Existence of financial equilibria with endogenous borrowing restrictions and real assets

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May 14, 2010

Abstract

We show existence of equilibria for a general class of economies in a model with real assets and restricted participation described by household specific endogenous borrowing constraints.

Keywords: General equilibrium; Restricted participation; Financial markets; Real and numeraire assets.

JEL classification: D50, D52, D53.

1 Introduction

We analyze a general financial equilibrium model with real assets and restricted participation. Assets are said to be real if they promise to deliver bundles of commodities. Participation is said to be restricted if each asset demand has to belong to a household specific set, called restriction set.

For a recent, brief survey of the literature on restricted participation see Carosi, Gori and Villanacci (2009). The standard main reference on financial equilibrium with real assets is still Duffie and Shafer (1985). As it is well known, in the case of incomplete markets with real assets, since changes in prices may change the rank of the return matrix causing a discontinuity in the demand function, equilibria may not exist. On the other hand, using a description of the financial structure in terms of Grassmannian manifolds, Duffie and Shafer (1985) show existence for every utility function vector and every endowment and yield vectors in an open and full measure set.

At the best of our knowledge, the only contribution which combines both real assets and restricted participation is the paper by Polemarchakis and Siconolfi (1997), whose restriction sets are not immediate to interpret. Each household is exogenously associated with a vector subspace of the possible wealth transfers; her actual restriction set is then described by the orthogonal projection of that subspace on the (price dependent) image of the return matrix.

In the present paper, we combine the presence of real assets with simple, economically sound participation restrictions on financial markets. Indeed, each household faces an asset specific borrowing constraint which depends upon prices, the existence of exogenously given bound being quite hard to justify on economic ground.

Main characteristics of our existence result can be described as follows. First of all, we do not use Grassmannian manifolds to describe the asset return structure. Moreover, we show existence of equilibria for all economies described by strictly positive endowments, standard smooth utility functions, general borrowing constraints and a quite large class of assets. In fact, our result holds true if assets are of the numeraire type, or if assets deliver general bundles of commodities, as long as yield vectors are sufficiently diversified in each state. Our result is not a generic one as in the model of Duffie and Shafer (1985). There, knowing the characteristics of a specific economy does not allow to conclude the desired existence result. In our case, the set of economies for which existence is obtained is described by the properties imposed on each exogenous variable, properties which can be immediately checked to hold.

In the remainder of the paper, we first describe the set-up of the model and the main results and we finally present proofs.

2 Set-up of the model and main results

Our model builds up on the very standard two-period, pure exchange economy with uncertainty and financial markets. We consider a commodity market in which $C \geq 2$ types of different commodities, denoted by $c \in \mathcal{C} = \{1, 2, \ldots, C\}$, are traded both today and tomorrow. We assume that tomorrow only one among $S \geq 1$ possible states of the world, denoted by $s \in \{1, \ldots, S\}$, will occur. We denote today by $s = 0$ and we define $\mathcal{S} = \{0, 1, \ldots, S\}$. Asset markets open in the first period, and there are $A \geq 1$ assets traded, denoted by $a \in \mathcal{A} = \{1, 2, \ldots, A\}$. We assume $A \leq S$. Finally, there are $H \geq 2$ households, denoted by $h \in \mathcal{H} = \{1, 2, \ldots, H\}$. The time structure of the model is as follows: today, households exchange commodities and assets, and consumption takes place. Then, tomorrow, uncertainty is resolved, households honor their financial obligations, and they again exchange and then consume commodities.

We denote by[1] $x_h^c(s) \in \mathbb{R}_{++}$ the consumption of commodity c in state s by household h and by $e_h^c(s) \in \mathbb{R}_{++}$ the endowment of commodity c in state s owned by household h. We define

$$\begin{aligned} x_h(s) &= (x_h^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad x_h = (x_h(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad x = (x_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}, \\ e_h(s) &= (e_h^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad e_h = (e_h(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad e = (e_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}, \end{aligned}$$

where $G = C(S+1)$. Household h's preferences are represented by a utility function $u_h : \mathbb{R}^G_{++} \to \mathbb{R}$. As in most of the literature on smooth economies we assume that, for every $h \in \mathcal{H}$,

$$u_h \in C^2(\mathbb{R}^G_{++}); \tag{1}$$

$$\text{for every } x_h \in \mathbb{R}^G_{++},\ Du_h(x_h) \gg 0\,; \tag{2}$$

$$\text{for every } v \in \mathbb{R}^G \setminus \{0\} \text{ and } x_h \in \mathbb{R}^G_{++},\ Du_h(x_h)\, v = 0 \text{ implies } v\, D^2u_h(x_h)\, v < 0\,; \tag{3}$$

$$\text{for every } \underline{x}_h \in \mathbb{R}^G_{++},\ \left\{x_h \in \mathbb{R}^G_{++} : u_h(x_h) \geq u_h(\underline{x}_h)\right\} \text{ is closed in the topology of } \mathbb{R}^G. \tag{4}$$

Let us denote by $\mathcal{U}$ the set of vectors $u = (u_h)_{h\in\mathcal{H}}$ of such utility functions. We denote by $p^c(s) \in \mathbb{R}_{++}$ the price of commodity c at spot s, by $q^a \in \mathbb{R}$ the price of asset a and by $z_h^a \in \mathbb{R}$ the quantity of asset a held by household h. Moreover we define

$$\begin{aligned} p(s) &= (p^c(s))_{c\in\mathcal{C}} \in \mathbb{R}^C_{++}, \quad p = (p(s))_{s\in\mathcal{S}} \in \mathbb{R}^G_{++}, \quad q = (q^a)_{a\in\mathcal{A}} \in \mathbb{R}^A, \\ z_h &= (z_h^a)_{a\in\mathcal{A}} \in \mathbb{R}^A, \qquad\quad z = (z_h)_{h\in\mathcal{H}} \in \mathbb{R}^{AH}. \end{aligned}$$

We denote by $y^{a,c}(s) \in \mathbb{R}$ the units of commodity c delivered by one unit of asset a in state s and we define

$$y^a(s) = (y^{a,c}(s))_{c\in\mathcal{C}} \in \mathbb{R}^C, \quad y(s) = (y^a(s))_{a\in\mathcal{A}} \in \mathbb{R}^{CA}, \quad y = (y(s))_{s\in\{1,\ldots,S\}} \in \mathbb{R}^{CAS}.$$

[1] For every positive integer N, we define the binary relations $\gg$, $\geq$ and $>$ over $\mathbb{R}^N$ as follows: given $v = (v_1, \ldots, v_N)$ and $w = (w_1, \ldots, w_N) \in \mathbb{R}^N$, we write

$$\begin{array}{lll} v \gg w & \text{if} & v_i > w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v \geq w & \text{if} & v_i \geq w_i, \quad \forall\, i \in \{1, \ldots, N\}\,; \\ v > w & \text{if} & v \geq w \text{ and } v \neq w. \end{array}$$

We define also the sets $\mathbb{R}^N_+ = \{v \in \mathbb{R}^N : v \geq 0\}$ and $\mathbb{R}^N_{++} = \{v \in \mathbb{R}^N : v \gg 0\}$.

Note in particular that, in state s, asset a promises to deliver a vector $y^a(s)$ of commodities. Define the return matrix function as follows

$$\mathcal{R} : \mathbb{R}^G_{++} \times \mathbb{R}^{CAS} \to \mathbb{M}(S, A),$$

$$(p, y) \mapsto [p(s)y^a(s)]_{s\in\{1,\dots,S\},a\in\mathcal{A}} = \begin{bmatrix} p(1)y^1(1) & \dots & p(1)y^a(1) & \dots & p(1)y^A(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(s)y^1(s) & \dots & p(s)y^a(s) & \dots & p(s)y^A(s) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(S)y^1(S) & \dots & p(S)y^a(S) & \dots & p(S)y^A(S) \end{bmatrix}$$

where $\mathbb{M}(S, A)$ is the space of the $S \times A$ matrices with real elements.

Consistently with our restricted participation framework, we assume that each household h has only partial access, in a personalized manner to the asset market. In particular we assume that each household cannot sell more than a fixed quantity, depending on commodity and asset prices, of each asset. More precisely, we assume that, for every $h \in \mathcal{H}$, there is a function

$$\sigma_h : \mathbb{R}^G_{++} \times \mathbb{R}^A \to \mathbb{R}^A, \quad (p, q) \mapsto (\sigma^a_h(p, q))_{a\in\mathcal{A}}$$

such that, for every $(p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A$, $\sigma^a_h(p, q)$ represents the largest quantity of asset a that household h can sell at prices (p, q), i.e., the maximum amount she can borrow using asset a. We then each function σ_h borrowing function. We assume that,

$$\text{for every } h \in \mathcal{H},\ \sigma_h \in C^2(\mathbb{R}^G_{++} \times \mathbb{R}^A, \mathbb{R}^A); \tag{5}$$

$$\text{for every } h \in \mathcal{H},\ a \in \mathcal{A},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A, \sigma^a_h(p, q) \geq 0; \tag{6}$$

$$\text{for every } a \in \mathcal{A},\ (p, q) \in \mathbb{R}^G_{++} \times \mathbb{R}^A,\ \sum_{h=1}^{H} \sigma^a_h(p, q) > 0. \tag{7}$$

Let us denote by Σ the set of vectors $\sigma = (\sigma_h)_{h\in\mathcal{H}}$ of such functions. The meaning of the above properties is simply described. Assumption (5) allows to use differential techniques. Assumption (6) permits no participation on financial markets. Assumption (7) insures that each asset is nontrivially exchanged.

We define the set of economies as

$$\mathcal{E} = \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathbb{R}^{CAS} \times \Sigma,$$

with generic element $E = (e, u, y, \sigma)$, and for given $(p, q, E) \in \mathbb{R}^G_{++} \times \mathbb{R}^A \times \mathcal{E}$, we assume household $h \in \mathcal{H}$ has to solve the following maximization problem

$$\max_{(x_h,z_h)} u_h(x_h) \quad \text{s.t.}$$

$$\begin{cases} p(0)x_h(0) + qz_h = p(0)e_h(0) \\ p(s)x_h(s) = p(s)e_h(s) + (p(s)y^a(s))^A_{a=1}\, z_h, \quad s \in \{1, \dots, S\} \\ z^a_h + \sigma^a_h(p, q) \geq 0 \end{cases} \tag{8}$$

We are now ready to give the definition of equilibrium we use.

Definition 1. $\theta = \big((x_h, z_h)_{h\in\mathcal{H}}, p, q\big) \in \mathbb{R}^{GH}_{++} \times \mathbb{R}^{AH} \times \mathbb{R}^G_{++} \times \mathbb{R}^A = \Theta$ *is an equilibrium for E if, for every $h \in \mathcal{H}$, (x_h, z_h) solves Problem (8) at (p, q, E) and (x, z) satisfies market clearing conditions, that is,*

$$\sum_{h=1}^{H} (x_h - e_h) = 0 \quad \text{and} \quad \sum_{h=1}^{H} z_h = 0. \tag{9}$$

We denote by $\Theta(E)$ the set of equilibria for E and we set

$$\Theta_{\mathrm{n}}(E) = \left\{\theta \in \Theta(E) : \forall s \in \mathcal{S},\ p^C(s) = 1\right\}.$$

We prove existence of equilibria using a homotopy argument. Two key steps in that strategy are the proof of compactness of the equilibrium set along the chosen homotopy path, and the regularity of the so called test economy. Our modeling of restrictions allows to get the first result with some work. To get regularity of the test economy, a rank property on the return matrix turns out to be sufficient - see (10) below - and of course, it needs to be translated in terms of fundamentals. In fact, assumptions on the yield structure which guarantee that rank property are proved to be relatively general: numeraire assets and real assets for which yield vectors are sufficiently diversified in each state do belong to the family of assets for which we show existence. In what follows, we formalize the above described second step.

We say that the allocation $(x_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}$ is Pareto Optimal for $u \in \mathcal{U}$ if there is no allocation $(\widetilde{x}_h)_{h\in\mathcal{H}} \in \mathbb{R}^{GH}_{++}$ such that

$$\sum_{h=1}^{H} \widetilde{x}_h \leq \sum_{h=1}^{H} x_h \quad \text{and} \quad (u_h(\widetilde{x}_h))_{h\in\mathcal{H}} > (u_h(x_h))_{h\in\mathcal{H}}\,.$$

Denote the set of Pareto Optimal allocations for $u \in \mathcal{U}$ by $\mathcal{P}(u) \subseteq \mathbb{R}^{GH}_{++}$ and define the set

$$\mathcal{E}^{\diamond} = \Big\{(e,u,y,\sigma) \in \mathcal{E} : \exists x \in \mathcal{P}(u) \text{ such that } \operatorname{rank}\left(\mathcal{R}(Du_1(x_1),y)\right) = A\Big\}. \tag{10}$$

We now state our existence result.

Theorem 2. *For every $E \in \mathcal{E}^{\diamond}$, $\Theta_{\mathrm{n}}(E) \neq \varnothing$.*

Conditions on yields which insure that the associated economies are indeed contained in $\mathcal{E}^{\diamond}$ are presented below. We say that $y \in \mathbb{R}^{CAS}$ belongs to $\mathcal{Y}_1$ if there exists $\mathcal{K} \subseteq \{1,\ldots,S\}$, with[2] $|\mathcal{K}| = A$, and a function $\alpha : \{1\ldots,S\} \to \mathcal{C}$ such that

$$\forall (s,a,c) \in \mathcal{K}\times\mathcal{A}\times\mathcal{C},\ c \neq \alpha(s) \ \Rightarrow\ y^{a,c}(s) = 0, \tag{11}$$

$$\operatorname{rank}\Big(\ \Big[y^{a,\alpha(s)}(s)\Big]_{s\in\mathcal{K},a\in\mathcal{A}}\Big) = A, \tag{12}$$

while we say that $y \in \mathbb{R}^{CAS}$ belongs to $\mathcal{Y}_2$ if there exists $\mathcal{K} \subseteq \{1,\ldots,S\}$, with $|\mathcal{K}| = A$, such that

$$\forall\, s \in \mathcal{K},\ \operatorname{rank}\Big([y^{a,c}(s)]_{c\in\mathcal{C},a\in\mathcal{A}}\Big) = A. \tag{13}$$

Yields in $\mathcal{Y}_1$ are associated with assets which deliver units of a unique, state dependent, commodity in each state in a subset of cardinality A of all possible states. Yields of numeraire assets belong to $\mathcal{Y}_1$: in that case, assets pay in units of the same commodity in each state. Yields in $\mathcal{Y}_2$ are associated with assets whose yield vectors are sufficiently diversified, in fact, linearly independent in A states. For example, assumptions defining $\mathcal{Y}_2$ are satisfied if there are A available commodities and in each state each asset a delivers units of commodity a only.

The following theorem explains the relationship between $\mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{E}^{\diamond}$.

Theorem 3. $\mathbb{R}^{GH}_{++}\times\mathcal{U}\times(\mathcal{Y}_1\cup\mathcal{Y}_2)\times\Sigma \subseteq \mathcal{E}^{\diamond}$. *Moreover, if $A \leq C$, then $\mathcal{Y}_2$ is an open and full measure subset of $\mathbb{R}^{CAS}$, else $\mathcal{Y}_2 = \varnothing$.*

3 Proof of the main results

Throughout we are going to use the two following results (see Villanacci et al. (2002)).

Theorem 4. *Let m, p, n and r be nonnegative integers, and let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $F : M\times\Omega \to N$ be a C^r function. Assume $r > \max\{m-n,0\}$. If y is a regular value for F, then there exists a full measure subset Ω^* of Ω such that, for every $\omega \in \Omega^*$, y is a regular value for $F_\omega : M \to N$, $x \mapsto F(x,\omega)$.*

[2]Given a set $\mathcal{B}$, we denote by $|\mathcal{B}|$ its cardinality.

Theorem 5. *Let M, N be C^2 boundaryless manifolds of the same dimension, $y \in N$ and $F, G : M \to N$ be continuous functions. Assume that G is C^1 in an open neighborhood U of $G^{-1}(y)$, y is a regular value for G restricted to U, $|G^{-1}(y)|$ is finite and odd, and there exists a continuous homotopy $H : M \times [0,1] \to N$ from F to G such that $H^{-1}(y)$ is compact. Then $F^{-1}(y) \neq \varnothing$.*

Define the vectors

$$\begin{array}{ll} x_h^{\setminus}(s) = (x_h^c(s))_{c\in\{1,\dots,C-1\}} \in \mathbb{R}^{C-1}_{++}, & x_h^{\setminus} = (x_h^{\setminus}(s))_{s\in\mathcal{S}} \in \mathbb{R}^{G-(S+1)}_{++}, \\ e_h^{\setminus}(s) = (e_h^c(s))_{c\in\{1,\dots,C-1\}} \in \mathbb{R}^{C-1}_{++}, & e_h^{\setminus} = (e_h^{\setminus}(s))_{s\in\mathcal{S}} \in \mathbb{R}^{G-(S+1)}_{++}. \end{array} \tag{14}$$

As $S+1$ Walras' laws hold true in our model, the significant market clearing conditions in Definition 1 are in fact

$$\sum_{h=1}^{H}(x_h^{\setminus} - e_h^{\setminus}) = 0 \quad \text{and} \quad \sum_{h=1}^{H} z_h = 0. \tag{15}$$

Since we are going to study equilibria in terms of first order conditions associated with households' maximization problems and (significant) market clearing conditions, define

$$\Xi = \mathbb{R}^{GH}_{++} \times \mathbb{R}^{(S+1)H}_{++} \times \mathbb{R}^{AH} \times \mathbb{R}^{AH} \times \mathbb{R}^{G}_{++} \times \mathbb{R}^{A}$$

with generic element

$$\xi = \left((x_h, \lambda_h, z_h, \mu_h)_{h\in\mathcal{H}}, p, q\right) = (x, \lambda, z, \mu, p, q),$$

and the function

$$\mathcal{F} : \Xi \times \mathcal{E} \to \mathbb{R}^{\dim(\Xi)},$$

$$\mathcal{F}(\xi, E) = \left[\begin{array}{ll} (16.1) & D_{x_h(s)}u_h(x_h) - \lambda_h(s)p(s) \\ (16.2) & \begin{array}{l} -p(0)(x_h(0) - e_h(0)) - qz_h \\ -p(s)(x_h(s) - e_h(s)) + (p(s)y^a(s))_{a=1}^{A} z_h, \quad s \in \{1,\dots,S\} \end{array} \\ (16.3) & -\lambda_h(0)q^a + \sum\limits_{s=1}^{S}\lambda_h(s)p(s)y^a(s) + \mu_h^a \\ (16.4) & \min\{\mu_h^a, z_h^a + \sigma_h^a(p,q)\} \\ (16.5) & \sum\limits_{h=1}^{H}(x_h^{\setminus} - e_h^{\setminus}) \\ (16.6) & \sum\limits_{h=1}^{H} z_h \\ (16.7) & p^C(s) - 1 \end{array}\right] \tag{16}$$

where $\dim(\Xi)$ denotes the dimension of the manifold (open set) Ξ.

Given now $E \in \mathcal{E}$, it is immediate to prove that if $\theta = \left((x_h, z_h)_{h\in\mathcal{H}}, p, q\right) \in \Theta_{\mathrm{n}}(E)$, then there exists $(\lambda_h, \mu_h)_{h\in\mathcal{H}} \in \mathbb{R}^{(S+1)H}_{++} \times \mathbb{R}^{AH}$ such that $\xi = \left((x_h, \lambda_h, z_h, \mu_h)_{h\in\mathcal{H}}, p, q\right) \in \Xi$ solves system $\mathcal{F}(\xi, E) = 0$. Vice versa, if $\xi = \left((x_h, \lambda_h, z_h, \mu_h)_{h\in\mathcal{H}}, p, q\right) \in \Xi$ solves system $\mathcal{F}(\xi, E) = 0$, then $\left((x_h, z_h)_{h\in\mathcal{H}}, p, q\right) \in \Theta_{\mathrm{n}}(E)$.

Of course, Theorem 2 is a consequence of the following result.

Theorem 6. *For every $E \in \mathcal{E}^{\diamond}$, there exists $\xi \in \Xi$ such that $\mathcal{F}(\xi, E) = 0$.*

Proof. Let $E = (e, u, y, \sigma) \in \mathcal{E}^{\diamond}$ be given, and define

$$F : \Xi \to \mathbb{R}^{\dim(\Xi)}, \quad \xi \mapsto \mathcal{F}(\xi, E).$$

Fix $\varepsilon > 0$ and $x^* \in \mathcal{P}(u)$ such that $\operatorname{rank}(\mathcal{R}(Du_1(x_1^*), y)) = A$. Consider then the system in the unknowns

$\xi = (x, \lambda, z, \mu, p, q) \in \Xi$ and $\tau \in [0, 1]$, given by

$$
\begin{cases}
(17.1) & D_{x_h(s)} u_h(x_h) - \lambda_h(s) p(s) = 0 \\
(17.2) & \begin{array}{l} -p(0)(x_h(0) - ((1-\tau)e_h(0) + \tau x_h^*(0))) - q z_h = 0 \\ -p(s)(x_h(s) - ((1-\tau)e_h(s) + \tau x_h^*(s))) + (p(s) y^a(s))_{a=1}^A z_h = 0, \quad s \in \{1, \ldots, S\} \end{array} \\
(17.3) & -\lambda_h(0) q^a + \sum_{s=1}^{S} \lambda_h(s) p(s) y^a(s) + \mu_h^a = 0 \\
(17.4) & \min \{\mu_h^a, z_h^a + (1-\tau)\sigma_h^a(p, q) + \tau\varepsilon\} = 0 \\
(17.5) & \sum_{h=1}^{H} \left(x_h^{\setminus} - \left((1-\tau)e_h^{\setminus} + \tau x_h^{*\setminus}\right)\right) = 0 \\
(17.6) & \sum_{h=1}^{H} z_h = 0 \\
(17.7) & p^C(s) - 1 = 0
\end{cases}
\tag{17}
$$

and define the functions

$$H : \Xi \times [0, 1] \to \mathbb{R}^{\dim(\Xi)}, \quad (\xi, \tau) \mapsto \text{ left hand side of System (17)},$$

and

$$G : \Xi \to \mathbb{R}^{\dim(\Xi)}, \quad \xi \mapsto H(\xi, 1).$$

Observe that, for every $\xi \in \Xi$, $H(\xi, 0) = F(\xi)$, and that F, H, G are C^0 in Ξ. Note also that, for every (p, q, τ) and $h \in \mathcal{H}$, if $(x_h, \lambda_h, z_h, \mu_h)$ solves equations (17.1)-(17.4), then (x_h, z_h) is optimal solution to the maximization problem

$$\max_{(x_h, z_h)} u_h(x_h) \quad \text{s.t.}$$

$$
\begin{cases}
p(0)x_h(0) + q z_h = p(0)((1-\tau)e_h(0) + \tau x_h^*(0)) \\
p(s)x_h(s) = p(s)((1-\tau)e_h(s) + \tau x_h^*(s)) + (p(s)y^a(s))_{a=1}^A z_h, \quad s \in \{1, \ldots, S\} \\
z_h^a + (1-\tau)\sigma_h^a(p, q) + \tau\varepsilon \geq 0
\end{cases}
\tag{18}
$$

and, vice versa, if (x_h, z_h) is optimal solution to (18), then there exists (λ_h, μ_h) such that $(x_h, \lambda_h, z_h, \mu_h)$ solves equations (17.1)-(17.4). Moreover, if $(\xi, \tau) \in \Xi \times [0, 1]$ is such that $H(\xi, \tau) = 0$, then indeed

$$\sum_{h=1}^{H} (x_h - ((1-\tau)e_h + \tau x_h^*)) = 0. \tag{19}$$

Finally, if we prove that

$$G^{-1}(0) = \{\xi^*\} \text{ and } G \text{ is } C^1 \text{ in an open neighborhood of } \xi^*, \tag{20}$$

$$D_\xi G(\xi^*) \text{ is not singular}, \tag{21}$$

$$H^{-1}(0) \text{ is compact}, \tag{22}$$

for some $\xi^* \in \Xi$, then Theorem 5 can be applied to get $F^{-1}(0) \neq \varnothing$. In what follows, we are going to prove that conditions (20), (21) and (22) hold true.

In order to prove (20), let us show that

$$G^{-1}(0) = \{\xi^*\} = \left\{\left((x_h^*, \lambda_h^*, z_h^*, \mu_h^*)_{h \in \mathcal{H}}, p^*, q^*\right)\right\},$$

where $(x_h^*)_{h \in \mathcal{H}}$ is the element of $\mathcal{P}(u)$ used in the definition of System (17) and

$$
\begin{array}{rcl}
\forall h \in \mathcal{H}, \quad \lambda_h^* & = & \left(\frac{D_{x_h^C(s)} u_h(x_h^*)}{D_{x_1^C(s)} u_1(x_1^*)} D_{x_1^C(s)} u_1(x_1^*)\right)_{s \in \mathcal{S}} \in \mathbb{R}_{++}^{S+1}, \\
\forall h \in \mathcal{H}, \quad z_h^* & = & 0 \in \mathbb{R}^A, \\
\forall h \in \mathcal{H}, \quad \mu_h^* & = & 0 \in \mathbb{R}^A, \\
p^* & = & \left(\frac{D_{x_1(s)} u_1(x_1^*)}{D_{x_1^C(s)} u_1(x_1^*)}\right)_{s \in \mathcal{S}} \in \mathbb{R}_{++}^G, \\
q^* & = & \left(\sum_{s=1}^{S} \frac{D_{x_1(s)} u_1(x_1^*)}{D_{x_1^C(0)} u_1(x_1^*)} \, y^a(s)\right)_{a \in \mathcal{A}} \in \mathbb{R}^A.
\end{array}
$$

Indeed, as $x^* = (x_h^*)_{h\in\mathcal{H}} \in \mathcal{P}(u)$, we know that there exists $(\theta^*, \gamma^*) = ((\theta_h^*)_{h\in\mathcal{H}}, \gamma^*) \in \mathbb{R}^H \times \mathbb{R}^G$ such that $(x^*, \theta^*, \gamma^*)$ solves the system

$$\begin{cases} \theta_h Du_h(x_h) - \gamma = 0 \\ \theta_1 - 1 = 0 \\ \sum_{h=1}^{H}(x_h - x_h^*) = 0 \end{cases} \tag{23}$$

Then it is immediate to prove that $G(\xi^*) = 0$, that is, ξ^* solves the system

$$\begin{cases} (24.1) & D_{x_h(s)}u_h(x_h) - \lambda_h(s)p(s) = 0 \\ (24.2) & \begin{array}{l} -p(0)(x_h(0) - x_h^*(0)) - qz_h = 0 \\ -p(s)(x_h(s) - x_h^*(s)) + (p(s)y^a(s))_{a=1}^A z_h = 0, \quad s \in \{1, \ldots, S\} \end{array} \\ (24.3) & -\lambda_h(0)q^a + \sum_{s=1}^{S} \lambda_h(s)p(s)y^a(s) + \mu_h^a = 0 \\ (24.4) & \min\{\mu_h^a, z_h^a + \varepsilon\} = 0 \\ (24.5) & \sum_{h=1}^{H} (x_h^\setminus - x_h^{*\setminus}) = 0 \\ (24.6) & \sum_{h=1}^{H} z_h = 0 \\ (24.7) & p^C(s) - 1 = 0 \end{cases} \tag{24}$$

Consider now

$$\xi^{**} = \left((x_h^{**}, \lambda_h^{**}, z_h^{**}, \mu_h^{**})_{h\in\mathcal{H}}, p^{**}, q^{**}\right) \in \Xi$$

such that $G(\xi^{**}) = 0$, and prove that $\xi^{**} = \xi^*$. Let us show at first that $x^{**} = x^*$. Suppose by contradiction $x^{**} \neq x^*$ and consider $\widetilde{x} = \frac{1}{2}(x^* + x^{**})$. Since $G(\xi^{**}) = 0$, then

$$\sum_{h=1}^{H} \widetilde{x}_h = \frac{1}{2}\sum_{h=1}^{H}(x_h^* + x_h^{**}) = \sum_{h=1}^{H} x_h^*. \tag{25}$$

Moreover we have also that, for every $h \in \mathcal{H}$, (x_h^{**}, z_h^{**}) is the maximum for Problem (18), considered at prices p^{**} and q^{**} and $\tau = 1$. As (x_h^*, z_h^*) is feasible for the same problem, we obtain $u_h(x_h^{**}) \geq u_h(x_h^*)$ and then, by (3) and $x^{**} \neq x^*$, it follows

$$(u_h(\widetilde{x}_h))_{h\in\mathcal{H}} > (u_h(x_h^*))_{h\in\mathcal{H}}\,. \tag{26}$$

Since (25) and (26) imply that $x^* \notin \mathcal{P}(u)$, the contradiction is finally found.

Let us prove now that $\lambda^{**} = \lambda^*$. From (24.7) we know that, for every $s \in \mathcal{S}$,

$$p^{**C}(s) = 1 = p^{*C}(s),$$

and then from (24.1) we find that, for every $h \in \mathcal{H}$, $s \in \mathcal{S}$,

$$\lambda_h^{**}(s) = D_{x_h^C(s)}u_h(x_h^*) = \lambda_h^*(s).$$

From the above relationship we immediately deduce $p^{**} = p^*$, as (24.1) implies that, for every $s \in \mathcal{S}$,

$$p^{**}(s) = \frac{D_{x_h(s)}u_h(x_h^*)}{\lambda_h^*(s)} = p^*(s).$$

In order to prove that $z^{**} = z^*$, observe that the definition of p^* implies $\operatorname{rank}(\mathcal{R}(p^*, y)) = A$. Moreover, from (24.2) and the equalities $x^{**} = x^*$ and $p^{**} = p^*$, we have that, for every $h \in \mathcal{H}$, $\mathcal{R}(p^*, y)z_h^{**} = 0$, that is, $z_h^{**} = 0$. Then we get the equality $z^{**} = z^*$.

Clearly, from (24.4), $\mu^{**} = \mu^* = 0$. Finally, from (24.3), we obtain that, for every $a \in \mathcal{A}$, $h \in \mathcal{H}$,

$$q^{**a} = \frac{1}{\lambda_h^*(0)} \left(\sum_{s=1}^{S} \lambda_h^*(s) p^*(s) y^a(s) + \mu_h^{*a} \right) = q^{*a}.$$

and then $q^{**} = q^*$, as well. Then we have obtained that $\xi^{**} = \xi^*$ and, since it is immediate to show that G is C^1 in an open neighborhood of ξ^*, (20) is proved.

The proof of (21) is indeed a very simple modification of the argument in Villanacci et al. (2002), Chapter 11, Lemma 18. More precisely, the computation of $D_\xi G(\xi^*)$ is described in the table below. Note that the components of the function G are listed in the first column, the variables with respect to which derivatives are taken are listed in the first row, and in the remaining bottom right corner the Jacobian matrix is displayed.

	x_h	λ_h	z_h	μ_h	p	q
(24.1)	$D^2 u_h(x_h^*)$	$-\Phi(p^*)^T$			$-\Lambda_h^*$	
(24.2)	$-\Phi(p^*)$		$\begin{bmatrix} -q^* \\ \mathcal{R}(p^*, y) \end{bmatrix}$			
(24.3)		$\begin{bmatrix} -q^* \\ \mathcal{R}(p^*, y) \end{bmatrix}^T$		I_A		$-\lambda_h^*(0) I_A$
(24.4)				I_A		
(24.5)	L					
(24.6)			I_A			
(24.7)					M	

where, for every positive integer N, I_N is the identity matrix of dimension N, and

$$\Phi(p) = \begin{bmatrix} p^1(0) & \cdots & p^{C-1}(0) & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & p^1(S) & \cdots & p^{C-1}(S) & 1 \end{bmatrix}_{(S+1)\times G}$$

$$M = \begin{bmatrix} 0 & \cdots & 0 & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \cdots & 0 & 1 \end{bmatrix}_{(S+1)\times G}$$

$$\Lambda_h^* = \begin{bmatrix} \lambda_h^*(0) I_C & & \\ & \ddots & \\ & & \lambda_h^*(S) I_C \end{bmatrix}_{G\times G}$$

$$L = \begin{bmatrix} I_{C-1} & 0 & & & \\ & & \ddots & & \\ & & & I_{C-1} & 0 \end{bmatrix}_{G-(S+1)\times G}$$

In order to simplify our argument, we make some preliminary observations. Thanks to the presence of I_A in correspondence to μ_h and the fact that there are no other nonnull elements on its super-row, we are left with proving it is not singular the matrix obtained from $D_\xi G(\xi^*)$ by erasing the super-row and the super-column corresponding to μ_h. Moreover, with the nonnull elements of M we can clean the corresponding columns and erase them, and finally erase the super-row of M, too. Define now

$$\widetilde{\Lambda}_h^* = \begin{bmatrix} \lambda_h^*(0) I_{C-1} & & \\ 0 & & \\ & \ddots & \\ & & \lambda_h^*(S) I_{C-1} \\ & & 0 \end{bmatrix}_{G\times G-(S+1)}$$

and introduce $p^\setminus = (p^\setminus(s))_{s\in\mathcal{S}}$, where $p^\setminus(s) = (p^1(s), \dots, p^{C-1}(s)) \in \mathbb{R}^{C-1}_{++}$. Hence, we are led to prove that the following matrix, that we call $N(\xi^*)$, is not singular

	x_h	λ_h	z_h	$p^\setminus$	q
(24.1)	$D^2 u_h(x_h^*)$	$-\Phi(p^*)^T$		$-\widetilde{\Lambda}_h^*$	
(24.2)	$-\Phi(p^*)$		$\begin{bmatrix} -q^* \\ \mathcal{R}(p^*,y) \end{bmatrix}$		
(24.3)		$\begin{bmatrix} -q^* \\ \mathcal{R}(p^*,y) \end{bmatrix}^T$			$-\lambda_h^*(0) I_A$
(24.5)	L				
(24.6)			I_A		

Let us show that if $N(\xi^*)\Delta\zeta = 0$, for some

$$\Delta\zeta = \left((\Delta x_h, \Delta\lambda_h, \Delta z_h)_{h\in\mathcal{H}}, \Delta p^\setminus, \Delta q \right) \in \mathbb{R}^{GH}_{++} \times \mathbb{R}^{(S+1)H}_{++} \times \mathbb{R}^{AH} \times \mathbb{R}^{G-(S+1)}_{++} \times \mathbb{R}^{A},$$

then $\Delta\zeta = 0$. We rewrite $N(\xi^*)\Delta\zeta = 0$ as

$$\begin{cases} (27.1) \quad D^2 u_h(x_h^*)\Delta x_h - \Phi(p^*)^T \Delta\lambda_h - \widetilde{\Lambda}_h^* \Delta p^\setminus = 0 \\ (27.2) \quad -\Phi(p^*)\Delta x_h + \begin{bmatrix} -q^* \\ \mathcal{R}(p^*,y) \end{bmatrix} \Delta z_h = 0 \\ (27.3) \quad \begin{bmatrix} -q^* \\ \mathcal{R}(p^*,y) \end{bmatrix}^T \Delta\lambda_h - \lambda_h^*(0)\Delta q = 0 \\ (27.4) \quad \sum_{h=1}^{H} \Delta x_h^\setminus = 0 \\ (27.5) \quad \sum_{h=1}^{H} \Delta z_h = 0, \end{cases} \tag{27}$$

where

$$\Delta x_h^\setminus(s) = (\Delta x_h^c(s))_{c\in\{1,\ldots,C-1\}} \in \mathbb{R}^{C-1}_{++} \quad \text{and} \quad \Delta x_h^\setminus = (\Delta x_h^\setminus(s))_{s\in\mathcal{S}} \in \mathbb{R}^{G-(S+1)}_{++}.$$

Recall that, as $(x_h^*)_{h\in\mathcal{H}} \in \mathcal{P}(u)$, there exists $(\theta^*, \gamma^*) \in \mathbb{R}^H \times \mathbb{R}^G$ such that $(x^*, \theta^*, \gamma^*)$ solves System (23). Then, defined

$$\Gamma^* = \begin{bmatrix} \gamma^{*C}(0) I_{C-1} & & \\ 0 & & \\ & \ddots & \\ & & \gamma^{*C}(S) I_{C-1} \\ & & 0 \end{bmatrix}_{G\times G-(S+1)}$$

where $\gamma^* = (\gamma^{*c}(s))_{c\in\mathcal{C}, s\in\mathcal{S}}$, we have

$$\widetilde{\Lambda}_h^* = \frac{1}{\theta_h^*}\Gamma^*. \tag{28}$$

From (27.1), for every $h \in \mathcal{H}$, we have in particular

$$\theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h - \theta_h^* \Delta x_h \Phi(p^*)^T \Delta\lambda_h - \theta_h^* \Delta x_h \widetilde{\Lambda}_h^* \Delta p^\setminus = 0,$$

and from (27.2) and (28), we obtain

$$\theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h = \theta_h^* \Delta z_h \begin{bmatrix} -q^* \\ \mathcal{R}(p^*,y) \end{bmatrix}^T \Delta\lambda_h + \Delta x_h \Gamma^* \Delta p^\setminus.$$

Using now (27.3) and observing that $\theta_h^* \lambda_h^*(0) = \gamma^{C*}(0)$, we get the equality

$$\theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h = \gamma^{C*}(0)\Delta z_h \Delta q + \Delta x_h \Gamma^* \Delta p^\setminus,$$

and summing up over $h \in \mathcal{H}$ and using (27.4) and (27.5), we obtain

$$\sum_{h=1}^{H} \theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h = 0. \tag{29}$$

Observe also that, for every $h \in \mathcal{H}$,

$$Du_h(x_h^*)\Delta x_h = 0. \tag{30}$$

Indeed, from (24.1), we obtain $Du_h(x_h^*)\Delta x_h = \lambda_h^* \Phi(p^*)\Delta x_h$, and then from (27.2), we get

$$\lambda_h^* \Phi(p^*)\Delta x_h = \lambda_h^* \begin{bmatrix} -q^* \\ \mathcal{R}(p^*, y) \end{bmatrix} \Delta z_h = -\mu_h^* \Delta z_h = 0,$$

where the last two equalities follow by (24.3) and $\mu_h^* = 0$, respectively. Define now the set

$$\mathcal{H}^* = \{h \in \mathcal{H} : \Delta x_h \neq 0\},$$

and assume by contradiction $\mathcal{H}^* \neq \varnothing$. Since $\theta^* \in \mathbb{R}^H_{++}$, we have that (3) and (30) imply

$$\sum_{h=1}^{H} \theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h = \sum_{h \in \mathcal{H}^*} \theta_h^* \Delta x_h D^2 u_h(x_h^*)\Delta x_h < 0. \tag{31}$$

As (31) contradicts (29), we get $\mathcal{H}^* = \varnothing$, that is, for every $h \in \mathcal{H}$, $\Delta x_h = 0$. Note also that, from (27.2) and since $\mathcal{R}(p^*, y)$ has full rank, we immediately obtain that, for every $h \in \mathcal{H}$, $\Delta z_h = 0$.

Let us show now that, for every $h \in \mathcal{H}$, $\Delta \lambda_h = 0$. From (27.1), we know that, for every $h \in \mathcal{H}$,

$$-\Phi(p^*)^T \Delta\lambda_h - \widetilde{\Lambda}_h^* \Delta p^{\setminus} = 0.$$

Then, for every $h \in \mathcal{H}$, $s \in \mathcal{S}$, we have $-p^{*C}(s)\Delta\lambda_h(s) = 0$ and hence $\Delta\lambda_h(s) = 0$, as desired.

Finally, as, for every $h \in \mathcal{H}$, $\Delta x_h = 0$ and $\Delta\lambda_h = 0$, from (27.1) and (27.3), we immediately obtain $\Delta p^{\setminus} = 0$ and $\Delta q = 0$, respectively. Then $\Delta\zeta = 0$ and the proof of (21) is complete.

In order to show (22), we show that each sequence

$$\left(\left(x_h^{[n]}, \lambda_h^{[n]}, z_h^{[n]}, \mu_h^{[n]}\right)_{h \in \mathcal{H}}, p^{[n]}, q^{[n]}, \tau^{[n]}\right)_{n \in \mathbb{N}} = \left(x^{[n]}, \lambda^{[n]}, z^{[n]}, \mu^{[n]}, p^{[n]}, q^{[n]}, \tau^{[n]}\right)_{n \in \mathbb{N}}$$

in $H^{-1}(0)$ admits a converging subsequence. As we are going to use a diagonal argument, every time we say that a sequence converges we mean it has a converging subsequence. Of course, since $\{\tau^{[n]} : n \in \mathbb{N}\} \subseteq [0,1]$, $(\tau^{[n]})_{n \in \mathbb{N}}$ converges to a certain $\overline{\tau} \in [0,1]$.

In order to prove the convergence of the sequence $(x^{[n]})_{n \in \mathbb{N}}$, let us notice at first that, defined,

$$\forall\, h \in \mathcal{H}, n \in \mathbb{N},\ \overline{e}_h^{[n]} = (1 - \tau^{[n]})e_h + \tau^{[n]} x_h^* \in \mathbb{R}^G_{++} \quad \text{and} \quad \overline{e}_h = (1 - \overline{\tau})e_h + \overline{\tau} x_h^* \in \mathbb{R}^G_{++},$$

we have that, for every $h \in \mathcal{H}$, $\overline{e}_h^{[n]} \to \overline{e}_h$. In particular, there exists $v_h \in \mathbb{R}^G_{++}$ such that, for every $n \in \mathbb{N}$, $\overline{e}_h^{[n]} \leq v_h$.

Fix now $h \in \mathcal{H}$. Since, for every $n \in \mathbb{N}$, $x_h^{[n]} \in \mathbb{R}^G_{++}$ and (19) holds, we have

$$\{x_h^{[n]} : n \in \mathbb{N}\} \subseteq \left\{ x_h \in \mathbb{R}^G : 0 \leq x_h \leq \sum_{h=1}^{H} v_h \right\} = C_h^1.$$

By (18), for every $n \in \mathbb{N}$, we have

$$u_h(x_h^{[n]}) \geq u_h(\overline{e}_h^{[n]}) \geq \min_{x_h \in \{\overline{e}_h^{[n]} : n \in \mathbb{N}\} \cup \{\overline{e}_h\}} u_h(x_h) = u_h(\underline{x}_h),$$

where $\underline{x}_h$ is a suitable element of the compact set $\{\overline{e}_h^{[n]} : n \in \mathbb{N}\} \cup \{\overline{e}_h\} \subseteq \mathbb{R}^G_{++}$, and then

$$\{x_h^{[n]} : n \in \mathbb{N}\} \subseteq \{x_h \in \mathbb{R}^G_{++} : u_h(x_h) \geq u_h(\underline{x}_h)\} = C_h^2.$$

From (4), C_h^2 is a closed subset of $\mathbb{R}^G$, and then $C_h^1 \cap C_h^2$ is a compact set contained in $\mathbb{R}^G_{++}$. As $\{x_h^{[n]} : n \in \mathbb{N}\} \subseteq C_h^1 \cap C_h^2$, we have that $(x_h^{[n]})_{n \in \mathbb{N}}$ converges to an element of $\mathbb{R}^G_{++}$, say $\overline{x}_h$, and then the convergence of $(x^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^{GH}_{++}$ is proved. From (17.1), (17.7) and (2) we find that, for every $h \in \mathcal{H}$, $s \in \mathcal{S}$,

$$\lambda_h^{[n]}(s) = D_{x_h^C(s)} u_h(x_h^{[n]}) \to D_{x_h^C(s)} u_h(\overline{x}_h) = \overline{\lambda}_h(s) \in \mathbb{R}_{++},$$

and, from (17.1) and (2), it follows, for every $s \in \mathcal{S}$,

$$p^{[n]}(s) = \frac{D_{x_h(s)} u_h(x_h^{[n]})}{\lambda_h^{[n]}(s)} \to \frac{D_{x_h(s)} u_h(\overline{x}_h)}{\overline{\lambda}_h(s)} = \overline{p}(s) \in \mathbb{R}^C_{++}.$$

Then $(\lambda^{[n]})_{n\in\mathbb{N}}$ converges to an element of $\mathbb{R}^{(S+1)H}_{++}$ and $(p^{[n]})_{n\in\mathbb{N}}$ converges to an element of $\mathbb{R}^G_{++}$.

Fix now $a \in \mathcal{A}$ and consider the sequence $(q^{a,[n]})_{n\in\mathbb{N}}$. We claim that it converges if there exist $h \in \mathcal{H}$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$,

$$z_h^{a,[n_k]} + (1 - \tau^{[n_k]})\sigma_h^a(p^{[n_k]}, q^{[n_k]}) + \tau^{[n_k]}\varepsilon > 0.$$

Indeed, if this is true, from (17.3) and (17.4) we get

$$q^{a,[n_k]} = \frac{1}{\lambda_h^{[n_k]}(0)} \sum_{s=1}^{S} \lambda_h^{[n_k]}(s) p^{[n_k]}(s) y^a(s) \to \frac{1}{\overline{\lambda}_h(0)} \sum_{s=1}^{S} \overline{\lambda}_h(s)\overline{p}(s) y^a(s) = \overline{q}^a.$$

In order to show the claim, assume by contradiction that there exists $\nu \in \mathbb{N}$ such that, for every $h \in \mathcal{H}$, $n \geq \nu$,

$$z_h^{a,[n]} + (1 - \tau^{[n]})\sigma_h^a(p^{[n]}, q^{[n]}) + \tau^{[n]}\varepsilon = 0.$$

Summing up on $h \in \mathcal{H}$, and using (17.6), we find

$$0 = \sum_{h=1}^{H} z_h^{a,[n]} + (1 - \tau^{[n]}) \sum_{h=1}^{H} \sigma_h^a(p^{[n]}, q^{[n]}) + \tau^{[n]} H\varepsilon = (1 - \tau^{[n]}) \sum_{h=1}^{H} \sigma_h^a(p^{[n]}, q^{[n]}) + \tau^{[n]} H\varepsilon.$$

From (7), the right hand side of the above equality has to be positive. Then the contradiction is found and the convergence of $(q^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^A$ follows. Fix now $h \in \mathcal{H}$, $a \in \mathcal{A}$ and consider the sequence $(z_h^{a,[n]})_{n\in\mathbb{N}}$. For every $n \in \mathbb{N}$, we have

$$-(1 - \tau^{[n]})\sigma_h^a(p^{[n]}, q^{[n]}) - \tau^{[n]}\varepsilon \leq z_h^{a,[n]} = -\sum_{h'\in\mathcal{H}, h'\neq h} z_{h'}^{a,[n]} \leq \sum_{h'\in\mathcal{H}, h'\neq h} \left((1 - \tau^{[n]})\sigma_h^a(p^{[n]}, q^{[n]}) + \tau^{[n]}\varepsilon\right),$$

and, since $\tau^{[n]} \to \overline{\tau}$, $p^{[n]} \to \overline{p}$, $q^{[n]} \to \overline{q}$ and (5) holds true, there exist $\underline{K}_h^a, \overline{K}_h^a \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$,

$$\underline{K}_h^a \leq z_h^{a,[n]} \leq \overline{K}_h^a.$$

Thus $z_h^{a,[n]} \to \overline{z}_h^a$ and the convergence of $(z^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^{AH}$ is proved. Finally, fix $h \in \mathcal{H}$, $a \in \mathcal{A}$ and study the convergence of the sequence $(\mu_h^{a,[n]})_{n\in\mathbb{N}}$. From (17.3) we immediately obtain

$$\mu_h^{a,[n]} = \lambda_h^{[n]}(0) q^{a,[n]} - \sum_{s=1}^{S} \lambda_h^{[n]}(s) p^{[n]}(s) y^a(s) \to \overline{\lambda}_h(0)\overline{q}^a - \sum_{s=1}^{S} \overline{\lambda}_h(s)\overline{p}(s) y^a(s) = \overline{\mu}_h^a,$$

and the convergence of $(\mu^{[n]})_{n\in\mathbb{N}}$ to an element of $\mathbb{R}^{AH}$ is proved. Then the proof of (22) is complete. □

Proof of Theorem 3. Let us consider at first the properties of $\mathcal{Y}_2$. If $A > C$, then we immediately have $\mathcal{Y}_2 = \varnothing$. Assume then $A \leq C$. The fact that $\mathcal{Y}_2$ is open can be proved as follows. Observe that

$$\mathcal{Y}_2 = \bigcup_{\mathcal{K}\subseteq\{1,\ldots,S\}, |\mathcal{K}|=A} \mathcal{Y}_2(\mathcal{K}) \tag{32}$$

where, for every $\mathcal{K} \subseteq \{1, \ldots, S\}$ such that $|\mathcal{K}| = A$,

$$\mathcal{Y}_2(\mathcal{K}) = \left\{ y \in \mathbb{R}^{CAS} : \forall s \in \mathcal{K},\ \operatorname{rank}\left([y^{a,c}(s)]_{c \in \mathcal{C}, a \in \mathcal{A}} \right) = A \right\}.$$

Note that, for every $\mathcal{K} \subseteq \{1, \ldots, S\}$ such that $|\mathcal{K}| = A$, the following equality holds

$$\mathbb{R}^{CAS} \setminus \mathcal{Y}_2(\mathcal{K}) = \bigcup_{s \in \mathcal{K}} \left(\bigcap_{\mathcal{I} \subseteq \mathcal{C}, |\mathcal{I}| = A} \left\{ y \in \mathbb{R}^{CAS} : \det \left([y^{c,a}(s)]_{c \in \mathcal{I},\, a \in \mathcal{A}} \right) = 0 \right\} \right). \tag{33}$$

As the right hand side of (33) is a closed subset of $\mathbb{R}^{CAS}$, it follows that $\mathcal{Y}_2(\mathcal{K})$ is an open subset of $\mathbb{R}^{CAS}$ and then, from (32), $\mathcal{Y}_2$ is an open subset of $\mathbb{R}^{CAS}$, as well.

In order to prove that $\mathcal{Y}_2$ is a full measure subset of $\mathbb{R}^{CAS}$, it is sufficient to show that the subset of $\mathcal{Y}_2$ defined as

$$\mathcal{Y}_3 = \left\{ y \in \mathbb{R}^{CAS} : \forall s \in \{1, \ldots, S\},\ \operatorname{rank}\left([y^{a,c}(s)]_{c \in \mathcal{C}, a \in \mathcal{A}} \right) = A \right\}$$

is a full measure subset of $\mathbb{R}^{CAS}$. To this end, we only need to show that, for every $s \in \{1, \ldots, S\}$, 0 is a regular value for the map $\Psi^s : \mathbb{R}^A \times \mathbb{R}^{CAS} \to \mathbb{R}^{C+1}$, defined as

$$\Psi^s(v, y) = \begin{bmatrix} [y^{a,c}(s)]_{c \in \mathcal{C}, a \in \mathcal{A}}\, v \\ \\ \frac{1}{2} v \cdot v - 1 \end{bmatrix}$$

In fact, if that is true, from Theorem 4 we have that, for every $s \in \{1, \ldots, S\}$, there exists a full measure subset $\mathcal{Y}_3(s)$ of $\mathbb{R}^{CAS}$ such that, for every $y \in \mathcal{Y}_3(s)$, 0 is a regular value for the function $\Psi^s_y : \mathbb{R}^A \to \mathbb{R}^{C+1}$, $v \mapsto \Psi^s(v, y)$. Since $A < C + 1$, that is equivalent to say that, for every $y \in \mathcal{Y}_3(s)$,

$$\{ v \in \mathbb{R}^A : \Psi^s(v, y) = 0 \} = \varnothing,$$

that is,

$$\operatorname{rank}\left([y^{a,c}(s)]_{c \in \mathcal{C}, a \in \mathcal{A}} \right) = A.$$

As $\cap_{s=1}^{S} \mathcal{Y}_3(s) \subseteq \mathcal{Y}_3$, $\mathcal{Y}_3$ has full measure in $\mathbb{R}^{CAS}$. Thus we are left with proving that, for every $s \in \{1, \ldots, S\}$, 0 is a regular value for Ψ^s, that is, if $(v, y) \in \mathbb{R}^A \times \mathbb{R}^{CAS}$ satisfies $\Psi^s(v, y) = 0$, then $D\Psi^s(v, y)$ has full rank. This fact immediately follows by the structure of $D\Psi^s(v, y)$.

Let us prove now the inclusion $\mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y}_1 \times \Sigma \subseteq \mathcal{E}^\diamond$. Consider then $(e, u, y, \sigma) \in \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y}_1 \times \Sigma$ and let $\mathcal{K} \subseteq \{1, \ldots, S\}$, with $|\mathcal{K}| = A$, and $\alpha : \{1, \ldots, S\} \to \mathcal{C}$ such that (11) and (12) hold. As $\mathcal{P}(u) \neq \varnothing$ and, for every $x_1 \in \mathbb{R}^G_{++}$,

$$\operatorname{rank}\left(\mathcal{R}(Du_1(x_1), y) \right) = \operatorname{rank}\left(\left[y^{a, \alpha(s)}(s) \right]_{s \in \mathcal{K}, a \in \mathcal{A}} \right) = A,$$

it immediately follows that $(e, u, y, \sigma) \in \mathcal{E}^\diamond$.

Finally, let us show the inclusion $\mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y}_2 \times \Sigma \subseteq \mathcal{E}^\diamond$. This is obvious if $A > C$, as $\mathcal{Y}_2 = \varnothing$. Assume then $A \leq C$, fix $(e, u, y, \sigma) \in \mathbb{R}^{GH}_{++} \times \mathcal{U} \times \mathcal{Y}_2 \times \Sigma$ and prove it belongs to $\mathcal{E}^\diamond$. First of all, let us recall that, given $(r, \underline{u}) = (r, \underline{u}_2, \ldots, \underline{u}_H) \in \mathbb{R}^G_{++} \times \mathbb{R}^{H-1}$, if the maximization problem

$$\max_{x \in \mathbb{R}^{GH}_{++}} u_1(x_1) \quad \text{s.t.} \quad \begin{cases} u_h(x_h) = \underline{u}_h, \quad h \geq 2 \\ \displaystyle\sum_{h=1}^{H} x_h = r \end{cases} \tag{34}$$

has a feasible solution, then it has a unique optimal solution x^* which is Pareto Optimal for u and there exists $(\theta^*, \gamma^*) \in \mathbb{R}^H \times \mathbb{R}^G$ such that $(x^*, \theta^*, \gamma^*)$ solves the system

$$\begin{cases} \theta_h Du_h(x_h) - \gamma = 0 \\ u_h(x_h) - \underline{u}_h = 0, \quad h \geq 2 \\ \theta_1 - 1 = 0 \\ -\displaystyle\sum_{h=1}^{H} x_h + r = 0 \end{cases} \tag{35}$$

Conversely, if there exists $(x^*, \theta^*, \gamma^*)$ solving System (35), then x^* is the unique optimal solution to Problem (34) and it is a Pareto Optimal allocation for u. Consider then $\mathcal{K} \subseteq \{1, \dots, S\}$, with $|\mathcal{K}| = A$, such that (13) holds and define the function

$$\Phi : \mathbb{R}^{GH}_{++} \times \mathbb{R}^H \times \mathbb{R}^G \times \mathbb{R}^A \times \mathbb{R}^G_{++} \times \mathbb{R}^{H-1} \to \mathbb{R}^{GH+H+G+A+1}$$

$$\Phi(x, \theta, \gamma, v, r, \underline{u}) = \begin{bmatrix} (36.1) & \theta_1 Du_1(x_1) - \gamma & \\ (36.2) & \theta_1 - 1 & \\ (36.3) & \theta_h Du_h(x_h) - \gamma, & h \geq 2 \\ (36.4) & u_h(x_h) - \underline{u}_h, & h \geq 2 \\ (36.5) & -\sum_{h=1}^{H} x_h + r & \\ (36.6) & \mathcal{R}_{\mathcal{K}}(\gamma, y)v & \\ (36.7) & \frac{1}{2}vv - 1 & \end{bmatrix} \tag{36}$$

where $\mathcal{R}_{\mathcal{K}}(\gamma, y)$ is the matrix built by considering only the rows of $\mathcal{R}(\gamma, y)$ whose index is in $\mathcal{K}$. Of course $\mathcal{R}_{\mathcal{K}}(\gamma, y) \in \mathbb{M}(A, A)$. Let us prove now that 0 is a regular value of Φ.

The Jacobian of Φ is described by the table below, where $h \geq 2$. The rank computation is performed through row and column operations. The starred matrices are full row rank matrices that we employ to "clean up" its super-row, since all the other elements of its super-column are null. A suitable order in which the appropriate elementary super-column operations have to be performed is the one indicated in the last column of the table.

	x_1	θ_1	x_h	θ_h	$\underline{u}_h$	r	γ	v	
(36.1)	$\theta_1 D^2 u_1(x_1)*$	$Du_1(x_1)$					$-I_G$		2
(36.2)		$1*$							3
(36.3)			$\theta_h D^2 u_h(x_h)*$	$Du_h(x_h)$			$-I_G$		2
(36.4)			$Du_h(x_h)$		$-1*$				3
(36.5)	$-I_G$		$-I_G$			I_G*			1
(36.6)							$M(y,v)*$	$\mathcal{R}_{\mathcal{K}}(\gamma, y)$	4
(36.7)								$v*$	5

Of course, we are left with showing that the matrix $M(y, v)$ indeed has full row rank. If we write

$$M(y, v) = \left[\left(m^0_{s'}(y, v), m^1_{s'}(y, v), \dots, m^S_{s'}(y, v)\right)\right]_{s' \in \mathcal{K}} \in \mathbb{M}(A, G),$$

where

$$\begin{aligned} &\forall\, s' \in \mathcal{K}, s \in \mathcal{S}, \quad m^s_{s'}(y, v) \in \mathbb{R}^C, \\ &\forall\, s' \in \mathcal{K}, \qquad\quad \left(m^0_{s'}(y, v), m^1_{s'}(y, v), \dots, m^S_{s'}(y, v)\right) \in \mathbb{R}^G, \end{aligned}$$

then a simple computation shows that

$$m^s_{s'}(y, v) = \begin{cases} \sum_{a \in \mathcal{A}} y^a(s') v^a & \text{if } s' = s \\ 0 & \text{if } s' \neq s \end{cases}$$

Then, using Assumption (13), the full row rank property of $M(y, v)$ immediately follows.

Applying now Theorem 4, we find there exists a full measure subset $\mathcal{D}$ of $\mathbb{R}^G_{++} \times \mathbb{R}^{H-1}$ such that, for every $(r, \underline{u}) \in \mathcal{D}$, 0 is a regular value of the function

$$\Phi_{(r,\underline{u})} : \mathbb{R}^{GH}_{++} \times \mathbb{R}^H \times \mathbb{R}^G \times \mathbb{R}^A \to \mathbb{R}^{GH+H+G+A+1}, \quad (x, \theta, \gamma, v) \mapsto \Phi(x, \theta, \gamma, v, r, \underline{u}).$$

As a consequence, for every $(r, \underline{u}) \in \mathcal{D}$,

$$\{(x, \theta, \gamma, v) : \Phi(x, \theta, \gamma, v, r, \underline{u}) = 0\} = \varnothing.$$

From the above considerations, we are done if we prove that $\Psi(\mathbb{R}^{GH}_{++}) \cap \mathcal{D} \neq \varnothing$, where

$$\Psi : \mathbb{R}^{GH}_{++} \to \mathbb{R}^{G}_{++} \times \mathbb{R}^{H-1}, \quad x \mapsto \left(\sum_{h=1}^{H} x_h, (u_h(x_h))_{h=2}^{H} \right).$$

It is immediate to prove that, for every $x \in \mathbb{R}^{GH}_{++}$, $D\Psi(x)$ is surjective. Then $\Psi(\mathbb{R}^{GH}_{++})$ is an open subset of $\mathbb{R}^{G}_{++} \times \mathbb{R}^{H-1}$ and indeed $\Psi(\mathbb{R}^{GH}_{++}) \cap \mathcal{D} \neq \varnothing$. $\square$

References

Carosi, L., Gori, M. and A. Villanacci, (2009), Endogenous restricted participation in general financial equilibrium, Journal of Mathematical Economics; 45; 787-806.

Polemarchakis, H.M. and P. Siconolfi, (1997), Generic Existence of Competitive Equilibria with Restricted Participation, Journal of Mathematical Economics; 28; 289-311.

Duffie, D. and W. Shafer, (1985), Equilibrium with incomplete markets: I. A basic model of generic existence, Journal of Mathematical Economics; 14; 3; 285-300.

Villanacci, A., Carosi, L., Benevieri, P. and A. Battinelli, (2002), Differential Topology and General Equilibrium with Complete and Incomplete Markets, Kluwer Academic Publishers.