Rethinking Alesina and Spolaore's "uni-dimensional world": existence of migration proof country structures for arbitrary distributed populations

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Abstract. In this note, we prove that for population arbitrarily distributed on the unit segment with density which is continuous and bounded away from zero, there exists a consecutive partition into countries that is migration-proof, i.e. such that no individual is willing to change his/her country. This was implicitly conjectured by Alesina and Spolaore in the conclusion to their seminal paper (Alesina, Spolaore 1997). The proof utilizes the celebrated Gale and Nikaido's lemma, which is known from the proof of the existence of competitive Walrasian equilibrium.

Keywords: Country Formation, Alesina and Spolaore's World, Migration, Stable Partitions, Gale and Nikaido's Lemma.

1. Introduction

In the seminal paper (Alesina and Spolaore 1997), questions of political economy were addressed within a simple model of the world: the whole population was assumed to be uniformly distributed over the segment of the unit length. Then, the population have to split into several "countries", the cost of financing each being a constant. The assumption which is usually maintained in this strand of literature is that once a country is formed, its center is to be placed necessarily at the location of its median citizen.

Authors characterized stable and efficient divisions into countries, where stability meant immunity against several types of possible reorganizations of the world. Later on, their conclusions were challenged in a number of papers which relax various assumptions of (Alesina and Spolaore 1997), as well as addressed some related questions of stability (Bogomolnaia et al (2005,2007), Dreze et al (2009)).

One of the most important issues was agreed to be that of migration stability, where every individual is allowed to change his country of residence. Needless to say that this issue is extremely relevant in the current world. Still, nothing was known about the conditions on the population distribution under which such migration-proof organization of the world can be guaranteed to exist. The only original case considered by Alesina and Spolaore where the distribution of population is uniform, is trivial: for every prescribed number of countries, there surely exists the division into equal-sized countries, which is migration-proof. [1] The number of countries being given, it is also efficient.

Problems arise once we depart from uniformity assumption. First of all, it is no longer true that Nash equilibrium (= migration-proof) partition should necessarily be efficient. Indeed, consider any individual living on the border between the two neighboring countries. In a Nash equilibrium, this individual is necessarily indifferent between the residence in the two neighboring countries; and it may well happen (in a non-uniform case) that the distance to one of the centers is smaller than that to the other one. In this case, efficiency is violated since the border should be drawn in the middle between those centers; however then, the median location of centers could be disturbed.

Overall, it is not at all clear how efficiency is related to migration-proofness. Moreover, straightforward ideas how to prove the existence of Nash equilibrium partitions all fail, due to the essential discontinuity of naturally arising mappings. Having believed, however, to the general existence result, we were looking into the Kakutani's fixpoint theorem as the source of the proof. Again, did not succeed — non-convexities came to play.

This final product constructs a mapping which satisfies all the assumptions of the celebrated Gale and Nikaido's lemma (see Mas-Colell et al, Chapter 17), stating the existence of a solution to a specific system of algebraic equations. At the same time, nullpoints of this mapping trivially coincide with the set of Nash equilibria.

2. Model

Consider a population P of a unit mass located over the segment $[0, 1]$ according to a distribution function $F(\cdot)$ (where $F(x)$ equals the population located within the segment $[0, x]$). We will restrict ourselves to the

[1] Migration usually means moving from one place to another. In the Alesina and Spolaore world, agents do not change locations when switching between countries. The natural way to think of this is to interpret location in the sense of preferences, not a geographical address.

regular case where F possesses a density f which is continuous and bounded away from zero (hence, "every point on the segment is populated").[2]

We will study *consecutive* partitions of P into intervals-countries $K_{l,r}$ each containing agents living within the limits $l < r$ on the line.[3] We can thus talk of partitions of the segment $[0,1] = K_1 \cup K_2 \cup \ldots \cup K_n$.

Due to the regularity assumption, every n-partition one-to-one corresponds to a point $\bar{a} = (a_1, \ldots, a_n)$ in the interior of the $(n-1)$-simplex Δ_{n-1} consisting of all $(a_1, \ldots, a_n)$ such that $\forall u\ a_u > 0$, and $\sum_{u=1}^{n} a_u = 1$. Here a_u equals the measure (=population) of the u-th country, enumerated from left to right.

This means that u-th country is $K_u = K_{w_{u-1},w_u}$, where $F(w_u) - F(w_{u-1}) = a_u$. Such a partition will be called $P(\bar{a})$ in what follows.

Next, we should specify a rule that assigns cost burden to any citizen of $K_{l,r}$ located at any point $x \in [0,1]$.[4] We will assume that this cost burden is additively separable and consist of two parts: *monetary cost* $\frac{g}{F(r)-F(l)}$ of maintaining the government (implicitly, total cost of government g is the universal constant, and it is to be allocated equally among all citizens of a country), and *transport cost* of being distant from the government, which may take a form of any increasing and bounded function of a distance between the agent and his government. For simplicity of exposition only, we let this function simply to be just equal to the distance: $|x - m[K_{l,r}]|$.

Now, to close the model, one has to specify the location $m[K_{l,r}$ to each country-to-form, which we denote by $m(l,r)$. In the mainstream case, the location is a *median* of a segment $[l,r]$, that is, the point $m(l,r)$ such that $F(r) - F(\mathrm{m(l,r)}) = F(\mathrm{m(l,r)}) - F(l)$; however, there exist a number of other specifications. The only claim we pose is that this median moves continuously:

Assumption C**:** The function $m(l,r)$ is continuous in its arguments.

Obviously, Assumption C is satisfied for the median specification in the regular case. Notice however that once the population density approaches zero, the assumption C tends to be violated.

Summarizing what we assumed above, we have the following

Assumption M**:** The total cost of a citizen i located at $x = x(i)$, who is willing to be a resident of a country $K(l,r)$ with the government located at $m(l,r)$, is equal to

$$c(x,l,r) = \frac{g}{F(r) - F(l)} + |m(l,r) - x|.$$

Given a partition $P(\bar{a})$, every citizen i located in $x(i) \in K_u$ would experience total costs $c(i, P(\bar{a}))$ equal to $c(x(i), w_{u-1}, w_u)$. Therefore, we can ask any such individual $i \in [0,1]$ whether he would like to change its place of residence from K_u to K_v: this would happen if $c(x(i), w_{v-1}, w_v) < c(i, P(\bar{a}))$. Our central definition next comes:

Migration-proofness: A (consecutive) partition $P(\bar{a})$ is migration-proof if there is no one citizen of the world who is willing to change his/her residence.

And our main result:

Theorem 1. *For every world P satisfying Assumptions C, M, and every positive integer n, there exists a consecutive migration-proof partition $P(\bar{a})$ of this world.*

Comment 1: There could not exist non-consecutive migration-proof partitions.

Comment 1: There are several very simple *quasiproofs* of this fact; all of them seem to be not only wrong but also hopeless to correct. I skip them here, for the sake of space.

Relevance to the real world: Obviously, this gives a portion of optimism, since in every conflict situation where there is only one parameter of disagreement, one would expect a solution which would satisfy individual rationality. (Compare to our previous findings (Bogomolnaia et al 2005,2007): there exist worlds which do not admit for partitions immune against *collective* actions!)

Proof (of the main result). In what follows, denote $b_0 = 0$, and for all $u = 1, \ldots, n$ we define $b_u = \sum_{j=1}^{i} a_u$ (hence $b_n = 1$). Now, under the Assumptions C, M, for every country $K(l,r)$ there could be uniquely defined two values, $c_L(l,r)$ and $c_R(l,r)$ as $c(l,l,r)$ and $c(r,l,r)$. Now, for any consecutive partition $P(\bar{a})$, there arise two collections of costs, left extremes $(c_{1L}, \ldots, c_{nL})$ defined as $c_{uL} = c_L(w_{u-1}, w_u)$ and right extremes $(c_{1R}, \ldots, c_{nR})$ alike. The following lemma characterizes migration-proof allocations:

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[2] We conjecture that our main result holds for *arbitrary* distribution F.

[3] As will be clear from below, it totally does not matter how to assign border agents living in points l and r: to include or exclude each of them from $K_{l,r}$.

[4] This is crucial even for $x \notin [l,r]$ since we will taste deviations of agents to *any* possible countries, not only the adjacent ones.

Lemma C: *For a partition $P(\bar{a})$ to be migration-proof, it is necessary and sufficient to observe that* $c_{1R} = c_{2L}, \ldots, c_{n-1,R} = c_{nL}$.

Comment: Lemma C just says that a function $c(i, P(\bar{a}))$ is continuous.
Hence, the main result states that for every regular distribution of the world P, one can design its division into n countries such that the total cost $c(x)$ of an individual located in $x \in [0,1]$ becomes a continuous function, producing no envy from one individuals to others as for where they are resided.

The proof, we believe, is elegant enough to present it here, in the main body of the paper. We construct the following vector-function (mapping) Ψ from the interior of the $(n-1)$-simplex Δ_n to $\mathbf{R}^n$:

$$\Psi(a_1, \ldots, a_n) =$$

$$[a_2(c_{1R} - c_{2L}), a_1(c_{2L} - c_{1R}) + a_3(c_{2R} - c_{3L}), \ldots, a_{n-2}(c_{n-1,L} - c_{n-2,R}) + a_n(c_{n-1,R} - c_{nL}), a_{n-1}(c_{nL} - c_{n-1,R})].$$

This mapping, being extended to $\mathbf{R}^n_{++}$ by homogeneity of degree 0, satisfies the following 5 properties, as listed in the celebrated Gale and Nikaido's Lemma:

(i) Homogeneity of degree 0. Just by construction.

(ii) Continuity. Immediate consequence of Assumptions C, M.

(iii) Walras Law: $\sum\limits_{u=1}^{n} a_u \Psi_u(a_1, \ldots, a_n) = 0$ for ALL combinations $(a_1, \ldots, a_n)$. Again, direct calculation — everything cancels.

(iv) Uniform boundedness from below: $a_{i-1}(c_{iL} - c_{i-1,R}) + a_{i+1}(c_{iR} - c_{i+1,L}) \geq -a_{i-1}c_{i-1,R} - a_{i+1}c_{i+1,L} \geq -1 - g - 1 - g = -(2+g)$.

(v) If a sequence $(a_1^m, \ldots, a_n^m)$ approaches some point $(a_1, \ldots, a_n)$ on the boundary of a simplex as $m \to +\infty$, which means that for some (but not all) u we have $a_u^m \to 0$, then for some v (probably $v \neq u$) we have that $\Psi_v(a_1^m, \ldots, a_n^m)$ approaches $+\infty$.

This is little bit more tricky. Still, it is quite straightforward. Due to already proved (iv) (in fact that negative parts are bounded from $-\infty$!), it is well enough to prove that positive parts sum to something going to $+\infty$ as $m \to +\infty$. This positive part is greater than $\frac{a_2^m}{a_1^m} + \frac{a_1^m}{a_2^m} + \ldots + \frac{a_{n-1}^m}{a_n^m} + \frac{a_n^m}{a_{n-1}^m}$. This expression goes to $+\infty$ when $(a_1^m, \ldots, a_n^m)$ approaches the boundary!

Now we simply apply Gale and Nikaido's lemma which states that under these conditions, there exists a collection $(a_1, \ldots, a_n)$ of nonzero numbers, such that we have

$$\Psi_u(a_1, \ldots, a_n) = 0 \quad \forall u = 1, \ldots, n.$$

For this collection, as $\forall u \; a_u \neq 0$, it is immediate that $c_{1R} = c_{2L}$, then from the second component $c_{2R} = c_{3L}$, and et cetera to $c_{n-1,R} = c_{nL}$.

Theorem 1 is completely proved.

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