\Gamma(\xi^*) & 0 \end{bmatrix} = D_{(\xi,t)}\begin{bmatrix} \mathcal{F}^{\mathbf{P}(\xi^*,E^*)} \\ \Gamma \end{bmatrix}(\xi^*, E^*, 0)$$

has full row rank. But this holds true thanks to the definition of $\mathcal{D}^\diamondsuit$, since $\mathcal{F}^{\mathbf{P}(\xi^*,E^*)}(\xi^*, E^*, 0) = 0$. Then we are left with proving that $\mathcal{D}^\diamondsuit$ is open and dense in $\mathcal{E}$. Set, for every $\mathbf{P} \in \mathfrak{P}_H$,

$$\mathcal{D}^{\diamondsuit,\mathbf{P}} = \left\{E \in \mathcal{D} : \left\{\begin{array}{l} \mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0 \\ \mathcal{F}_0(\xi, E) = 0 \end{array}\right. \Rightarrow \quad \text{rank } D_{(\xi,t)}\begin{bmatrix} \mathcal{F}^{\mathbf{P}} \\ \Gamma \end{bmatrix}(\xi, E, 0) = \dim(\Xi) + H\right\}.$$

Since

$$\mathcal{D}^\diamondsuit = \bigcap_{\mathbf{P}\in\mathfrak{P}_H} \mathcal{D}^{\diamondsuit,\mathbf{P}}$$

and since $\mathcal{D}$ is dense in $\mathcal{E}$, it is sufficient to show that, for every $\mathbf{P} \in \mathfrak{P}_H$, $\mathcal{D}^{\diamondsuit,\mathbf{P}}$ is open and dense in $\mathcal{D}$. Let us then fix $\mathbf{P} \in \mathfrak{P}_H$ and consider $\mathcal{D}^{\diamondsuit,\mathbf{P}}$.

As regards the openness of $\mathcal{D}^{\diamondsuit,\mathbf{P}}$, we notice that it is the complement in $\mathcal{D}$ of $\widetilde{\pi}_0(M)$, where

$$M = \left\{(\xi, E) \in \Xi \times \mathcal{D} : \begin{array}{l} \mathcal{F}^{\mathbf{P}}(\xi, E, 0) = 0,\ \mathcal{F}_0(\xi, E) = 0 \ \text{ and} \\ \text{rank } D_{(\xi,t)}\begin{bmatrix} \mathcal{F}^{\mathbf{P}} \\ \Gamma \end{bmatrix}(\xi, E, 0) < \dim(\Xi) + H \end{array}\right\},$$

and

$$\widetilde{\pi}_0 : \mathcal{F}_0^{-1}(0) \cap (\Xi \times \mathcal{D}) \to \mathcal{D},\ (\xi, E) \mapsto \widetilde{\pi}_0(\xi, E) = E.$$

Since the properness of $\widetilde{\pi}_0$ easily follows by Lemma 9, we just have to show that M is closed in $\Xi \times \mathcal{D}$. But it comes from the fact that $\mathcal{F}^{\mathbf{P}}$ is continuous on $\Xi \times \mathcal{D}$, and thus $(\mathcal{F}^{\mathbf{P}})^{-1}(0)$ is closed in $\Xi \times \mathcal{D}$, and by observing that, in order to have

$$\text{rank } D_{(\xi,t)}\begin{bmatrix} \mathcal{F}^{\mathbf{P}} \\ \Gamma \end{bmatrix}(\xi, E, 0) < \dim(\Xi) + H,$$

the determinants of all square submatrices of

$$D_{(\xi,t)}\begin{bmatrix} \mathcal{F}^{\mathbf{P}} \\ \Gamma \end{bmatrix}(\xi, E, 0)$$

of dimension $\dim(\Xi) + H$ have to be zero. Recall indeed that such determinants are continuous functions on $\Xi \times \mathcal{D}$.

We prove the density of $\mathcal{D}^{\diamondsuit,\mathbf{P}}$ in $\mathcal{D}$ by showing that for every $E^* \in \mathcal{D}$ there exists a sequence $\left(E^{[n]}\right)_{n\in\mathbb{N}}$ in $\mathcal{D}^{\diamondsuit,\mathbf{P}}$ such that $E^{[n]} \to E^*$. Fix then $E^* = (e^*, u^*, Y^*, \sigma^*) \in \mathcal{D}$ and consider the set

$$\left\{\xi^{i*}\right\}_{i=1}^k = \{\xi \in \Xi : \mathcal{F}_0(\xi, E^*) = 0\}.$$

[12]We recall that a function $f : M \to N$, with M and N topological spaces, is essentially surjective at $x \in M$ if the image of any open neighborhood of x in M contains an open neighborhood of $f(x)$ in N.

For every $i \in \{1, \ldots, k\}$, let $O\left(\xi^{i*}\right) \subseteq \Xi$ be open sets such that $\xi^{i*} \in O\left(\xi^{i*}\right)$ and $O\left(\xi^{i*}\right) \cap O\left(\xi^{j*}\right) = \varnothing$, for $i \neq j$. Moreover, with any of such ξ^{i*} it is uniquely associated a partition $\mathbf{P}(\xi^{i*}, E^*)$. Let us define

$$\mathfrak{P}(E^*) = \left\{\mathbf{P}(\xi^{i*}, E^*)\right\}_{i=1}^{k}.$$

Notice that, by continuity, in a suitable neighborhood $V(E^*) \subseteq \mathcal{D}$ of E^* we have that

$$(\xi, E) \in \Xi \times V(E^*),\ \mathcal{F}_0(\xi, E) = 0 \ \Rightarrow \ \xi \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \text{ and } \mathbf{P}(\xi, E) \in \mathfrak{P}(E^*).$$

If $\mathbf{P} \notin \mathfrak{P}(E^*)$, then it is immediate to conclude that $E^* \in \mathcal{D}^{\lozenge,\mathbf{P}}$.
If $\mathbf{P} \in \mathfrak{P}(E^*)$, we have to argue as follows. For simplicity, we assume that, for every $h \in \mathcal{H}$, $\mathcal{P}_h^{(2)} \neq \varnothing$. The case in which there exists $\widehat{h} \in \mathcal{H}$ such that $\mathcal{P}_{\widehat{h}}^{(2)} = \varnothing$ can be handled similarly.
Let us select exactly a tool for each consumer: for instance, for any $h \in \mathcal{H}$, we can set

$$a_h = \min\{a \in \mathcal{A} : a \in \mathcal{P}_h^{(2)}\} \tag{32}$$

and work with the H tools $t_1^{a_1}, \ldots, t_H^{a_H}$.
For $i \in \{1, \ldots, k\}$ and $h \in \mathcal{H}$, let $x_h^{i*} \in \mathbb{R}_{++}^G$ be the consumption of household h associated with ξ^{i*} and let $\left\{B_h^i\right\}_{i=1}^k$ be a set of open balls of $\mathbb{R}_{++}^G$ such that $x_h^{i*} \in B_h^i$ and $B_h^i \cap B_h^j = \varnothing$, for $i \neq j$. In particular we can choose $\left\{B_h^i\right\}_{i=1}^k$ so that there exist pairwise disjoint open balls $\left\{\widehat{B}_h^i\right\}_{i=1}^k$ of $\mathbb{R}_{++}^G$ with

$$B_h^i \subseteq \mathrm{Cl}\left(B_h^i\right) \subseteq \widehat{B}_h^i,\ \forall i \in \{1, \ldots, k\},$$

where $\mathrm{Cl}\left(B_h^i\right)$ denotes the closure of B_h^i in $\mathbb{R}_{++}^G$. Let then $\rho_h \in C^\infty(\mathbb{R}_{++}^G, [0,1])$ be such that, for every $x_h \in \bigcup_{i=1}^k B_h^i$, $\rho_h(x_h) = 1$ while, for every $x_h \in \mathbb{R}_{++}^G \setminus \bigcup_{i=1}^k \widehat{B}_h^i$, $\rho_h(x_h) = 0$.
Let $\mathbb{S}_G$ be the set of the $G \times G$ symmetric matrices with real elements[13] and let $\mathcal{N}(0)$ be a neighborhood of $0 \in (\mathbb{S}_G)^H$ such that, for every $A = (A_h)_{h \in \mathcal{H}} \in \mathcal{N}(0)$, it holds that:

- for every $h \in \mathcal{H}$, the function

  $$u_h^{A_h}(x_h) = u_h^*(x_h) + \frac{1}{2}\rho_h(x_h) \sum_{i=1}^{k} \left[\left(x_h - x_h^{i*}\right) A_h \left(x_h - x_h^{i*}\right)\right]$$

  satisfies Assumptions (1)–(4);
- the economy $(e^*, u^A, Y^*, \sigma^*) \in V(E^*)$, where $u^A = \left(u_h^{A_h}\right)_{h \in \mathcal{H}}$.

Note that the first condition follows from Villanacci et al. (2002), Chapter 15, Lemma 25, and the second one is trivial.

For any $\mathbf{P} \in \mathfrak{P}(E^*)$, we can then introduce the functions

$$\widehat{\mathcal{F}}^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathcal{N}(0) \times \mathcal{T} \to \mathbb{R}^{\dim \Xi},$$

[13] Notice that $\mathbb{S}_G$ can be identified with $\mathbb{R}^{\frac{G(G+1)}{2}}$.

$$
\widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, t) = \left[\begin{array}{ll}
(33.1) & D_{x_h(s)}\left(u_h^*(x_h) + \frac{1}{2}\rho_h(x_h)\sum_{i=1}^{k}\left[\left(x_h - x_h^{i*}\right) A_h \left(x_h - x_h^{i*}\right)\right]\right) - \lambda_h(s)p(s) \\
(33.2) & \begin{array}{l} -p(0)(x_h(0) - e_h^*(0)) - qz_h \\ -p(s)(x_h(s) - e_h^*(s)) + p^C(s)y^*(s)z_h, \quad s \in \{1, \ldots, S\} \end{array} \\
(33.3) & -\lambda_h(0)q^a + \sum_{s=1}^{S}\lambda_h(s)p^C(s)y^{*a}(s) + \mu_h^a \\
(33.4) & \mu_h^a, \quad h \in \mathcal{H},\, a \in \mathcal{P}_h^{(1)} \\
(33.5) & z_h^a + (1 + t_h^a)\sigma_h^{*a}(p, q), \quad h \in \mathcal{H},\, a \in \mathcal{P}_h^{(2)} \\
(33.6) & \sum_{h=1}^{H}\left(x_h^{\backslash} - e_h^{*\backslash}\right) \\
(33.7) & \sum_{h=1}^{H} z_h \\
(33.8) & p^C(s) - 1
\end{array}\right] \tag{33}
$$

and

$$
\widehat{\Gamma} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathcal{N}(0) \times \mathcal{T} \to \mathbb{R}^H,
$$

$$
\widehat{\Gamma}(\xi, A, t) = \left[u_h^*(x_h) + \frac{1}{2}\rho_h(x_h)\sum_{i=1}^{k}\left[\left(x_h - x_h^{i*}\right) A_h \left(x_h - x_h^{i*}\right)\right]\right] \tag{34}
$$

Consider then the function

$$
\psi^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{\dim(\Xi)+H} \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi)+\dim(\Xi)+AH+1},
$$

$$
\psi^{\mathbf{P}}(\xi, c, A) = \left[\begin{array}{l}
\widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) \\
\left(D_{(\xi,t)}\left[\begin{array}{c}\widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma}\end{array}\right](\xi, A, 0)\right)^T c \\
\frac{1}{2}cc - 1
\end{array}\right]
$$

If we show that, for almost all $A \in \mathcal{N}(0)$, there is no (ξ, c) such that

$$
\psi^{\mathbf{P}}(\xi, c, A) = 0,
$$

we are done. In fact, if it is the case, there exists a sequence $\left(A^{[n]}\right)_{n\in\mathbb{N}}$ in $\mathcal{N}(0)$ converging to 0 such that, for every $n \in \mathbb{N}$, when $\widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A^{[n]}, 0) = 0$ then $D_{(\xi,t)}\left[\begin{array}{c}\widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma}\end{array}\right](\xi, A^{[n]}, 0)$ has full rank. As $\left(A^{[n]}\right)_{n\in\mathbb{N}}$ is in $\mathcal{N}(0)$, then the sequence of economies $E^{[n]} = (e^*, u^{A^{[n]}}, Y^*, \sigma^*) \to E^*$ and, since $\widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A^{[n]}, 0) = \mathcal{F}^{\mathbf{P}}(\xi, E^{[n]}, 0)$ and $\widehat{\Gamma}(\xi, A^{[n]}, 0) = \Gamma(\xi, E^{[n]}, 0)$, we obtain that $\left(E^{[n]}\right)_{n\in\mathbb{N}}$ is in $\mathcal{D}^{\Diamond,\mathbf{P}}$, as desired.
We are left with showing that, for almost all $A \in \mathcal{N}(0)$, there is no (ξ, c) such that

$$
\psi^{\mathbf{P}}(\xi, c, A) = 0.
$$

Let

$$
c = \left((c_{x_h}, c_{\lambda_h}, c_{z_h}, c_{\mu_h})_{h\in\mathcal{H}}, c_{p^{\backslash}}, c_q, c_{p^C}, c_t\right)
$$
$$
\in \mathbb{R}^{GH} \times \mathbb{R}^{(S+1)H} \times \mathbb{R}^{AH} \times \mathbb{R}^{AH} \times \mathbb{R}^{G-(S+1)} \times \mathbb{R}^{A} \times \mathbb{R}^{S+1} \times \mathbb{R}^{H}.
$$

Since $(e^*, u^A, Y^*, \sigma^*) \in V(E^*) \subseteq \mathcal{D}_2$, then (ξ, c, A) satisfies $\psi^{\mathbf{P}}(\xi, c, A) = 0$ if and only if it satisfies $\widetilde{\psi}^{\mathbf{P}}(\xi, c, A) = 0$, where

$$\widetilde{\psi}^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{\dim(\Xi)+H} \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi)+\dim(\Xi)+AH+1},$$

$$\widetilde{\psi}^{\mathbf{P}}(\xi, c, A) = \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) \\ \left(D_{(\xi,t)} \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma} \end{bmatrix} (\xi, A, 0)\right)^T c \\ \frac{1}{2} c_t c_t - 1 \end{bmatrix}$$

Then we have to prove that, for almost all $A \in \mathcal{N}(0)$, there is no (ξ, c) such that $\widetilde{\psi}^{\mathbf{P}}(\xi, c, A) = 0$. Notice that we can obtain the desired result showing that, for almost all $A \in \mathcal{N}(0)$, there is no (ξ, c) such that $\check{\psi}^{\mathbf{P}}(\xi, c, A) = 0$, where

$$\check{\psi}^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi_i^*\right) \times \mathbb{R}^{\dim(\Xi)+H} \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi)+\dim(\Xi)+H+1},$$

$$\check{\psi}^{\mathbf{P}}(\xi, c, A) = \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) \\ \left(D_{(\xi,\widetilde{t})} \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma} \end{bmatrix} (\xi, A, 0)\right)^T c \\ \frac{1}{2} c_t c_t - 1 \end{bmatrix}$$

and $\widetilde{t} = (t_1^{a_1}, \ldots, t_H^{a_H}) \in \mathbb{R}^H$.

In what follows, for every $h \in \mathcal{H}$, $i = 1, 2$, we define $\mu_h^{(i)} = (\mu_h^a)_{a \in \mathcal{P}_h^{(i)}}$.

The computation of the matrix $D_{(\xi,\widetilde{t})} \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma} \end{bmatrix} (\xi, A, 0)$ is the following:

	x_1	λ_1	z_1	$\mu_1^{(1)}$	$\mu_1^{(2)}$	$\dots$	x_H	λ_H	z_H	$\mu_H^{(1)}$	$\mu_H^{(2)}$	$p^{\backslash}$	p^C	q	$t_1^{a_1}$	$\dots$	$t_H^{a_H}$
$(33.1)_1$	$D^2u_1 + A_1$	$-\Phi(p)^T$										$-\tilde{\Lambda}_1$	$\circledcirc$				
$(33.2)_1$	$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$									$\circledcirc$	$\circledcirc$	$\begin{bmatrix}-z_1\\0\end{bmatrix}$			
$(33.3)_1$		$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$\circledcirc$	$I(\mathcal{A}, \mathcal{P}_1^{(2)})$								$\circledcirc$	$-\lambda_1(0)I$			
$(33.4)_1$				I													
$(33.5)_1$			$I(\mathcal{A}, \mathcal{P}_1^{(2)})^T$									$\circledcirc$	$\circledcirc$	$\circledcirc$	Σ_{a_1}		
$\vdots$						$\ddots$						$\vdots$	$\vdots$	$\vdots$		$\ddots$	
$(33.1)_H$							$D^2u_H + A_H$	$-\Phi(p)^T$				$-\tilde{\Lambda}_H$	$\circledcirc$				
$(33.2)_H$							$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$			$\circledcirc$	$\circledcirc$	$\begin{bmatrix}-z_H\\0\end{bmatrix}$			
$(33.3)_H$								$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$\circledcirc$	$I(\mathcal{A}, \mathcal{P}_H^{(2)})$		$\circledcirc$	$-\lambda_H(0)I$			
$(33.4)_H$										I							
$(33.5)_H$									$I(\mathcal{A}, \mathcal{P}_H^{(2)})^T$			$\circledcirc$	$\circledcirc$	$\circledcirc$			Σ_{a_H}
(33.6)	L					$\dots$	L										
(33.7)			I			$\dots$			I								
(33.8)													I				
$(34)_1$	Du_1																
$\vdots$						$\ddots$											
$(34)_H$							Du_H										

where

$$\widetilde{\Lambda}_h = \begin{bmatrix} \lambda_h(0)I_{C-1} & & \\ 0 & & \\ & \ddots & \\ & & \lambda_h(S)I_{C-1} \\ & & 0 \end{bmatrix}_{G\times(G-(S+1))}$$

and $\Sigma_{a_h} \in \mathbb{R}^{P_h^2}$ is a vector whose components are all null except for the term $\sigma_h^{a_h}(p,q)$ in position a_h, as defined in (32), which is positive by (12). The other symbols have the same meaning as in (24). In particular, we recall that, given two finite sets S and T, with $S \supseteq T$, $I(S,T) = (\delta_{st})_{s\in S,\, t\in T} \in \mathbb{R}^{|S|} \times \mathbb{R}^{|T|}$ is a matrix with generic element $\delta_{st} = 1$, if $s = t$, and $\delta_{st} = 0$, else.

By neglecting some rows and columns, we find that $D_{(\xi,\widetilde{t})}\begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}} \\ \widehat{\Gamma} \end{bmatrix}(\xi, A, 0)$ has full row rank if and only if the following matrix, that we call $M(\xi, A)$, has full row rank.

In the first row and the second column of the table we indicate the dimension of the corresponding super-columns and super-rows.

		G	$S+1$	A	$P_1^{(2)}$	$\dots$	G	$S+1$	A	$P_H^{(2)}$	$G-(S+1)$	A	1	$\dots$	1
		x_1	λ_1	z_1	$\mu_1^{(2)}$	$\dots$	x_H	λ_H	z_H	$\mu_H^{(2)}$	$p\setminus$	q	$t_1^{a_1}$	$\dots$	$t_H^{a_H}$
$(33.1)_1$	G	$D^2u_1+A_1$	$-\Phi(p)^T$								$-\tilde{\Lambda}_1$				
$(33.2)_1$	$S+1$	$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$							$\circledcirc$	$\begin{bmatrix}-z_1\\0\end{bmatrix}$			
$(33.3)_1$	A		$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$I(\mathcal{A},\mathcal{P}_1^{(2)})$							$-\lambda_1(0)I$			
$(33.5)_1$	$P_1^{(2)}$			$I(\mathcal{A},\mathcal{P}_1^{(2)})^T$							$\circledcirc$	$\circledcirc$	Σ_{a_1}		
$\vdots$	$\vdots$					$\ddots$					$\vdots$	$\vdots$		$\ddots$	
$(33.1)_H$	G						$D^2u_H+A_H$	$-\Phi(p)^T$			$-\tilde{\Lambda}_H$				
$(33.2)_H$	$S+1$						$-\Phi(p)$		$\begin{bmatrix}-q\\Y\end{bmatrix}$		$\circledcirc$	$\begin{bmatrix}-z_H\\0\end{bmatrix}$			
$(33.3)_H$	A							$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$I(\mathcal{A},\mathcal{P}_H^{(2)})$		$-\lambda_H(0)I$			
$(33.5)_H$	$P_H^{(2)}$								$I(\mathcal{A},\mathcal{P}_H^{(2)})^T$		$\circledcirc$	$\circledcirc$			Σ_{a_H}
(33.6)	$G-(S+1)$	L				$\dots$	L								
(33.7)	A			I		$\dots$			I						
$(34)_1$	1	Du_1													
$\vdots$	$\vdots$					$\ddots$									
$(34)_H$	1						Du_H								

(35)

Define now

$$\mathbb{R}^{T(\mathbf{P})} = \mathbb{R}^{GH} \times \mathbb{R}^{(S+1)H} \times \mathbb{R}^{AH} \times \mathbb{R}^{\sum_h P_h^{(2)}} \times \mathbb{R}^{G-(S+1)} \times \mathbb{R}^{A} \times \mathbb{R}^{H},$$

with generic element

$$\widetilde{c} = \left((c_{x_h}, c_{\lambda_h}, c_{z_h}, c_{\mu_h^{(2)}})_{h\in\mathcal{H}}, c_{p^{\setminus}}, c_q, c_t\right).$$

Then we conclude if we prove that, for almost all $A \in \mathcal{N}(0)$, there is no $(\xi, \widetilde{c})$ such that $\phi^{\mathbf{P}}(\xi, \widetilde{c}, A) = 0$, where

$$\phi^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{T(\mathbf{P})} \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi)+T(\mathbf{P})+1},$$

$$\phi^{\mathbf{P}}(\xi, \widetilde{c}, A) = \begin{bmatrix} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) \\ M(\xi, A)^T \widetilde{c} \\ \frac{1}{2} c_t c_t - 1 \end{bmatrix}$$

For any $\mathcal{K} \subseteq \mathcal{H}$, consider now the following system in the variables $(\xi, \widetilde{c}, A) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{T(\mathbf{P})} \times \mathcal{N}(0)$

$$\begin{cases} \phi^{\mathbf{P}}(\xi, \widetilde{c}, A) = 0 \\ c_{x_j} = 0, \; j \in \mathcal{H} \setminus \mathcal{K} \\ c_{x_j} \neq 0, \; j \in \mathcal{K}. \end{cases} \tag{36}$$

It is simple to verify that the proof is complete if we show that, for every $\mathcal{K} \subseteq \mathcal{H}$, there exists a full measure subset $\mathcal{N}_{\mathcal{K}}$ of $\mathcal{N}(0)$ such that, for every $A \in \mathcal{N}_{\mathcal{K}}$, there is no $(\xi, \widetilde{c}) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{T(\mathbf{P})}$ such that $(\xi, \widetilde{c}, A)$ solves (36).
Define then,

$$\forall \varnothing \neq \mathcal{K} \subseteq \mathcal{H}, \; O(\mathcal{K}) = \{\widetilde{c} \in \mathbb{R}^{T(\mathbf{P})} : c_{x_j} \neq 0, \forall j \in \mathcal{K}\} \;\text{ and }\; O(\varnothing) = \mathbb{R}^{T(\mathbf{P})}.$$

Such sets are clearly open subsets of $\mathbb{R}^{T(\mathbf{P})}$.
Moreover, for every $\mathcal{K} \subseteq \mathcal{H}$, define $\phi^{\mathbf{P}}_{\mathcal{K}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K}) \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi)+T(\mathbf{P})+1+G|\mathcal{H}\setminus\mathcal{K}|}$ as

$$\phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) = \begin{bmatrix} \phi^{\mathbf{P}}(\xi, \widetilde{c}, A) \\ c_{x_j}, \; j \in \mathcal{H} \setminus \mathcal{K} \end{bmatrix}$$

It is immediate to prove that we get the result if, for every $\mathcal{K} \subseteq \mathcal{H}$, there exists a full measure subset $\widetilde{\mathcal{N}}_{\mathcal{K}}$ of $\mathcal{N}(0)$ such that, for every $A \in \widetilde{\mathcal{N}}_{\mathcal{K}}$, there is no $(\xi, \widetilde{c}) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K})$ with $\phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) = 0$.
Consider then, for every $\mathcal{K} \subseteq \mathcal{H}$, the system

$$\phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) = 0 \tag{37}$$

on $\bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K}) \times \mathcal{N}(0)$.
We prove the desired property analyzing several cases:

Case 1. Assume $\mathcal{K} = \mathcal{H}$. We are going to apply Theorem 7. More precisely, if we show that

$$D_{(\xi,\widetilde{c},A)} \phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) \tag{38}$$

has full row rank for every $(\xi, \widetilde{c}, A) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K}) \times \mathcal{N}(0)$ solution to $\phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) = 0$, then, as desired, there exists a full measure subset $\widetilde{\mathcal{N}}_{\mathcal{K}}$ of $\mathcal{N}(0)$ such that, for every $A \in \widetilde{\mathcal{N}}_{\mathcal{K}}$, there is no $(\xi, \widetilde{c}) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K})$ with $\phi^{\mathbf{P}}_{\mathcal{K}}(\xi, \widetilde{c}, A) = 0$. It is easy to see (cf. Villanacci et al. (2002), Chapter 15, Condition 8) that the matrix in (38) has full row rank if its submatrix

$$B(\xi, \widetilde{c}, A) = \begin{bmatrix} M(\xi, A)^T & N(\xi, \widetilde{c}, A) \\ (0, \ldots, 0, c_t) & 0 \end{bmatrix}$$

has full row rank, where $M(\xi, A)$ has been defined in (35) and $N(\xi, \widetilde{c}, A) = D_A \left[M(\xi, A)^T \widetilde{c}\right]$.
The computation of $B(\xi, \widetilde{c}, A)$ is described below. In the first column of the table we indicate the dimension of the corresponding super-rows.

	c_{x_1}	c_{λ_1}	c_{z_1}	$c_{\mu_1^{(2)}}$	$\cdots$	c_{x_H}	c_{λ_H}	c_{z_H}	$c_{\mu_H^{(2)}}$	$c_{p\backslash}$	c_q	c_{t_1}	$\cdots$	c_{t_H}	A_1	$\cdots$	A_H	
G	$D^2u_1 + A_1$	$-\Phi^T(p)$								L^T		$D^T u_1$			$N_1 *$			1
$S+1$	$-\Phi(p)*$		$\begin{bmatrix} -q \\ Y \end{bmatrix}$															3
A		$\begin{bmatrix} -q \\ Y \end{bmatrix}^T *$		$I(\mathcal{A}, \mathcal{P}_1^{(2)})$							I							5
$P_1^{(2)}$			$I(\mathcal{A}, \mathcal{P}_1^{(2)})^T *$															7
$\vdots$					$\ddots$					$\vdots$	$\vdots$		$\ddots$			$\ddots$		$\vdots$
G						$D^2u_H + A_H$	$-\Phi^T(p)$			L^T				$D^T u_H$			$N_H *$	1
$S+1$						$-\Phi(p)*$		$\begin{bmatrix} -q \\ Y \end{bmatrix}$										3
A							$\begin{bmatrix} -q \\ Y \end{bmatrix}^T *$		$I(\mathcal{A}, \mathcal{P}_H^{(2)})$		I							5
$P_H^{(2)}$								$I(\mathcal{A}, \mathcal{P}_H^{(2)})^T *$										7
$G-(S+1)$	$-\widetilde{\Lambda}_1^T *$	$\circledcirc$		$\circledcirc$	$\cdots$	$-\widetilde{\Lambda}_H^T$	$\circledcirc$		$\circledcirc$									4
A		$\begin{bmatrix} -z_1 \\ 0 \end{bmatrix}^T$	$-\lambda_1(0)I*$	$\circledcirc$	$\cdots$		$\begin{bmatrix} -z_H \\ 0 \end{bmatrix}^T$	$-\lambda_H(0)I*$	$\circledcirc$									6
1				$\Sigma_{a_1}^T *$														8
$\vdots$					$\ddots$													$\vdots$
1									$\Sigma_{a_H}^T *$									8
1												$c_{t_1} *$	$\cdots$	$c_{t_H} *$				2

(39)

We stress that, by Villanacci et al. (2002), Chapter 15, Lemma 28, the matrices $N_h = N_h(\xi, \widetilde{c}, A)$, with $h \in \{1, \ldots, H\}$, have the following property:

$$c_{x_h} \neq 0 \quad \Rightarrow \quad N_h \text{ has full row rank}\,; \tag{40}$$

$$c_{x_h} = 0 \quad \Rightarrow \quad \text{all the components of } N_h \text{ are } 0\,. \tag{41}$$

Then, by performing the column operations according to the order in the last column of Table (39), we conclude. Notice that in Step 1 we have exploited (40), while in Step 6 we have used the fact that, by (25), for every $a \in \mathcal{A}$ there exists $h \in \mathcal{H}$ such that $a \in \mathcal{P}_h^{(1)}$. Hence, the matrix, whose columns are the ones of the matrices $-\lambda_h(0)I$, for $h \in \mathcal{H}$, that match well with $I(\mathcal{A}, \mathcal{P}_h^{(2)})^T$, has full rank. In Step 2 all elements are starred because we don't know which c_{t_h}'s are nonnull: however, since $\frac{1}{2}c_t c_t - 1 = 0$, $c_t \neq 0$.

Case 2. Let us consider $\mathcal{K} = \varnothing$, so that (37) becomes

$$\begin{cases} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) = 0 \\ M(\xi, A)^T \widetilde{c} = 0 \\ \frac{1}{2} c_t c_t - 1 = 0 \\ c_{x_1} = 0 \\ \quad \vdots \\ c_{x_H} = 0. \end{cases} \tag{42}$$

In this case we are going to prove that $\left(\phi_{\mathcal{K}}^{\mathbf{P}}\right)^{-1}(0) = \varnothing$ by showing that, considering just the following subsystem of (42),

$$\begin{cases} M(\xi, A)^T \widetilde{c} = 0 \\ c_{x_1} = 0 \\ \quad \vdots \\ c_{x_H} = 0, \end{cases} \tag{43}$$

if $(\xi, \widetilde{c}, A) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{T(\mathbf{P})} \times \mathcal{N}(0)$ solves (43), then it has to be $\widetilde{c} = 0$. Hence, that solution is not admissible for (42).

System (43) can be represented by the following table

		c_{λ_1}	c_{z_1}	$c_{\mu_1^{(2)}}$	$\cdots$	c_{λ_H}	c_{z_H}	$c_{\mu_H^{(2)}}$	$c_{p\setminus}$	c_q	c_{t_1}	$\cdots$	c_{t_H}
$(44.1)_1$	G	$-\Phi^T(p)$							L^T		$D^T u_1$		
$(44.2)_1$	$S+1$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$										
$(44.3)_1$	A	$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T$		$I(\mathcal{A},\mathcal{P}_1^{(2)})$						I			
$(44.4)_1$	$P_1^{(2)}$		$I(\mathcal{A},\mathcal{P}_1^{(2)})^T$										
$\vdots$	$\vdots$				$\ddots$				$\vdots$	$\vdots$		$\ddots$	
$(44.1)_H$	G					$-\Phi^T(p)$			L^T				$D^T u_H$
$(44.2)_H$	$S+1$						$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$						
$(44.3)_H$	A					$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}^T$		$I(\mathcal{A},\mathcal{P}_H^{(2)})$		I			
$(44.4)_H$	$P_H^{(2)}$						$I(\mathcal{A},\mathcal{P}_H^{(2)})^T$						
(44.5)	$G-(S+1)$	$\circledcirc$		$\circledcirc$	$\cdots$	$\circledcirc$		$\circledcirc$					
(44.6)	A	$\begin{bmatrix}-z_1\\0\end{bmatrix}^T$	$-\lambda_1(0)I$	$\circledcirc$	$\cdots$	$\begin{bmatrix}-z_H\\0\end{bmatrix}^T$	$-\lambda_H(0)I$	$\circledcirc$					
$(44.7)_1$	1			$\Sigma_{a_1}^T$									
$\vdots$	$\vdots$				$\ddots$								
$(44.7)_H$	1							$\Sigma_{a_H}^T$					

(44)

where the components of the vector $\widetilde{c}$ in the upper row have to be intended as multiplying the matrices in the corresponding column, not as the variables with respect to which derivatives are taken. Then, taking the sums along each super-row and setting them equal to 0, this becomes an alternative way of writing a linear system. To such equations we associate the labels in the first column.
By $(44.2)_h$ we thus find $c_{z_h} = 0$. Moreover, recalling (26.1), from $(44.1)_h$ we get

$$c_{p\setminus} = 0 \quad \text{and} \quad c_{\lambda_h} = c_{t_h}\lambda_h.$$

Finally, from this, (26.3) and $(44.3)_h$, we obtain

$$\begin{aligned} 0 &= c_{\lambda_h}\begin{bmatrix} -q \\ Y \end{bmatrix} + c_{\mu_h^{(2)}} I(\mathcal{A}, \mathcal{P}_h^{(2)})^T + c_q I = \\ &= c_{t_h}\lambda_h \begin{bmatrix} -q \\ Y \end{bmatrix} + c_{\mu_h^{(2)}} I(\mathcal{A}, \mathcal{P}_h^{(2)})^T + c_q = -c_{t_h}\begin{bmatrix} \mu_h^1 \\ \vdots \\ \mu_h^A \end{bmatrix}^T + c_{\mu_h^{(2)}} I(\mathcal{A}, \mathcal{P}_h^{(2)})^T + c_q. \end{aligned}$$

Since, by (25), for every $a \in \mathcal{A}$ there exists $h \in \mathcal{H}$ with $\mu_h^a = 0$, then $c_q = 0$.
System (43) can thus be represented as

		c_{λ_1}	$c_{\mu_1^{(2)}}$	$\cdots$	c_{λ_H}	$c_{\mu_H^{(2)}}$	c_{t_1}	$\cdots$	c_{t_H}
$(44.1)_1$	G	$-\Phi^T(p)$					$D^T u_1$		
$(44.3)_1$	A	$\begin{bmatrix} -q \\ Y \end{bmatrix}^T$	$I(\mathcal{A}, \mathcal{P}_1^{(2)})$						
$\vdots$	$\vdots$			$\ddots$				$\ddots$	
$(44.1)_H$	G				$-\Phi^T(p)$				$D^T u_H$
$(44.3)_H$	A				$\begin{bmatrix} -q \\ Y \end{bmatrix}^T$	$I(\mathcal{A}, \mathcal{P}_H^{(2)})$			
(44.5)	$G-(S+1)$	$\circledcirc$	$\circledcirc$	$\ldots$	$\circledcirc$	$\circledcirc$			
(44.6)	A	$\begin{bmatrix} -z_1 \\ 0 \end{bmatrix}^T$	$\circledcirc$	$\ldots$	$\begin{bmatrix} -z_H \\ 0 \end{bmatrix}^T$	$\circledcirc$			
$(44.7)_1$	1		$\Sigma_{a_1}^T$						
$\vdots$	$\vdots$			$\ddots$					
$(44.7)_H$	1					$\Sigma_{a_H}^T$			

Since there are more rows than columns, we can erase some further rows. This corresponds to consider a subsystem of (44), and thus of (43), for which we will show that, for any solution $(\xi, \widetilde{c}, A) \in \bigcup_{i=1}^k O\left(\xi^{i*}\right) \times \mathbb{R}^{T(\mathbf{P})} \times \mathcal{N}(0)$, it has to be $\widetilde{c} = 0$, as already explained.
More precisely, for every $h \in \mathcal{H}$, using (26.1) we reduce the dimension of $(44.1)_h$ from G to $S+1$ by rewriting it as

$$\left(-c_{\lambda_h} + c_{t_h}\lambda_h\right)\Phi(p) = 0$$

and by considering just the $C(s+1)$-th components, for $s \in \{0, \ldots, S\}$, i.e.,

$$-c_{\lambda_h(s)} + c_{t_h}\lambda_h(s) = 0.$$

Moreover we reduce the dimension of $(44.3)_h$ from A to $P_h^{(2)}$, by considering just the rows corresponding to $\mathcal{P}_h^{(2)}$. In particular, we call $\begin{bmatrix} -q^{(2)}_{h} \\ Y^{(2)}_{h} \end{bmatrix}^T$ the matrix so obtained by $\begin{bmatrix} -q \\ Y \end{bmatrix}^T$. Finally we erase the super-rows corresponding to (44.5) and (44.6). In the first column of the next table we indicate the matching variables:

			c_{λ_1}	$c_{\mu_1^{(2)}}$	$\cdots$	c_{λ_H}	$c_{\mu_H^{(2)}}$	c_{t_1}	$\cdots$	c_{t_H}
c_{λ_1}	$(44.1)_1$	$S+1$	$-I$					λ_1		
$c_{\mu_1^{(2)}}$	$(44.3)_1$	$P_1^{(2)}$	$\begin{bmatrix} -q^{(2)}_{1} \\ Y^{(2)}_{1} \end{bmatrix}^T$	I						
$\vdots$	$\vdots$	$\vdots$			$\ddots$				$\ddots$	
c_{λ_H}	$(44.1)_H$	$S+1$				$-I$				λ_H
$c_{\mu_H^{(2)}}$	$(44.3)_H$	$P_H^{(2)}$				$\begin{bmatrix} -q^{(2)}_{H} \\ Y^{(2)}_{H} \end{bmatrix}^T$	I			
c_{t_1}	$(44.7)_1$	1		$\Sigma_{a_1}^T$						
$\vdots$	$\vdots$	$\vdots$			$\ddots$					
c_{t_H}	$(44.7)_H$	1					$\Sigma_{a_H}^T$			

(45)

Since the matrix above is square, in order to get the desired result, it is sufficient to show that it has full row rank.
By elementary row and column operations, for $h \in \mathcal{H}$, we can cancel $\begin{bmatrix} -q^{(2)}_{h} \\ Y^{(2)}_{h} \end{bmatrix}^T$ from $(44.3)_h$, but then

in the same row the term $-\mu_h^{(2)}$ appears in the column corresponding to c_{t_h}.

		c_{λ_1}	$c_{\mu_1^{(2)}}$	$\cdots$	c_{λ_H}	$c_{\mu_H^{(2)}}$	c_{t_1}	$\cdots$	c_{t_H}
$(44.1)_1$	$S+1$	$-I$					λ_1		
$(44.3)_1$	$P_1^{(2)}$		I				$-\mu_1^{(2)}$		
$\vdots$	$\vdots$			$\ddots$				$\ddots$	
$(44.1)_H$	$S+1$				$-I$				λ_H
$(44.3)_H$	$P_H^{(2)}$					I			$-\mu_H^{(2)}$
$(44.7)_1$	1		$\Sigma_{a_1}^T$						
$\vdots$	$\vdots$			$\ddots$					
$(44.7)_H$	1					$\Sigma_{a_H}^T$			

We are thus led to consider the simpler matrix:

		$c_{\mu_1^{(2)}}$	$\cdots$	$c_{\mu_H^{(2)}}$	c_{t_1}	$\cdots$	c_{t_H}
$(44.3)_1$	$P_1^{(2)}$	I			$-\mu_1^{(2)}$		
$\vdots$	$\vdots$		$\ddots$			$\ddots$	
$(44.3)_H$	$P_H^{(2)}$			I			$-\mu_H^{(2)}$
$(44.7)_1$	1	$\Sigma_{a_1}^T$					
$\vdots$	$\vdots$		$\ddots$				
$(44.7)_H$	1			$\Sigma_{a_H}^T$			

By elementary row and column operations, for $h \in \mathcal{H}$, we can erase $\Sigma_{a_h}^T$ from $(44.7)_h$, but then in the same row the positive term $\mu_h^{a_h}\sigma_h^{a_h}(p,q)$ appears in the column corresponding to c_{t_h}. Notice that here we are using Assumption (12).

		$c_{\mu_1^{(2)}}$	$\cdots$	$c_{\mu_H^{(2)}}$	c_{t_1}	$\cdots$	c_{t_H}
$(44.3)_1$	$P_1^{(2)}$	I			$-\mu_1^{(2)}$		
$\vdots$	$\vdots$		$\ddots$			$\ddots$	
$(44.3)_H$	$P_H^{(2)}$			I			$-\mu_H^{(2)}$
$(44.7)_1$	1				$\mu_1^{a_1}\sigma_1^{a_1}(p,q)$		
$\vdots$	$\vdots$					$\ddots$	
$(44.7)_H$	1						$\mu_H^{a_H}\sigma_H^{a_H}(p,q)$

The above matrix has clearly full rank.

Case 3. Consider $\emptyset \neq \mathcal{K} \neq \mathcal{H}$. Then (37) becomes

$$\begin{cases} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) = 0 \\ M(\xi, A)^T \widetilde{c} = 0 \\ \frac{1}{2} c_t c_t - 1 = 0 \\ c_{x_j} = 0, \; j \in \mathcal{H} \setminus \mathcal{K}. \end{cases} \tag{46}$$

Let us show that a suitable subsystem of (46) and, more precisely, of

$$\begin{cases} \widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0) = 0 \\ M(\xi, A)^T \widetilde{c} = 0 \\ c_{x_j} = 0, \; j \in \mathcal{H} \setminus \mathcal{K} \end{cases} \tag{47}$$

has generically no solution by a transversality argument.
As already explained in Case 2, a suitable subsystem of (47) can be represented by the following table,

where we assume $i \in \mathcal{K}$ and $j \in \mathcal{H} \setminus \mathcal{K}$.

		c_{x_i}	c_{λ_i}	c_{z_i}	$c_{\mu_i^{(2)}}$	c_{λ_j}	c_{z_j}	$c_{\mu_j^{(2)}}$	$c_{p\backslash}$	c_q	c_{t_i}	c_{t_j}
$(48.1)_i$	G	$D^2 u_i + A_i$	$-\Phi^T(p)$						L^T		$D^T u_i$	
$(48.2)_i$	$S+1$	$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$								
$(48.3)_i$	A		$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$I(\mathcal{A}, \mathcal{P}_i^{(2)})$					I		
$(48.4)_i$	$P_i^{(2)}$			$I(\mathcal{A}, \mathcal{P}_i^{(2)})^T$								
$(48.1)_j$	G					$-\Phi^T(p)$			L^T			$D^T u_j$
$(48.2)_j$	$S+1$						$\boxed{\begin{smallmatrix}-q\\Y\end{smallmatrix}}$					
$(48.3)_j$	A					$\begin{bmatrix}-q\\Y\end{bmatrix}^T$		$I(\mathcal{A}, \mathcal{P}_j^{(2)})$		I		
$(48.4)_j$	$P_j^{(2)}$						$I(\mathcal{A}, \mathcal{P}_j^{(2)})^T$					
(48.5)	$G-(S+1)$	$-\tilde{\Lambda}_i^T$	$\circledcirc$		$\circledcirc$	$\circledcirc$		$\circledcirc$				
(48.6)	A		$\begin{bmatrix}-z_i\\0\end{bmatrix}^T$	$-\lambda_i(0)I$	$\circledcirc$	$\begin{bmatrix}-z_j\\0\end{bmatrix}^T$	$-\lambda_j(0)I$	$\circledcirc$				
$(48.7)_i$	1				$\Sigma_{a_i}^T$							
$(48.7)_j$	1							$\Sigma_{a_j}^T$				

(48)

Exactly as in Case 2, from System (48) we obtain

$$c_{z_j} = 0, \quad \forall j \in \mathcal{H} \setminus \mathcal{K}$$

$$c_{p\backslash} = 0$$

and

$$c_{\lambda_j} = c_{t_j} \lambda_j, \quad \forall j \in \mathcal{H} \setminus \mathcal{K}.$$

Thus, erasing the null terms in System (48), it becomes

		c_{x_i}	c_{λ_i}	c_{z_i}	$c_{\mu_i^{(2)}}$	c_{λ_j}	$c_{\mu_j^{(2)}}$	c_q	c_{t_i}	c_{t_j}
$(48.1)_i$	G	$D^2u_i + A_i$	$-\Phi^T(p)$						D^Tu_i	
$(48.2)_i$	$S+1$	$-\Phi(p)$		$\begin{bmatrix} -q \\ Y \end{bmatrix}$						
$(48.3)_i$	A		$\begin{bmatrix} -q \\ Y \end{bmatrix}^T$		$I(\mathcal{A}, \mathcal{P}_i^{(2)})$			I		
$(48.4)_i$	$P_i^{(2)}$			$I(\mathcal{A}, \mathcal{P}_i^{(2)})^T$						
$(48.1)_j$	G					$-\Phi^T(p)$				D^Tu_j
$(48.3)_j$	A					$\begin{bmatrix} -q \\ Y \end{bmatrix}^T$	$I(\mathcal{A}, \mathcal{P}_j^{(2)})$	I		
(48.5)	$G-(S+1)$	$-\tilde{\Lambda}_i^T$	$\circledcirc$		$\circledcirc$	$\circledcirc$	$\circledcirc$			
(48.6)	A		$\begin{bmatrix} -z_i \\ 0 \end{bmatrix}^T$	$-\lambda_i(0)I$	$\circledcirc$	$\begin{bmatrix} -z_j \\ 0 \end{bmatrix}^T$	$\circledcirc$			
$(48.7)_i$	1				$\Sigma_{a_i}^T$					
$(48.7)_j$	1						$\Sigma_{a_j}^T$			

Since there are more rows than columns, we can erase some further rows, provided the number of rows still exceeds the number of columns. This corresponds to consider a subsystem of (48), for which we will show that it generically has no solutions.
More precisely, for every $j \in \mathcal{H} \setminus \mathcal{K}$, using (26.1), we reduce the dimension of $(48.1)_j$ from G to $S+1$ by rewriting it as

$$(-c_{\lambda_j} + c_{t_j}\lambda_j)\Phi(p) = 0$$

and by considering just the $C(s+1)$-th components, for $s \in \{0, \ldots, S\}$, i.e.,

$$-c_{\lambda_j(s)} + c_{t_j}\lambda_j\,(s) = 0.$$

Moreover, for every $j \in \mathcal{H} \setminus \mathcal{K}$, we reduce the dimension of $(48.3)_j$ from A to $P_j^{(2)}$, by considering only the rows corresponding to $\mathcal{P}_j^{(2)}$. In particular, we call $\begin{bmatrix} -q_{\,j}^{(2)} \\ Y_{\,j}^{(2)} \end{bmatrix}^T$ the matrix so obtained by $\begin{bmatrix} -q \\ Y \end{bmatrix}^T$.
Finally we erase the super-row corresponding to (48.6).
In this way, we obtain the following table, whose first column indicates the matching variables of each super-row. The square around c_q in the super-row corresponding to (48.5) means that we match that super-row with the lower dimensional variable c_q.

			c_{x_i}	c_{λ_i}	c_{z_i}	$c_{\mu_i^{(2)}}$	c_{λ_j}	$c_{\mu_j^{(2)}}$	c_q	c_{t_i}	c_{t_j}
c_{x_i}	$(48.1)_i$	G	$D^2 u_i + A_i$	$-\Phi^T(p)$						$D^T u_i$	
c_{λ_i}	$(48.2)_i$	$S+1$	$-\Phi(p)$		$\boxed{\begin{smallmatrix}-q\\ Y\end{smallmatrix}}$						
c_{z_i}	$(48.3)_i$	A		$\boxed{\begin{smallmatrix}-q\\ Y\end{smallmatrix}}^T$		$I(\mathcal{A}, \mathcal{P}_i^{(2)})$			I		
$c_{\mu_i^{(2)}}$	$(48.4)_i$	$P_i^{(2)}$			$I(\mathcal{A}, \mathcal{P}_i^{(2)})^T$						
c_{λ_j}	$(48.1)_j$	$S+1$					$-I$				λ_j
$c_{\mu_j^{(2)}}$	$(48.3)_j$	$P_j^{(2)}$					$\begin{bmatrix}-q_j^{(2)}\\ Y_j^{(2)}\end{bmatrix}^T$	I	$I(\mathcal{A}, \mathcal{P}_j^{(2)})^T$		
$\boxed{c_q}$	(48.5)	$G-(S+1)$	$-\tilde{\Lambda}_i^T$	$\circledcirc$		$\circledcirc$	$\circledcirc$	$\circledcirc$			
c_{t_i}	$(48.7)_i$	1				$\Sigma_{a_i}^T$					
c_{t_j}	$(48.7)_j$	1						$\Sigma_{a_j}^T$			

Let us call $\delta_{\mathcal{K}}^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})} \times \mathcal{N}(0) \to \mathbb{R}^{M_{\mathcal{K}}(\mathbf{P})}$, the function describing the left hand side of the above linear system[14], where

$$\mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})} = \mathbb{R}^{G|\mathcal{K}|} \times \mathbb{R}^{(S+1)|\mathcal{K}|} \times \mathbb{R}^{A|\mathcal{K}|} \times \mathbb{R}^{\sum_{i\in\mathcal{K}} P_i^{(2)}} \times \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{(S+1)|\mathcal{H}\setminus\mathcal{K}|} \times \mathbb{R}^{\sum_{j\in\mathcal{H}\setminus\mathcal{K}} P_j^{(2)}} \times \mathbb{R}^{|\mathcal{H}\setminus\mathcal{K}|} \times \mathbb{R}^{A}$$

and

$$\mathbb{R}^{M_{\mathcal{K}}(\mathbf{P})} = \mathbb{R}^{G|\mathcal{K}|} \times \mathbb{R}^{(S+1)|\mathcal{K}|} \times \mathbb{R}^{A|\mathcal{K}|} \times \mathbb{R}^{\sum_{i\in\mathcal{K}} P_i^{(2)}} \times \mathbb{R}^{|\mathcal{K}|} \times \mathbb{R}^{(S+1)|\mathcal{H}\setminus\mathcal{K}|} \times \mathbb{R}^{\sum_{j\in\mathcal{H}\setminus\mathcal{K}} P_j^{(2)}} \times \mathbb{R}^{|\mathcal{H}\setminus\mathcal{K}|} \times \mathbb{R}^{(G-(S+1))}.$$

More precisely, for $(\xi, \check{c}, A) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})} \times \mathcal{N}(0)$, with

$$\check{c} = \left(\left(c_{x_i}, c_{\lambda_i}, c_{z_i}, c_{\mu_i^{(2)}}, c_{t_i}\right)_{i\in\mathcal{K}}, \left(c_{\lambda_j}, c_{\mu_j^{(2)}}, c_{t_j}\right)_{j\in\mathcal{H}\setminus\mathcal{K}}, c_q\right),$$

[14]Notice that we have suitably permuted the rows and columns of the above table.

$$
\delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A) = \left[\begin{array}{l}
c_{x_i} \left(D^2 u_i(x_i) + A_i \right) - c_{\lambda_i} \Phi(p) + c_{t_i} D u_i(x_i), \quad i \in \mathcal{K} \\
-c_{x_i} \Phi(p)^T + c_{z_i} \left[\begin{array}{c} -q \\ Y \end{array} \right]^T, \quad i \in \mathcal{K} \\
c_{\lambda_i} \left[\begin{array}{c} -q \\ Y \end{array} \right] + c_{\mu_i^{(2)}} I(\mathcal{A}, \mathcal{P}_i^{(2)})^T + c_q, \quad i \in \mathcal{K} \\
c_{z_i} I(\mathcal{A}, \mathcal{P}_i^{(2)}), \quad i \in \mathcal{K} \\
c_{\mu_i^{(2)}} \Sigma_{a_i}, \quad i \in \mathcal{K} \\
-c_{\lambda_j} + c_{t_j} \lambda_j, \quad j \in \mathcal{H} \setminus \mathcal{K} \\
c_{\lambda_j} \left[\begin{array}{c} -q^{(2)}_{j} \\ Y^{(2)}_{j} \end{array} \right] + c_{\mu_j^{(2)}} + c_q I(\mathcal{A}, \mathcal{P}_j^{(2)}), \quad j \in \mathcal{H} \setminus \mathcal{K} \\
c_{\mu_j^{(2)}} \Sigma_{a_j}, \quad j \in \mathcal{H} \setminus \mathcal{K} \\
-c_{x_i} \widetilde{\Lambda}_i + c_{\lambda_i} \circledcirc + c_{\mu_i^{(2)}} \circledcirc + c_{\lambda_j} \circledcirc + c_{\mu_j^{(2)}} \circledcirc, \quad i \in \mathcal{K},\ j \in \mathcal{H} \setminus \mathcal{K}
\end{array} \right]
$$

Then we are going to apply Theorem 7 to

$$
\Delta_{\mathcal{K}}^{\mathbf{P}} : \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})} \times \mathcal{N}(0) \to \mathbb{R}^{\dim(\Xi) + M_{\mathcal{K}}(\mathbf{P})},
$$

$$
(\xi, \check{c}, A) \mapsto \Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A) = \left(\widehat{\mathcal{F}}^{\mathbf{P}}(\xi, A, 0), \delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A) \right).
$$

Since, by our assumptions, $A < G-(S+1)$, and thus $L_{\mathcal{K}}(\mathbf{P}) < M_{\mathcal{K}}(\mathbf{P})$, if we prove that, for every solution to $\Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A) = 0$,

$$
D_{(\xi, \check{c}, A)} \Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A)
$$

has full row rank, then there exists a full measure subset $\widetilde{\mathcal{N}}_{\mathcal{K}}$ of $\mathcal{N}(0)$ such that, for every $A \in \widetilde{\mathcal{N}}_{\mathcal{K}}$, there is no $(\xi, \check{c}) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times \check{O}(\mathcal{K})$ such that $\Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A) = 0$, where

$$
\check{O}(\mathcal{K}) = \{ \check{c} \in \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})} : c_{x_j} \neq 0, \forall j \in \mathcal{K} \} \subseteq \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P})}.
$$

Then, a fortiori, for every $A \in \widetilde{\mathcal{N}}_{\mathcal{K}}$, there is no $(\xi, \widetilde{c}) \in \bigcup_{i=1}^{k} O\left(\xi^{i*}\right) \times O(\mathcal{K})$ such that $\phi_{\mathcal{K}}^{\mathbf{P}}(\xi, \widetilde{c}, A) = 0$, as desired.
In particular we are going to show that $D_{(\xi, \check{c}, A)} \Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A)$ has full row rank by checking that a suitable submatrix of $D_{(\widehat{c}, A)} \Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A)$ has full row rank, where

$$
\widehat{c} = \left(\left(c_{x_i}, c_{\lambda_i}, c_{z_i}, c_{\mu_i^{(2)}}, c_{t_i} \right)_{i \in \mathcal{K}}, \left(c_{\lambda_j}, c_{\mu_j^{(2)}}, c_{t_j} \right)_{j \in \mathcal{H} \setminus \mathcal{K}} \right) \in \mathbb{R}^{L_{\mathcal{K}}(\mathbf{P}) - A}.
$$

Recalling (41), the computation of the submatrix of $D_{(\widehat{c}, A)} \Delta_{\mathcal{K}}^{\mathbf{P}}(\xi, \check{c}, A)$ is described in the following table. The numbers in the last column indicate the order of the steps of the perturbation technique.

	c_{x_i}	c_{λ_i}	c_{z_i}	$c_{\mu_i^{(2)}}$	c_{λ_j}	$c_{\mu_j^{(2)}}$	c_{t_i}	c_{t_j}	A_i	
G	$D^2 u_i + A_i$	$-\Phi^T(p)$					$D^T u_i$		$N_i *$	1
$S+1$	$-\Phi(p)*$		$\boxed{\begin{matrix}-q\\ Y\end{matrix}}$							2
A		$\boxed{\begin{matrix}-q\\ Y\end{matrix}}^T *$		$I(\mathcal{A}, \mathcal{P}_i^{(2)})$						4
$P_i^{(2)}$			$I(\mathcal{A}, \mathcal{P}_i^{(2)})^T *$							5
$S+1$					$-I$			λ_j		
$P_j^{(2)}$					$\begin{bmatrix} -q_j^{(2)} \\ Y_j^{(2)} \end{bmatrix}^T$	I				
$G-(S+1)$	$-\tilde{\Lambda}_i^T *$	$\circledcirc$		$\circledcirc$	$\circledcirc$	$\circledcirc$				3
1				$\Sigma_{a_i}^T *$						6
1						$\Sigma_{a_j}^T$				

Hence we are left with showing that the following smaller matrix has full row rank:

		c_{λ_j}	$c_{\mu_j^{(2)}}$	c_{t_j}
c_{λ_j}	$S+1$	$-I$		λ_j
$c_{\mu_j^{(2)}}$	$P_j^{(2)}$	$\begin{bmatrix} -q_j^{(2)} \\ Y_j^{(2)} \end{bmatrix}^T$	I	
c_{t_j}	1		$\Sigma_{a_j}^T$	

Assume $\mathcal{H} \setminus \mathcal{K} = \{1, \ldots, \widehat{H}\}$, for some $\widehat{H} \geq 1$. The matrix above then becomes:

		c_{λ_1}	$c_{\mu_1^{(2)}}$	$\cdots$	$c_{\lambda_{\widehat{H}}}$	$c_{\mu_{\widehat{H}}^{(2)}}$	c_{t_1}	$\cdots$	$c_{t_{\widehat{H}}}$
c_{λ_1}	$S+1$	$-I$					λ_1		
$c_{\mu_1^{(2)}}$	$P_1^{(2)}$	$\begin{bmatrix} -q_1^{(2)} \\ Y_1^{(2)} \end{bmatrix}^T$	I						
$\vdots$	$\vdots$			$\ddots$				$\ddots$	
$c_{\lambda_{\widehat{H}}}$	$S+1$				$-I$				$\lambda_{\widehat{H}}$
$c_{\mu_{\widehat{H}}^{(2)}}$	$P_{\widehat{H}}^{(2)}$				$\begin{bmatrix} -q_{\widehat{H}}^{(2)} \\ Y_{\widehat{H}}^{(2)} \end{bmatrix}^T$	I			
c_{t_1}	1		$\Sigma_{a_1}^T$						
$\vdots$	$\vdots$			$\ddots$					
$c_{t_{\widehat{H}}}$	1					$\Sigma_{a_{\widehat{H}}}^T$			

The matrix above can then be shown to have full rank following the steps used for the matrix in (45). The only difference between the two matrices is indeed the dimension, but the structure is exactly the same. $\square$

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