Ex-Post Individually Rational, Budget-Balanced Mechanisms and Allocation of Surplus*

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Abstract

We investigate the issue of implementation via ex-post individually rational budget-balanced Bayesian mechanisms and allocation of surplus. We set up an auxiliary problem for minimized information rent where an agent is committed to a certain mixed deviation strategy before the agent's own type is known to himself. This formulation allows us to provide necessary and sufficient implementability conditions. We also develop an algorithm to determine whether a decision rule is implementable or not and to compute the informational rents earned by the players. We provide full characterization of the optimal mechanisms and implementability conditions in a number of special, but common cases.

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1 Introduction

In this paper, we investigate mechanism design when the players can withdraw from the mechanism ex-post, after learning the allocation assigned to them. In technical terms, we study mechanisms with ex-post individual rationality. Such mechanisms have intuitive appeal. Indeed, it is plausible that a player has to decide what to report to the mechanism (e.g. which bid to submit or what goods to order) without knowing much about the other players (e.g. competitors). Yet, she can withdraw from the mechanism ex-post, after the mechanism announces the allocation. This descriptions fits a number of situations. For example, workers have an option to resign or retire from their employment, rather than to continue in a job that is not beneficial. A limited liability corporation has an option to declare bankruptcy, which becomes attractive if the market turns against it, e.g., the value of the corporation's investments decreases significantly. Examples of such strategic bankruptcy abound. In particular, a number of firms who have apparently overbid in wireless spectrum auctions in the U.S. have declared bankruptcy to avoid large payments for the spectrum. Recent financial crisis in the U.S. shows that a significant number of homeowners are willing to abandon their houses when their mortgage indebtedness exceeds the value of the house. Other examples of environments where ex-post individual rationality is relevant include various trading situations, especially with opt-out options (such options are legally required in several countries and industries), and also in internet auction environments, since payments are made after winning an auction over the internet.

Despite its apparent relevance, mechanism design with ex-post individual rationality has not been sufficiently explored in the literature, especially in multiperson situations. (In some single-agent situations, such as principal-agent model with adverse selection, the notions of interim and ex-post individual rationality coincide.) Kosmopoulou (1999) establishes an equivalence between efficient implementation in ex-post individually rational, ex ante budget balanced dominant strategy mechanism and efficient implementation in Bayesian ex-post budget balanced mechanisms with interim individual rationality, under independent type distribution. In contrast, we focus on Bayesian mechanisms with ex-post individual rationality, which have not been studied in the literature, and allow non-independent type distributions.

In this paper we do not restrict the environment under consideration in any way, except for the quasi-linearity of the players' utility functions with respect to money. We allow for arbitrary decision rules which generate nonnegative expected social surplus. Our goal is to characterize the conditions under which a decision rule can be implemented via a Bayesian mechanism with ex-post individual rationality and (ex-ante) budget balance.

The implementability conditions for our problem turn out to have both similarities and also significant differences with the implementability conditions under interim incentive and individual rationality constraints with budget balance. In the latter framework, sufficient expected surplus is necessary when the types are distributed independently (Makowski and Mezzetti 1994) and this amount of surplus does not depend on the type distribution and is determined by the structure of the incentive problem. However, with non-independent type distribution, availability of surplus is not required, as all social decision rules generating a positive expected social surplus are implementable if the joint type distribution satisfies generic identifiability condition (Severinov and Kosenok 2008).

In our ex-post problem, a sufficient amount of expected surplus is needed even with non-independent type distributions. However, the necessary amount of expected surplus does depend on the joint type distribution, since the sum of expected informational rents of the agents -which has to be covered by the surplus - is sensitive to the type distribution.

In the first step in our analysis, we establish an important dichotomization property of Bayesian implementation with ex-post individual rationality and ex-ante budget balance. Precisely, a decision rule can be implemented in such mechanism if and only if the expected social surplus generated by the decision rule exceeds the expected sum of the surpluses that the mechanism designer needs to pay to the individual players to guarantee truth-telling. Clearly, this is the weakest possible condition. For, with budget balance, it is impossible to provide more surplus to the players than the mechanism generates. Thus, one of the contributions of the paper is to establish that the conditions required for budget-balanced implementation under ex-post individual rationality are the minimal possible one in this environment.

Although computing the expected social surplus in the mechanism is an easy task, determining the value of the surpluses (informational rents ensuring incentive compatibility) is more challenging. We provide a significant simplification here by establishing that the expected informational rent of each individual player can be computed by finding the "best" deviation for this player and determining the associated profits that she can get from this deviation. In other words, a player's best deviation combines in itself all other profitable deviations. So, by allocating sufficient surplus to prevent this deviation the mechanism designer prevents the player from engaging in any other deviation and ensures truth-telling. Thus, the task of computing the informational rents earned by a player boils down to finding her best deviation. However, this is sufficiently complex since the best deviation for a particular type of a player depends on the allocation of the informational rents of its other types, which in turn, depend on the best deviations of those other types.

For computing the best deviation and the expected informational rent associated with it, the main tasks is to allocate the informational rents (surpluses) across player types in a way that minimizes the incentives of the other types to imitate the type receiving the surplus. This is essential for minimizing the expected informational rents that the mechanism designer needs to pay. This problem has a simple solution when only one type wishes to imitate the type getting an informational rent. In this case, the latter should be paid in the "state of the world" with the highest likelihood ratio i.e., when the ratio of the probabilities of the announced type profile of the other players conditional on the imitated type and on the imitator type reaches its maximal value. When two types want to imitate each other, giving information rent to one type will increase the information rent given to the other type; thus, it will increase the information rent to the former one even further. This vicious cycle may leads to infinite information rent. However, we show that the information rent is finite if and only if the highest likelihood ratios are different for the two types: the product of the two likelihood ratios is larger than unity, and it plays the role of a discount factor to make the infinite accumulation of information rent a finite value. In general, if there a cycle in which each type wants to imitate its neighborhood in one direction, the information rent is finite if and only if at least two of the highest likelihood ratios are different.

The problem is more complex when two or more types wish to imitate another type. Still, the basic intuition of paying the informational rents to each type in the "state of the world" with the highest likelihood ratio, appropriately defined, survives. The appropriate definition of the likelihood ratio in this case is a linear combination of individual likelihood ratios of the other players type profile conditional on the imitated type and each of its imitator types. In the paper, we develop a fairly simple and tractable algorithm for solving this problem. We then demonstrate how our method works by characterizing the set of implementable allocations in some special, yet common, cases.

Thus, our paper shows that the optimal mechanism and implementability conditions under ex-post individual rationality remain sensitive to the joint probability distribution from which the players' types are drawn. This feature of the optimal mechanisms, originally established by Crémer and McLean (1985) and (1988) for Bayesian mechanisms, had been subject to scrutiny in the literature from the point of view of the robustness of the mechanisms and their sensitivity to the players' beliefs. Yet, it is worth noting that our mechanisms utilize the players beliefs in a particular fashion, via likelihood ratio: the designer needs to know which profile of the other players types is seen as more likely by a particular type relative (in ratio to) the beliefs of the other types of this player. Likelihood ratio techniques are intuitive and commonly used in the principal-agency theory and in econometrics. There are simple methods to identify and discover them by asking the players to engage in simple bets. Furthermore, requiring the knowledge of how the likelihood ratio of a state of the world varies with type of a player does not appear to be any more stringent than requiring the knowledge of a player's utility function and its dependence on her type.

The rest of the paper is organized as follows. In section 2, we develop the model. In section 3, we present the main analytical results. In section 4 we discuss a computational algorithm. In section 5 we derive the solution in some special cases. Section 6 concludes. All proofs are relegated to an Appendix.

2 The Model

We consider an economy with n players, $N = \{1, 2, \ldots, n\}$. Player $i \in N$ has privately known type drawn from the type space $\Theta_i \equiv \{\theta_i^1, ..., \theta_i^{m_i}\}$ of cardinality m_i, $2 \leq m_i < \infty$. The set of type profiles is given by $\Theta \equiv \prod_{i=1,\ldots,n} \Theta_i$, with cardinality $L \equiv \prod_{i=1,\ldots,n} m_i$. A state of the world is equivalent to the realized type profile. When focussing on player i, we will use the notation (θ_i, θ_{-i}) to denote a profile of types where θ_{-i} is the profile of types of players other than i. Let $\Theta_{-i} = \prod_{l \neq i} \Theta_l$, $L_{-i} = \prod_{l \neq i} m_l$, $\Theta_{-i-j} = \prod_{l \notin \{i,j\}} \Theta_l$, and $L_{-i-j} = \prod_{l \notin \{i,j\}} m_l$. A generic element of Θ_{-i-j} is denoted by θ_{-i-j}.

The probability distribution from which the players' type profile θ is drawn is denoted by $p(\theta)$, with $p_i(\theta_i)$ and $p_{i,j}(\theta_i, \theta_j)$ denoting the corresponding marginal probability distribution of agent i's type and the marginal probability distribution of types of players i and j, respectively. We assume that $p(\theta)$ is common knowledge. We also assume that $p_{i,j}(\theta_i, \theta_j) > 0$ for any $\theta_i \in \Theta_i, \theta_j \in \Theta_j$ of any two agents i and j. Further, let $p_{-i}(\theta_{-i}|\theta_i)$ denote the probability distribution of type profiles of players other than i conditional on the type of agent i. We use a similar system of notation for other probability distributions over Θ that we use below. The set of all probability distributions over Θ is denoted by $\mathcal{P}(\Theta)$.

A mechanism designer does not have any private information. She chooses a public decision x from the feasible set X and allocates transfers to and from players. Player i's utility function is given by $u_i(x, \theta) + t_i$ where t_i is a transfer that i receives from the mechanism. Without loss of generality, a player's reservation utility is normalized to zero.[1] A (social) decision rule $x(\cdot)$ is a function mapping the type space Θ into the set of public decisions X.[2] Also, $t(\cdot) = (t_1(\cdot), ..., t_n(\cdot))$ is a collection of transfer functions to all agents, where $t_i(\cdot) : \Theta \mapsto \mathbf{R}$ is a transfer function to agent i. An allocation profile is a combination of a decision rule $x(\cdot)$ with a collection of transfer functions $t(\cdot)$.

By the Revelation Principle, we can restrict the analysis to direct mechanisms in which the mechanism designer offers an allocation profile to the agents. If the agents, informed of their types, decide to participate in this mechanism, they report their types to the mechanism designer, and the allocation corresponding to the reported type profile is implemented.

Our main goal is to provide necessary and sufficient conditions for the existence of ex-

[1] Suppose that agent i's utility from her outside option is equal to $w_i(\theta_i, \theta_{-i})$. Such environment is equivalent to the environment where i's utility function is given by $u_i(x, \theta) - w_i(\theta) + t_i$ and her outside option is 0. Note that the sets of ex-post efficient decision rules and the notions of social surplus are the same in both environments.

[2] Note that randomization in public decisions is implicitly allowed, since X can be regarded as a set of probability distributions over some set of "pure" outcomes.

post individually rational and ex-ante budget-balanced Bayesian mechanisms implementing desirable decision rules. Let us describe these properties formally.

First, decision rule $(x(\cdot), t(\cdot))$ is incentive compatible if the following *Interim Incentive Constraint* $IC_i(\theta_i, \theta_i')$ holds for all $i \in \{1, ..., n\}$ and $\theta_i, \theta_i' \in \Theta_i$:

$$\sum_{\theta_{-i}} \Big[u_i(x(\theta_{-i}, \theta_i), (\theta_{-i}, \theta_i)) + t_i(\theta_{-i}, \theta_i) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - t_i(\theta_{-i}, \theta_i')\Big] p_{-i}(\theta_{-i}|\theta_i) \geq 0. \quad (1)$$

A decision rule $x(\cdot)$ is said to be *implementable* if there exists a profile of transfer functions $t(\cdot)$ such that $(x(\cdot), t(\cdot))$ is incentive compatible.

Ex-post Individual Rationality (EPIR) requires the following constraint to hold for all $i \in \{1, ..., n\}$ and $\theta \in \Theta$:

$$u_i(x(\theta), \theta) + t_i(\theta) \geq 0. \quad (2)$$

Ex-ante Budget Balancing (EABB) constraint can be written as follows:

$$\sum_{\theta \in \Theta} \sum_{i=1}^{n} t_i(\theta) p(\theta) = 0. \quad (3)$$

A decision rule $x(\cdot)$ is *ex-post efficient* if $x(\theta) \in \arg\max_{x \in X} \sum_{i=1}^{n} u_i(x, \theta)$ for all $\theta \in \Theta$, i.e. $x(\theta)$ maximizes ex-post *social surplus* $\sum_{i=1}^{n} u_i(x, \theta)$. Since the principal always has an option to disband the mechanism and cause the agents to take their outside options, we assume without loss of generality that $\max_{x \in X} \sum_{i=1}^{n} u_i(x, \theta) \geq 0$ for all $\theta \in \Theta$. Finally, *EPIR* and *EABB* together imply the following *Ex-Ante Social Rationality (EASR)* condition:

$$S \equiv \sum_{\theta \in \Theta} \sum_{i=1}^{n} u_i(x(\theta), \theta) p(\theta) \geq 0. \quad (4)$$

EASR simply says that a decision rule must generate a nonnegative (ex ante) expected surplus. Clearly, this is a very weak requirement. It is satisfied by a large variety of decision rules, including the ex-post efficient ones. Having established *EASR* as a necessary condition, in the next section we characterize necessary and sufficient conditions for *EPIR* and *EABB* implementation of *EASR* decision rules which include ex-post efficient ones.

3 Analysis

Under truth-telling agent i gets net utility

$$U_i(\theta) = u_i(x, \theta) + t_i(\theta). \tag{5}$$

Since for given decision rule $x(\cdot)$, the net surplus $U_i(\theta)$ uniquely determines the transfer $t_i(\theta)$, and vice versa, we can use $U_i(\cdot)$ and $t_i(\cdot)$ interchangeably. In particular, rewriting (1), (2) and (3), we can state:

Definition 1 *A decision rule $x(\cdot)$ is implementable via an ex-post individually rational, ex-ante budget balanced Bayesian mechanism if there exists a profile $\{U_i(\theta)\}_{i=1,\dots,n,\ \theta_i\in\Theta_i}$ s.t. for all $i = 1, ..., n$, $\theta_i, \theta_i' \in \Theta_i$ and $\theta \in \Theta$, the following conditions hold:*

$$IC_i(\theta_i, \theta_i') : \quad \sum_{\theta_{-i}\in\Theta_{-i}} p_{-i}(\theta_{-i}|\theta_i) \left\{U_i(\theta_{-i}, \theta_i) - U_i(\theta_{-i}, \theta_i')\right\}$$

$$\geq \sum_{\theta_{-i}\in\Theta_{-i}} \left\{u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta_i'), (\theta_{-i}, \theta_i'))\right\} p_{-i}(\theta_{-i}|\theta_i), \tag{6}$$

$$EPIR_i(\theta) : \quad U_i(\theta) \geq 0, \tag{7}$$

$$EABB : \quad \sum_{\theta\in\Theta} \sum_{i=1}^{n} U_i(\theta)p(\theta) = \sum_{\theta\in\Theta} \sum_{i=1}^{n} u_i(x(\theta), \theta)p(\theta) = S. \tag{8}$$

Next, consider the following problem of minimizing agent i's ax-ante expected surplus.

$$V_i = \min_{\{U_i(\theta)\geq 0:\theta\in\Theta\}} \sum_{\theta\in\Theta} U_i(\theta)p(\theta) \quad s.t. \quad (6) \ \text{ holds for all } \quad \theta_i, \theta_i' \in \Theta_i \tag{9}$$

The solution to problem (9) subject to (6) determines the minimal ex-ante surplus V_i necessary to ensure truth-telling by agent i and her voluntary participation in the mechanism. Specifically, if the constraint set of this problem is compatible i.e., there is a profile $\{U_i(\theta)\}_{\theta\in\Theta}$ s.t. all inequalities in (7) and (6) hold, then there exists a solution $\{U_i^*(\theta)\}_{\theta\in\Theta}$ to Problem (9):

$$V_i = \sum_{\theta\in\Theta} U_i^*(\theta)p(\theta) < \infty. \tag{10}$$

By construction, $V_i \geq 0$.

If the constraint set is empty, i.e. there is no $\{U_i(\theta)\}_{\theta\in\Theta}$ satisfying (6) and (7), then we take the value of the problem (9) to be infinite i.e., $V_i = +\infty$.

Next, we establish the following important result:

Proposition 1 *Decision rule $x(\cdot)$ is implementable via an ex-post individually rational, ex-ante budget balanced Bayesian mechanism if and only if the solution to (9) for all i induces such $V_1, ..., V_n$ via (10) that*

$$\sum_{i=1}^{n} V_i \leq S. \tag{11}$$

Proposition 1 establishes an important decomposition property: it is sufficient to consider each agent's problem separately. This result will allow us to find minimal informational rents required for implementability. If social surplus is sufficient to cover the sum of agents' individual rents, then a mechanism exists. From the proof of Proposition 1, it can be seen that the excess of social surplus over the sum of required informational rents can be split among the agents in an arbitrary way.

3.1 Strategies and Informational Rent

In this section, we will first consider strategies chosen by the agents *in a direct mechanism.* Let us start by introducing some additional notation. Agent i's strategy s_i in a direct mechanism is a vector of size m_i^2 such that its entry $s_i(\theta_i, \theta_i')$ denotes the probability with which agent i of type θ_i reports type θ_i'. Note that $s_i(\theta_i, \theta_i') \in [0, 1]$ and $\sum_{\theta_i'\in\Theta_i} s_i(\theta_i, \theta_i') = 1$ for all $\theta_i \in \Theta_i$. Let S_i be the set of all such strategies. A truthful strategy s_i^* of agent i satisfies $s_i(\theta_i, \theta_i) = 1$ and $s_i(\theta_i, \theta_i') = 0$ for all $\theta_i, \theta_i' \in \Theta_i$, $\theta_i \neq \theta_i'$. A strategy profile $\mathbf{s} \equiv (s_1, ..., s_n)$ is a collection of strategies followed by all agents. A strategy profile such that agent i follows strategy s_i and all other agents follow truthful strategies is denoted by (s_i, s_{-i}^*).

Definition 2 *Say that the strategy profile $\mathbf{s} \equiv (s_1, ..., s_n)$ induces the probability distribution over the reported type profiles $q(.|\mathbf{s})$ if type profile $\theta' \in \Theta$ is reported with probability $q(\theta'|\mathbf{s})$ when the agents follow strategies $\mathbf{s} = (s_1, ..., s_n)$ and the types are drawn from the prior $p(\cdot)$.*

To compute $q(.|\mathbf{s})$, note that

$$q(\theta_1',...,\theta_n'|\mathbf{s}) = \sum_{(\theta_1,...,\theta_n)\in\Theta} \Big(p(\theta_1,...,\theta_n)\prod_{i=1}^n s_i(\theta_i,\theta_i')\Big) \text{ for any } (\theta_1',...,\theta_n')\in\Theta.$$

Also, let

$$g_i(s_i) \equiv \sum_{\theta_i,\theta_i'\in\Theta_i}\sum_{\theta_{-i}\in\Theta_{-i}} [u_i(x(\theta_{-i},\theta_i'),(\theta_{-i},\theta_i)) - u_i(x(\theta_{-i},\theta_i'),(\theta_{-i},\theta_i'))]s_i(\theta_i,\theta_i')p(\theta_{-i},\theta_i). \quad (12)$$

With this notation, incentive constraints (6) can be rewritten as follows

$$\sum_{\theta\in\Theta}(p(\theta) - q(\theta|s_i,s_{-i}^*))U_i(\theta) \geq g_i(s_i) \quad for\ any \quad \theta_i\in\Theta_i \ and \ \ s_i\in S_i. \quad (13)$$

The constraints (13) can be interpreted as follows. The right-hand side of (13) is the expected gain that agent i gets from a better allocation $x(\cdot)$ when he follows strategy s_i rather than reports truthfully. The left-hand side of (13) reflects expected loss of surplus for agent i when he follows strategy s_i, instead of reporting her true type.

It will be useful to consider the following problem. Suppose that agent i can follow only one deviation strategy $s_i \in S_i$. Let us consider how the principal can prevent her from this deviation. The corresponding problem and its dual are stated below.

$$V_i(s_i) = \min_{U_i(\theta)\geq 0}\sum_\theta U_i(\theta)p(\theta) \quad s.t. \quad \sum_\theta(p(\theta) - q(\theta|s_i))U_i(\theta) \geq g_i(s_i), \qquad [LP_P(s_i)]$$

$$V_i(s_i) = \max_{\alpha_i(s_i)\geq 0} g_i(s_i)\alpha_i(s_i) \quad s.t. \quad \alpha_i(s_i)[p(\theta) - q(\theta|s_i)] \leq p(\theta) \qquad [LP_D(s_i)]$$

with simplifying notation of $q(\cdot|s_i) := q(\cdot|s_i,s_{-i}^*)$.

The solution to $LP_P(s_i)$ and $LP_D(s_i)$ is characterized in the following Proposition:

Proposition 2 *If $q(.|s_i) \neq p(\cdot)$, then the solution to problems $LP_P(s_i)$ and $LP_D(s_i)$, $V_i(s_i)$, is given by:*

$$V_i(s_i) = \frac{g_i(s_i)}{1 - \min\limits_{\theta\in\Theta}\{q(\theta|s_i,s_{-i}^*))/p(\theta)\}}. \quad (14)$$

If $q(.|s_i) = p(\cdot)$, then $V_i(s_i) = \infty$ if $g_i(s_i) > 0$ and $V_i(s_i) = 0$ if $g_i(s_i) \leq 0$.

To understand Proposition 2, consider the dual problem $LP_D(s_i)$. First, suppose that $q(.|s_i) \neq p(\cdot)$. Then, since $q(\cdot|s_i)$ and $p(\cdot)$ are probability distributions, i.e. $\sum_\theta p(\theta) =$

$\sum_\theta q(\theta|s_i) = 1$, we must have $q(\theta|s_i) < p(\theta)$ for some $\theta \in \Theta$. The solution to $LP_D(s_i)$ involves setting $\alpha_i(s_i)$ at the maximum i.e., $\alpha(s_i) = \frac{1}{1-\min_{\tilde{\theta}\in\Theta} \frac{q(\tilde{\theta}|s_i)}{p(\tilde{\theta})}}$. However, if $q(.|s_i) = p(\cdot)$, then $V_i(s_i) = \infty$ if $g_i(s_i) < 0$ and $V_i(s_i) = 0$ if $g_i(s_i) \leq 0$.

Intuitively, the solution to $LP_P(s_i)$ can be explained as follows. Suppose that, in order to prevent the agent from using the deviation strategy s_i, the mechanism designer provides ex-ante surplus $V_i(s_i)$ to player i by making a transfer to her in the state of the world θ. Then the corresponding payment to i in the state of the world θ will be equal to $\frac{V_i(s_i)}{p(\theta)}$. If player i uses strategy s_i, her ex-ante expected payment loss will be equal to $V_i(s_i)(p(\theta) - q(\theta|s_i, s^*_{-i}))/p(\theta)$. Hence, deviation s_i would not be profitable if $V_i(s_i)(p(\theta) - q(\theta|s_i, s^*_{-i}))/p(\theta) \geq g_i(s_i)$. Clearly, the mechanism designer can minimize $V_i(s_i)$ by choosing to pay the agent in the state of the world θ where $(p(\theta) - q(\theta|s_i, s^*_{-i}))/p(\theta)$ is maximal. The latter observation implies (14). Also, it is clear from $LP_D(s_i)$ that $V_i(s_i) = \infty$, if $q(.|s_i) = p(\cdot)$ and $g_i(s_i) > 0$ and $V_i(s_i) = 0$, if $q(.|s_i) = p(\cdot)$ and $g_i(s_i) \leq 0$.

Next, let us consider the ex ante expected informational rent V_i necessary to induce truth-telling by agent i. The original problem and its dual are summarized as follows.

$$V_i = \min_{U_i(\theta)\geq 0} \sum_\theta U_i(\theta)p(\theta) \quad s.t. \quad \sum_{\theta_{-i}} p(\theta_i, \theta_{-i})[U_i(\theta) - U_i(\theta_{-i}, \theta'_i)] \geq \Delta u_i(\theta_i, \theta'_i)p_i(\theta_i) \qquad [LP_P]$$

$$V_i = \max_{\gamma_i(\theta_i,\theta'_i)\geq 0} \sum_{\theta_i,\theta'_i} \gamma_i(\theta_i, \theta'_i)\Delta u_i(\theta_i, \theta'_i)p_i(\theta_i)$$

$$s.t. \quad \sum_{\theta'_i} \gamma_i(\theta_i, \theta'_i)p_{-i}(\theta_{-i}, \theta_i) - \sum_{\theta'_i} \gamma_i(\theta'_i, \theta_i)p_{-i}(\theta_{-i}, \theta'_i) \leq p(\theta). \qquad [LP_D]$$

where $\Delta u_i(\theta_i, \theta'_i) = \sum_{\theta_{-i}\in\Theta_{-i}} \{u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta_i)) - u_i(x(\theta_{-i}, \theta'_i), (\theta_{-i}, \theta'_i))\} \, p_{-i}(\theta_{-i}|\theta_i)$, and $\gamma_i(\theta'_i, \theta_i)$ is a dual variables corresponding to the incentive constraint $IC(\theta, \theta')$.

Clearly, $V_i \geq V_i(s_i)$ for all $s_i \in S_i$. Moreover, the next Proposition shows that, in fact, $V_i = \sup_{s_i\in S_i} V_i(s_i)$.

Proposition 3 *The minimal informational rent of agent i, V_i, satisfies:*

$$V_i = \max_{s_i\in S_i} V_i(s_i).$$

According to Proposition 3, there exists "best" deviation strategy s_i for agent i which combines together all profitable deviations for this agent. By preventing such best deviation, the mechanism designer prevents the agent from engaging in any other deviation. Hence, the problem of determining the informational rents V_i for agent i boils down to finding such best deviation s_i and the corresponding informational rent $V_i(s_i)$.

Below, we will address this question as follows. First, we will provide an algorithm for computing the best s_i for agent i and the corresponding maximal $V_i(s_i)$. Second, we provide a full characterization for the cases $\#\Theta_i = 2$. Third, we provide a general two-step procedure to characterize more general cases of $\#\Theta_i > 2$, and apply it for a full characterization for the cases $\#\Theta_i = 3$. All along, we will focus on the conditions on the primitives determining the value of V_i.

4 Algorithm to calculate information rent

Let A be $\#\Theta \times \#\Theta_i(\#\Theta_i - 1)$ matrix where A's θ-th row and $(\tilde{\theta}_i, \tilde{\theta}'_i)$-th column element is

$$A[\theta, (\tilde{\theta}_i, \tilde{\theta}'_i)] = \begin{cases} p(\theta) & \text{if } \ \tilde{\theta}_i = \theta_i \\ -p(\theta) & \text{if } \ \tilde{\theta}'_i = \theta_i \\ 0 & \text{otherwise,} \end{cases}$$

$c_N = (\Delta u_i(\theta_i, \theta'_i) p_i(\theta_i))_{\theta_i \neq \theta'_i, (\theta_i, \theta'_i) \in \Theta_i^2}$ and $x_N = (\gamma_i(\theta_i, \theta'_i))_{\theta_i \neq \theta'_i, (\theta_i, \theta'_i) \in \Theta_i^2}$ be vectors of dimension $\#\Theta_i(\#\Theta_i - 1)$, and $b = (p(\theta))_{\theta \in \Theta}$ be a vector of dimension $\#\Theta$. Then we can write $[LP_D]$ as a matrix form: $\max_{x_N} c_N \cdot x_N$ s.t. $Ax_N \leq b_N$.

Let B be identity matrix of dimension $\#\Theta \times \#\Theta$ and c_B be zero vector of dimension $\#\Theta$. Then $[LP_D]$ is written as the following with auxiliary variable $x_B \geq 0$.

$$\max_{x_N, x_B} c_N \cdot x_N + c_B \cdot x_B \ \text{ s.t. } \ Ax_N + Bx_B = b \tag{15}$$

Plugging the constraint into the objective function, we derive:

$$c_N \cdot x_N + c_B \cdot x_B = c_N \cdot x_N + c_B \cdot (B^{-1}b - B^{-1}Ax_N) = c_B B^{-1} b + (c_N - c_B B^{-1} A) x_N. \tag{16}$$

We start from $(x_N = 0, x_B = b)$ that is feasible. To keep track of the changes we will make for the matrices, we denote the initial matrices by $c_N^{(0)}$, $x_N^{(0)}$, $c_B^{(0)}$, $x_B^{(0)}$, $A^{(0)}$, $B^{(0)}$, and $b^{(0)}$.

We repeat the following step from $k = 0$.

(Step $k+1$) Increase an element of $x_N^{(k)}$ (say, $({\theta_i}^k, {\theta'_i}^k)$-th element) such that $({\theta_i}^k, {\theta'_i}^k)$-th column of $(c_N^{(k)} - c_B^{(k)}[B^{(k)}]^{-1}A^{(k)})$ is the largest and positive ($c_B^{(k)} = 0$ when $k=0$; however, that is not generally the case for $k>0$). If there is no positive element in $(c_N^{(k)} - c_B^{(k)}[B^{(k)}]^{-1}A^{(k)})$, we end the procedure here. Note that $B^{(k)}$ is invertible in the ongoing steps, $k \geq 0$ (See Appendix A.7). In equation (16), ${c_N}^{(k)}$ is the (marginal) direct effect of the (marginal) change in ${x_N}^{(k)}$. The change in ${x_N}^{(k)}$ leads to change in $A^{(k)}{x_N}^{(k)}$. Accordingly, $B^{(k)}{x_B}^{(k)}$ changes in the opposite direction (see the constraint in (15)). The change of ${x_B}^{(k)}$ is measured by $[B^{(k)}]^{-1}(b^{(k)} - A^{(k)}{x_N}^{(k)})$, and its effect on the objective function is ${c_B}^{(k)}[B^{(k)}]^{-1}(b^{(k)} - A^{(k)}{x_N}^{(k)})$. Thus ${c_B}^{(k)}[B^{(k)}]^{-1}A^{(k)}$ is the (marginal) indirect effect of the (marginal) change in ${x_N}^{(k)}$. We interpret this as "externality". In short, ${c_N}^{(k)} - {c_B}^{(k)}[B^{(k)}]_{-1}A^{(k)}$ measures the direct benefit net of externality.

We can increase $({\theta_i}^k, {\theta'_i}^k)$-th element of $x_N^{(k)}$ only without breaking $x_B^{(k)} = [B^{(k)}]^{-1}(b^{(k)} - A^{(k)}x_N^{(k)}) \geq 0$, i.e., until a certain row in $x_B^{(k)}$ becomes zero. Once a certain row of $x_B^{(k)}$ (say, θ^k-th row) becomes zero, we form new matrices $c_N^{(k+1)}$, $x_N^{(k+1)}$, $c_B^{(k+1)}$, $x_B^{(k+1)}$, $A^{(k+1)}$, $B^{(k+1)}$, and $b^{(k+1)}$ by exchanging θ^k-th row of $x_B^{(k)}$ and $({\theta_i}^k, {\theta'_i}^k)$-th row of $x_N^{(k)}$, θ^k-th column of $c_B^{(k)}$ and $(\theta_i^k, {\theta'_i}^k)$-th column of $c_N^{(k)}$, and θ^k-th column of $B^{(k)}$ and $(\theta_i^k, {\theta'_i}^k)$-th column of $A^{(k)}$, respectively. Then the linear program still looks the same to (15) with the re-defined $c_N^{(k+1)}$, $x_N^{(k+1)}$, $c_B^{(k+1)}$, $x_B^{(k+1)}$, $A^{(k+1)}$, $B^{(k+1)}$, and $b^{(k+1)}$:

$$\max c_N^{(k+1)} \cdot x_N^{(k+1)} + c_B^{(k+1)} \cdot x_B^{(k+1)} \quad \text{s.t.} \quad A^{(k+1)}x_N^{(k+1)} + B^{(k+1)}x_B^{(k+1)} = b^{(k+1)} \quad \text{where} \quad x_N^{(k+1)} = 0.$$

Our algorithm is in fact an economic interpretation of simplex method in linear programming. The following two propositions are well-known in simplex method.

Proposition 4 *If* $(c_N - c_B B^{-1}A) \leq \mathbf{0}$ *after a certain number of steps, the basic feasible solution represented by* $\left({x_B}^T, {x_N}^T\right) = \left((B^{-1}b)^T, 0\right)$ *is optimal for* LP_D*.*

Proposition 5 *If the basic feasible solution* $B^{-1}b$ *is a strictly positive vector in each step, the algorithm finishes in finite steps to find the optimal value with an optimal* $\gamma_i(\cdot,\cdot)$*.*

However, the basic feasible solution $B^{-1}b$ is often not strictly positive in our context. Thus the above algorithm may fail to finish in finite steps, and cycle (See Beale (1955) and Marshall and Suurballe (1969)). There are a few additional tests preventing such possibility of a cycle, and the tests can be incorporated into the algorithm so that it finished in finite steps. We do not discuss the tests here as we do not see meaningful economic intuition behind them. Interested readers can refer to Dantzig, Orden, and Wolfe (1955) and Bland (1977).

Our problem shares similar features with *min cost flow problem* (Klein (1967)); however, our problem is neither a special case nor a general case of of the problem. See Appendix A.4 for the similarities and the differences.

5 Two-step procedure for characterization

Let the expected information rent be $U_i(\theta_i) := \sum_{\theta_{-i}} U_i(\theta_i, \theta_{-i}) p(\theta_{-i}|\theta_i)$. Let Γ_i be the set of binding incentive compatibility constraints. Thus, $IC_i(\theta_i, \theta_i') \in \Gamma_i$ if and only if $\gamma_i(\theta_i, \theta_i') > 0$. We visualize Γ_i as a directed graph: there is a directed arc from θ_i to θ_i' if and only if $IC_i(\theta_i, \theta_i') \in \Gamma_i$. Let Π_i be the set of all the possible directed graphs.

The primitives of an economy is (Θ_i, Θ_{-i}, $p(\theta)$, $\Delta u_i(\theta_i, \theta_i')$). Let $\mathcal{E}_i$ be the set of all the possible primitives. We mean by "characterization" that we partition $\mathcal{E}_i$ so that each partition represents necessary and sufficient conditions for a directed graph in Π_i. In other words, we find an *onto* mapping $f_i : \mathcal{E}_i \to \Pi_i$.

Our procedure of the characterization is as follows.

1. For given directed graph Γ_i, we provide the conditions for the binding incentive compatibility constraints and derive informational rent.

2. Then we provide the conditions under which all the other incentive constraints are non-binding or binding with zero shadow value.

The following lemma simplifies the first procedure.

Lemma 1 *For given Γ_i and given θ_i', if there are k different θ_is such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$, then there are* at most *k states $\theta_{-i} \in \Theta_{-i}$ such that $U_i(\theta_i', \theta_{-i})$ is strictly positive.*

Proof. For given θ_i' and for each θ_i such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$, the binding incentive compatibility constraint is: $U_i(\theta_i) = \Delta u_i(\theta_i, \theta_i') + \sum_{\theta_{-i}} U_i(\theta_i', \theta_{-i}) p(\theta_{-i}|\theta_i)$ where the right-hand side is the gain from imitating θ_i'.

There are k such constraints since there are k different θ_i such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$. Thus for given $U_i(\theta_i)$ such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$, the designer needs to solve the following program to minimize the information rent given to θ_i', i.e.,

$$\min_{U_i(\theta_i', \theta_{-i}) \geq 0} \quad \sum_{\theta_{-i}} U_i(\theta_i', \theta_{-i}) p(\theta_i', \theta_{-i}) \tag{17}$$

$$s.t. \;\; \Delta u_i(\theta_i, \theta_i') + \sum_{\theta_{-i}} U_i(\theta_i', \theta_{-i}) p(\theta_{-i}|\theta_i) = U_i(\theta_i), \forall \theta_i \; s.t. \; IC_i(\theta_i, \theta_i') \in \Gamma_i.$$

Clearly, the optimal solution can be attained with *at most* k positive $U_i(\theta_i', \theta_{-i})$ since there are k constraints that non-negative vector $(U_i(\theta_i))_{\theta_i \in \Theta_i}$ has to satisfy. ∎

Remark: Suppose there is only one θ_i such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$ for given θ_i'. Then the solution of (17) is $U_i(\theta_i', \theta_{-i}) = \frac{U_i(\theta_i')}{p(\theta_i', \theta_{-i})}$ for $\theta_{-i} = \underset{\theta_{-i}}{\operatorname{argmin}} \left[\frac{p(\theta_i, \theta_{-i})}{p(\theta_i', \theta_{-i})}\right]$, and $U_i(\theta_i) = \Delta u_i(\theta_i, \theta_i') + \min_{\theta_{-i}} \left[\frac{p(\theta_i, \theta_{-i})}{p(\theta_i', \theta_{-i})}\right] U_i(\theta_i')$. This finding will be used repeatedly in the following sections. ∎

For any θ_i' such that $IC_i(\theta_i, \theta_i') \in \Gamma_i$ with at least one $\theta_i \neq \theta_i'$, the first procedure determines how much information rent should be given at each state θ_{-i}. However, for θ_i' such that $IC_i(\theta_i, \theta_i') \notin \Gamma_i$ with any $\theta_i \in \Theta_i$, the first procedure does not dictate how the expected information rent $U_i(\theta_i')$ is distributed to $U_i(\theta_i', \theta_{-i})$. The second procedure will determine the range how the information rent should be distributed across states Θ_{-i}. The distribution of information rent $U_i(\theta_i')$ to different states θ_{-i}, $U_i(\theta_i', \theta_{-i})$, should deter other types $\theta_i (\neq \theta_i')$ to imitate θ_i', i.e., the distribution $U_i(\theta_i', \theta_{-i})$ should satisfy the following incentive compatibility constraints:

$$\Delta u_i(\theta_i, \theta_i') + \sum_{\theta_{-i}} U_i(\theta_i', \theta_{-i}) p(\theta_{-i}|\theta_i) \leq U_i(\theta_i), \;\; \forall \theta_i \;\; \text{s.t.} \;\; IC_i(\theta_i, \theta_i') \notin \Gamma_i.$$

We use this two-step procedure to characterize the case of $\#\Theta_i = 2$ and $\#\Theta_i = 3$ in the following sections.

6 Characterization of the case with $\#\Theta_i = 2$

Let $\Theta_i = \{\theta_i^1, \theta_i^2\}$. If $\Delta u_i(\theta_i^1, \theta_i^2) < 0$ and $\Delta u_i(\theta_i^2, \theta_i^1) < 0$, then $s_i(\theta_i, \theta_i) \equiv 1$ is optimal trivially.

Suppose $\Delta u_i(\theta_i^1, \theta_i^2) > 0$ and $\Delta u_i(\theta_i^1, \theta_i^2) p_i(\theta_i^1) \geq \Delta u_i(\theta_i^2, \theta_i^1) p_i(\theta_i^2)$ without loss of generality. There are two possibilities: $s_i(\theta_i^2, \theta_i^1) > 0$ or $s_i(\theta_i^2, \theta_i^1) = 0$ depicted below.

$$\begin{tikzcd} \theta_i^1 \arrow[loop left, "s_i(\theta_i^1,\theta_i^1)"] \arrow[r, bend left, "s_i(\theta_i^1,\theta_i^2)"] & \theta_i^2 \arrow[l, bend left, "s_i(\theta_i^2,\theta_i^1)"] \arrow[loop right, "s_i(\theta_i^2,\theta_i^2)"] \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} \theta_i^1 \arrow[loop left, "s_i(\theta_i^1,\theta_i^1)"] \arrow[r, "s_i(\theta_i^1,\theta_i^2)"] & \theta_i^2 \arrow[loop right, "s_i(\theta_i^2,\theta_i^2)"] \end{tikzcd}$$

$s_i(\theta_i^l, \theta_i^k) > 0$ implies that the incentive compatibility constraint for θ_i^l not to imitate θ_i^k is binding. Directed arcs in the diagrams represent the binding incentive compatibility constraints.

6.1 The case with $s_i(\theta_i^2, \theta_i^1) > 0$:

From the remark following Lemma 1, we derive:

$$U_i(\theta_i^1) = \Delta u_i(\theta_i^1, \theta_i^2) + U_i(\theta_i^2) \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}, \; U_i(\theta_i^2) = \Delta u_i(\theta_i^2, \theta_i^1) + U_i(\theta_i^1) \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})}.$$

Solving the simultaneous equation, we derive:

$$U_i(\theta_i^1) = \frac{\Delta u_i(\theta_i^1, \theta_i^2) + \Delta u_i(\theta_i^2, \theta_i^1) \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}}{1 - \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})} \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}}, \; U_i(\theta_i^2) = \frac{\Delta u_i(\theta_i^2, \theta_i^1) + \Delta u_i(\theta_i^1, \theta_i^2) \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})}}{1 - \min_{\theta_{-i}} \frac{p(\theta_i^2, \theta_{-i})}{p(\theta_i^1, \theta_{-i})} \min_{\theta_{-i}} \frac{p(\theta_i^1, \theta_{-i})}{p(\theta_i^2, \theta_{-i})}}.$$

The following is equivalent to $U_i(\theta_i^1)$ and shows better how information rent is accumulated.

$$\begin{aligned} U_i(\theta_i^1) =& \Delta u_i(\theta_i^1, \theta_i^2) + \min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)} \Delta u_i(\theta_i^2, \theta_i^1) + \min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)} \min \frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)} \Delta u_i(\theta_i^1, \theta_i^2) \\ &+ \left[\min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\right]^2 \min \frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)} \Delta u_i(\theta_i^2, \theta_i^1) + \left[\min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)}\right]^2 \left[\min \frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)}\right]^2 \Delta u_i(\theta_i^1, \theta_i^2) + \ldots \end{aligned}$$

The designer gives information rent $\Delta u_i(\theta_i^1, \theta_i^2)$ to type 1, then type 2 wants to imitate 1; so the designer gives $\Delta u_i(\theta_i^2, \theta_i^1)$ to type 2, then type 1 wants to imitate type 1 due to the increased benefit of imitating type 2; so the designer adds the benefit of imitation $\min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)} \Delta u_i(\theta_i^2, \theta_i^1)$ to type 1's information rent, then type 2 wants to imitate 1 due to the addition; so the designer add $\min \frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)} \Delta u_i(\theta_i^1, \theta_i^2)$ to type 2's information rent, then type 1 wants to imitate type 2 again; so the designer adds $\min \frac{p(\cdot|\theta_i^1)}{p(\cdot|\theta_i^2)} \min \frac{p(\cdot|\theta_i^2)}{p(\cdot|\theta_i^1)} \Delta u_i(\theta_i^1, \theta_i^2)$ to type 1's information rent; and so on.

In summary, we derive:

$$V_i = p_i(\theta_i^1) U_i(\theta_i^1) + p_i(\theta_i^2) U_i(\theta_i^2) = \frac{\Delta u_i(\theta_i^1, \theta_i^2) \left[1 + \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}\right] p_i(\theta_i^1) + \Delta u_i(\theta_i^2, \theta_i^1) \left[1 + \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}\right] p_i(\theta_i^2)}{1 - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)} \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}}.$$

6.2 The case with $s_i(\theta_i^2, \theta_i^1) = 0$:

Following the remark after Lemma 1, we trivially derive:

$$U_i(\theta_i^1) = \Delta u_i(\theta_i^1, \theta_i^2),\ U_i(\theta_i^2) = 0, \quad \text{and} \quad V_i = \Delta u_i(\theta_i^1, \theta_i^2) p(\theta_i^1).$$

6.3 Characterization

Let us compare the two optimal values of the two cases:

$$\Delta u_i(\theta_i^1, \theta_i^2) p(\theta_i^1) \lessgtr \frac{\Delta u_i(\theta_i^1, \theta_i^2)\Big(1 + \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)}\Big) p_i(\theta_i^1) + \Delta u_i(\theta_i^2, \theta_i^1)\Big(1 + \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}\Big) p_i(\theta_i^2)}{1 - \min \frac{p(\theta_i^2, \cdot)}{p(\theta_i^1, \cdot)} \min \frac{p(\theta_i^1, \cdot)}{p(\theta_i^2, \cdot)}}$$

$$\Leftrightarrow \ 0 \lessgtr \Delta u_i(\theta_i^2, \theta_i^1) + \Delta u_i(\theta_i^1, \theta_i^2) \min_{\theta_{-i}} \frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}. \tag{18}$$

Condition (18) means that if the θ_i^2's utility loss from imitating θ_i^1 (that is $-\Delta u_i(\theta_i^2, \theta_i^1)$) is smaller than the information rent from imitating (that is $\Big[\Delta u_i(\theta_i^1, \theta_i^2) \min_{\theta_{-i}} \frac{p(\theta_{-i}|\theta_i^2)}{p(\theta_{-i}|\theta_i^1)}\Big]$), then θ_i^2 will have incentive to imitate θ_i^1. Thus inequality (18) characterizes the case with $\#\Theta_i = 2$.

7 Characterization of the case with $\#\Theta_i = 3$

There are three types of agent i, $\theta_i \in \{\theta_i^1, \theta_i^2, \theta_i^3\}$, and M types of agent $(-i)$, $\theta_{-i} \in \{\theta_{-i}^1, \dots, \theta_{-i}^M\}$. For notational simplicity, we re-define notations: the probability distribution function is $p^{jk} := p(\theta_i^j, \theta_{-i}^k)$, type j's marginal distribution is $p_j = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_i^j, \theta_{-i})$, conditional probability of θ_{-i}^k given θ_i^j is $p_j^k = \frac{p^{jk}}{\sum_{1 \le l \le M} p^{jl}}$, $R_j := U_i(\theta_i^j)$, $R_j(k) := U_i(\theta_i^j, \theta_{-i}^k)$, and $\gamma_{jk} := \gamma_i(\theta_i^j, \theta_i^k)$ and $\Delta_{jk} := \Delta u_i(\theta_i^j, \theta_i^k)$. We also define $\mathcal{R}_j = (U_i(\theta_i^j, \theta_{-i}^k))_{k \in \Theta_{-i}}$, and $\mathcal{P}_j = (p_j^k)_{k \in \Theta_{-i}}$.

A binding incentive constraint between types θ_i^j to θ_i^k is represented by a directed arc from θ_i^j to θ_i^k in a graph with nodes of $\{\theta_i^1, \theta_i^2, \theta_i^3\}$. There are sixteen directed graphs to consider up to permutation of agent i's types (Figure 1).

Case 1: $\theta_i^3 \qquad \theta_i^2 \qquad \theta_i^1$

Information rents are trivially: $R_1 = 0,\ R_2 = 0,\ R_3 = 0$.

Six incentive compatibility constraints characterizing this case are:

$$R_1 = 0 \ge \Delta_{12}, R_1 = 0 \ge \Delta_{13}, R_2 = 0 \ge \Delta_{21}, R_2 = 0 \ge \Delta_{22}, R_3 = 0 \ge \Delta_{31}, R_3 = 0 \ge \Delta_{32}.$$

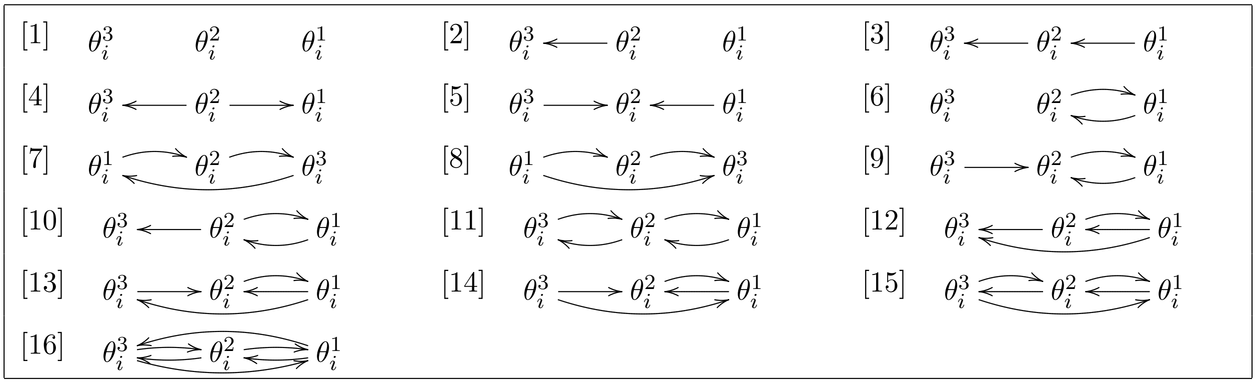

Figure 1:

Case 2: $\theta_i^3 \longleftarrow \theta_i^2 \qquad \theta_i^1$

Informational rents are trivially: $R_1 = 0$, $R_2 = \Delta_{23}$, $R_3 = 0$. Case 2 arises if and only if:

$$R_2 = \Delta_{23} \geq 0, \quad R_1 = 0 \geq \Delta_{13}, \; R_2 = \Delta_{23} \geq \Delta_{21}, \quad R_3 = 0 \geq \Delta_{31},$$
$$\exists (R_2(k) \geq 0)_{k \in \Theta_{-i}} \text{ s.t. } \Delta_{23} = \sum_k R_2(k) p_2^k, \quad 0 \geq \Delta_{12} + \sum_k R_2(k) p_1^k, \quad 0 \geq \Delta_{32} + \sum_k R_2(k) p_3^k.$$

The first constraint is non-negativity of type θ_i^2's information rent. The next three constraints are type θ_i^1's incentive compatibility constraint not to imitate θ_i^3, type θ_i^2's constraint not to imitate θ_i^3, and type θ_i^3's constraint not to imitate θ_i^1. In the second line, the first equality means that θ_i^2's expected information rent is Δ_{23}. The last two inequalities are type θ_i^1's incentive compatibility constraint not to imitate θ_i^2 and type θ_i^3's constraint not to imitate θ_i^2.

The second line of the condition can be written as $\mathcal{P}_2 \cdot \mathcal{R}_2 = \Delta_{23}$, $\mathcal{P}_1 \cdot \mathcal{R}_2 \leq -\Delta_{12}$, $\mathcal{P}_3 \cdot \mathcal{R}_2 \leq -\Delta_{32}$. We can show that the following is a necessary and sufficient condition for the existence of such $\mathcal{R}_2 \geq 0$ (See Appendix A.5 for proof).

$$\alpha \Delta_{12} + (1-\alpha)\Delta_{32} + \Delta_{23} \min_k \left[\alpha \frac{p_1^k}{p_2^k} + (1-\alpha) \frac{p_3^k}{p_2^k} \right] \leq 0, \; \forall \alpha \in [0,1].$$

For $\alpha = 0$, the last inequality implies $\Delta_{32} + \Delta_{23} \min_k \left[\frac{p_3^k}{p_2^k} \right] \leq 0$, which means that the misrepresentation of θ_i^3 can be deterred. For $\alpha = 1$, the inequality implies $\Delta_{12} + \Delta_{23} \min_k \left[\frac{p_1^k}{p_2^k} \right] \leq 0$, which means that θ_i^1's misrepresentation is deterred. The inequality could also be written as:

$$-\alpha \frac{\Delta_{12}}{\Delta_{23}} - (1-\alpha) \frac{\Delta_{32}}{\Delta_{23}} \geq \min_k \left[\alpha \frac{p_1^k}{p_2^k} + (1-\alpha) \frac{p_3^k}{p_2^k} \right], \; \forall \alpha \in [0,1].$$

For given α, the constraint is "easier" to satisfy as the structure of Θ_{-i} becomes "richer".

Case 3: $\theta_i^3 \longleftarrow \theta_i^2 \longleftarrow \theta_i^1$

We compute the informational rent as follows:

$$R_3 = 0, \quad R_2 = \Delta_{23}, \; R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23}$$

θ_i^2 receives positive rent at state $\theta_{-i}^{k_2}$ for $k_2 = \operatorname{argmin} p_1^k / p_2^k$. The conditions for this case are:

$$R_1 = \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} \geq 0, \; R_2 = \Delta_{23} \geq 0, \quad \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} \geq \Delta_{13},$$
$$0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} \Delta_{23}, \quad \Delta_{23} \geq \Delta_{21} + \sum R_1(k) p_2^k, \; 0 \geq \Delta_{31} + \sum R_1(k) p_2^k.$$

The first two are non-negativity conditions for R_1 and R_2. The last four conditions are for θ_i^1 not to imitate θ_i^3, for θ_i^3 not to imitate θ_i^2, for θ_i^2 not to imitate θ_i^1, and for θ_i^3 not to imitate θ_i^1.

Similarly to case 2, the following is a necessary and sufficient condition for case 3.

$$\Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} \geq 0, \; \Delta_{23} \geq 0, \quad \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23} \geq \Delta_{13}, \; 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} \Delta_{23}$$
$$\Big(\Delta_{12} + \min_k \frac{p_1^k}{p_2^k} \Delta_{23}\Big) \min_k \left[\alpha \frac{p_2^k}{p_1^k} + (1-\alpha) \frac{p_3^k}{p_1^k}\right] + \alpha(\Delta_{21} - \Delta_{23}) + (1-\alpha)\Delta_{31} \leq 0, \; \forall \alpha \in [0,1].$$

Case 4: $\theta_i^3 \longleftarrow \theta_i^2 \longrightarrow \theta_i^1$ (non-generic case)

Information rents are: $R_1 = 0, \;\; R_2 = \Delta_{23} = \Delta_{32}, \; R_3 = 0$. Note that this case is non-generic because it requires $\Delta_{23} = \Delta_{32}$. This case arises under the following conditions:

$$\Delta_{23} = \Delta_{32} = \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \; 0 \geq \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1, \; 0 \geq \Delta_{13}, \; 0 \geq \Delta_{31}, \; 0 \geq \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.$$

Similarly to case 2, the following is a necessary and sufficient condition for case 4.

$$\Delta_{23} = \Delta_{32} \geq 0, \; 0 \geq \Delta_{13}, \; 0 \geq \Delta_{31}, \; \Delta_{23} \min_k \left[\alpha \frac{p_1^k}{p_2^k} + (1-\alpha) \frac{p_3^k}{p_2^k}\right] + \alpha \Delta_{12} + (1-\alpha)\Delta_{32} \leq 0, \forall \alpha \in [0,1].$$

Case 5: $\theta_i^3 \longrightarrow \theta_i^2 \longleftarrow \theta_i^1$

The informational rents are: $R_1 = \Delta_{12}, \;\; R_2 = 0, \; R_3 = \Delta_{32}$. This case is characterized by:

$$\Delta_{12} \geq 0, \quad \Delta_{32} \geq 0, \; \Delta_{12} \geq \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1, \; 0 \geq \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2, \quad \Delta_{32} = \mathcal{R}_3 \cdot \mathcal{P}_3,$$
$$0 \geq \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2, \; \Delta_{32} \geq \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3, \quad \Delta_{12} = \mathcal{R}_1 \cdot \mathcal{P}_1$$

Similarly to case 2, the following is a necessary and sufficient condition for case 5.

$$\Delta_{12} \geq 0,\ \Delta_{32} \geq 0,\ \alpha(\Delta_{13} - \Delta_{12}) + (1-\alpha)\Delta_{23} + \Delta_{32} \min_k \left[\alpha \frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0,\ \forall \alpha \in [0,1],$$

$$\alpha\Delta_{21} + (1-\alpha)(\Delta_{31} - \Delta_{32}) + \Delta_{12} \min_k \left[\alpha \frac{p_2^k}{p_1^k} + (1-\alpha)\frac{p_3^k}{p_1^k}\right] \leq 0,\ \forall \alpha \in [0,1].$$

Case 6: $\theta_i^1 \qquad \theta_i^2 \rightleftarrows \theta_i^3$

From $R_1 = 0$, $R_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3$, $R_3 = \Delta_{32} + \min_k \frac{p_3^k}{p_2^k} R_2$, we derive informational rent:

$$R_1 = 0,\ R_2 = \frac{\Delta_{23} + \min_k \frac{p_2^k}{p_3^k}\Delta_{32}}{1 - \min_k \frac{p_2^k}{p_3^k} \min_k \frac{p_3^k}{p_2^k}},\ R_3 = \frac{\Delta_{32} + \min_k \frac{p_3^k}{p_2^k}\Delta_{23}}{1 - \min_k \frac{p_3^k}{p_2^k} \min_k \frac{p_2^k}{p_3^k}}.$$

Let $k_{23} = \operatorname*{argmin}_k \frac{p_2^k}{p_3^k}$, and $k_{32} = \operatorname*{argmin}_k \frac{p_3^k}{p_2^k}$. Type θ_i^2 receives positive rent at state $(\theta_i^2, \theta_{-i}^{k_{32}})$, and type θ_i^3 at $(\theta_i^3, \theta_{-i}^{k_{23}})$. For the information rent to be finite, k_{23} and k_{32} should be different.[3]

The necessary and sufficient conditions for this case are:

$$k_{23} \neq k_{32},\ R_2 \geq 0,\ R_3 \geq 0,\ R_1 \geq \Delta_{13} + \frac{p_1^{k_{23}}}{p_3^{k_{23}}} R_3,\ R_1 \geq \Delta_{12} + \frac{p_1^{k_{32}}}{p_2^{k_{32}}} R_2,\ R_2 \geq \Delta_{21},\ R_3 \geq \Delta_{32}.$$

Case 7: $\begin{tikzcd} \theta_i^1 \arrow[r] & \theta_i^2 \arrow[r] & \theta_i^3 \arrow[ll, bend left] \end{tikzcd}$

From $R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2$, $R_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3$, $R_3 = \Delta_{31} + \min_k \frac{p_3^k}{p_1^k} R_1$, we derive:

$$R_1 = \frac{\Delta_{12} + \frac{p_1^{k_1}}{p_2^{k_1}}\Delta_{23} + \frac{p_1^{k_1}}{p_2^{k_1}}\frac{p_2^{k_2}}{p_3^{k_2}}\Delta_{31}}{1 - \frac{p_1^{k_1}}{p_2^{k_1}}\frac{p_2^{k_2}}{p_3^{k_2}}\frac{p_3^{k_3}}{p_1^{k_3}}}, R_2 = \frac{\Delta_{23} + \frac{p_2^{k_2}}{p_3^{k_2}}\Delta_{31} + \frac{p_2^{k_2}}{p_3^{k_2}}\frac{p_3^{k_3}}{p_1^{k_3}}\Delta_{12}}{1 - \frac{p_1^{k_1}}{p_2^{k_1}}\frac{p_2^{k_2}}{p_3^{k_2}}\frac{p_3^{k_3}}{p_1^{k_3}}}, R_3 = \frac{\Delta_{31} + \frac{p_3^{k_3}}{p_1^{k_3}}\Delta_{12} + \frac{p_3^{k_3}}{p_1^{k_3}}\frac{p_1^{k_1}}{p_2^{k_1}}\Delta_{23}}{1 - \frac{p_1^{k_1}}{p_2^{k_1}}\frac{p_2^{k_2}}{p_3^{k_2}}\frac{p_3^{k_3}}{p_1^{k_3}}}$$

where $k_1 = \operatorname*{argmin}_k \frac{p_1^k}{p_2^k}$, $k_2 = \operatorname*{argmin}_k \frac{p_2^k}{p_3^k}$, and $k_3 = \operatorname*{argmin}_k \frac{p_3^k}{p_1^k}$.

Type θ_i^1 receives positive rent at state $(\theta_i^1, \theta_{-i}^{k_3})$, type θ_i^2 at $(\theta_i^2, \theta_{-i}^{k_1})$, and type θ_i^3 at $(\theta_i^3, \theta_{-i}^{k_2})$. For finite information rent, at least two of k_1, k_2, and k_3 should be different. The conditions for this case are:

$$R_1 \geq 0,\ R_2 \geq 0,\ R_3 \geq 0,\ R_1 \geq \Delta_{13} + \frac{p_1^{k_2}}{p_3^{k_2}} R_3,\ R_2 \geq \Delta_{21} + \frac{p_2^{k_3}}{p_1^{k_3}} R_1,\ R_3 \geq \Delta_{32} + \frac{p_3^{k_1}}{p_2^{k_1}} R_2.$$

[3]Note $\min_k p_2^k/p_3^k \min_k p_3^k/p_2^k < 1$ if and only if $k_{23} \neq k_{32}$ since $\min_k p_l^k/p_m^k < 1$ unless $p(\cdot|\theta_i^m) \equiv p(\cdot|\theta_i^l)$.

Assuming all of k_1, k_2, k_3 are all different, The last three inequalities can be simplified into

$$R_1 \geq \frac{\Delta_{13} + \frac{p_1^{k_2}}{p_3^{k_2}}\Delta_{31}}{1 - \frac{p_1^{k_2}}{p_3^{k_2}}\frac{p_3^{k_3}}{p_1^{k_3}}},\ R_2 \geq \frac{\Delta_{21} + \frac{p_2^{k_3}}{p_1^{k_3}}\Delta_{12}}{1 - \frac{p_2^{k_3}}{p_1^{k_3}}\frac{p_1^{k_1}}{p_2^{k_1}}},\quad R_3 \geq \frac{\Delta_{32} + \frac{p_3^{k_1}}{p_2^{k_1}}\Delta_{23}}{1 - \frac{p_3^{k_1}}{p_2^{k_1}}\frac{p_2^{k_2}}{p_3^{k_2}}}.$$

These conditions mean that any deviation to form a local cycle (as in case 6) is never profitable.

Case 8: $\begin{tikzcd}\theta_i^1 \arrow[r] \arrow[rr, bend right] & \theta_i^2 \arrow[r] & \theta_i^3\end{tikzcd}$ (non-generic case)

Information rents are:

$$R_3 = 0,\ R_2 = \Delta_{23},\ R_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k} R_2 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} = \Delta_{13}.$$

Note that $\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} = \Delta_{13}$ is a measure zero case. Thus this is a non-generic case.

For $k_2 = \operatorname{argmin} \frac{p_1^k}{p_2^k}$, type θ_i^2 receives positive rent at $(\theta_i^2, \theta_{-i}^{k_2})$. This case is characterized by the following conditions on the primitive:

$$\Delta_{23} \geq 0,\ \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} = \Delta_{13} \geq 0,$$

$$R_2 = \Delta_{23} \geq \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2,\ R_3 = 0 \geq \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3,\ R_3 = 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}} R_2 = \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}\Delta_{23}.$$

Similarly to case 2, the following is a necessary and sufficient condition for case 8.

$$\Delta_{23} \geq 0,\ \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{23} = \Delta_{13} \geq 0,\ 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}\Delta_{23},$$

$$\Delta_{13} \min_k \left[\alpha \frac{p_2^k}{p_1^k} + (1-\alpha)\frac{p_3^k}{p_1^k}\right] + \alpha(\Delta_{21} - \Delta_{23}) + (1-\alpha)\Delta_{31} \leq 0,\ \forall \alpha \in [0,1].$$

Case 9: $\begin{tikzcd}\theta_i^3 \arrow[r] & \theta_i^2 \arrow[r, bend left] & \theta_i^1 \arrow[l, bend left]\end{tikzcd}$

Information rents (Derivation is in Appendix A.6.) are:

$$R_1 = \frac{\Delta_{12} + \frac{p_1^{\bar k}}{p_2^{\bar k}}\Delta_{21}}{1 - \frac{p_1^{\bar k}}{p_2^{\bar k}} \min_k \frac{p_2^k}{p_1^k}},\ R_2 = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \frac{p_1^{\bar k}}{p_2^{\bar k}} \min_k \frac{p_2^k}{p_1^k}},\ R_3 = \Delta_{32} + \frac{p_3^{\bar k}}{p_2^{\bar k}} \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \frac{p_1^{\bar k}}{p_2^{\bar k}} \min_k \frac{p_2^k}{p_1^k}}.$$

where $\bar k = \operatorname{argmin}_{\tilde k} \left(1 + \frac{p^{1\tilde k} + p^{3\tilde k}}{p^{2\tilde k}}\right) \Big/ \left(1 - \frac{p_1^{\tilde k}}{p_2^{\tilde k}} \min_k \frac{p_2^k}{p_1^k}\right)$. Let $k_2 := \operatorname{argmin} \frac{p_2^k}{p_1^k}$. Information rent is finite if $\bar k \neq k_1$. Type θ_i^2 receives positive rent at state $(\theta_i^2, \theta_{-i}^{\bar k})$, and type θ_i^1 at state $(\theta_i^1, \theta_{-i}^{k_2})$.

This case is characterized by the following conditions on the primitive.

$$R_1 \geq 0,\ R_2 \geq 0,\ R_3 \geq 0,\ R_1 \geq \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1,\ R_2 \geq \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2,\ R_3 \geq \Delta_{31} + R_1 \frac{p_3^{k_2}}{p_1^{k_2}}.$$

Similarly to case 2, the following is a necessary and sufficient condition for case 9.

$$\frac{\Delta_{12} + \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}}\Delta_{21}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}}\frac{p_2^{k_2}}{p_1^{k_2}}} \geq 0,\ \frac{\Delta_{21} + \frac{p_2^{k_2}}{p_1^{k_2}}\Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}}\frac{p_2^{k_2}}{p_1^{k_2}}} \geq 0,\ \Delta_{32} + \frac{p_3^{\bar{k}}}{p_2^{\bar{k}}}\frac{\Delta_{21} + \frac{p_2^{k_2}}{p_1^{k_2}}\Delta_{12}}{1 - \frac{p_1^{\bar{k}}}{p_2^{\bar{k}}}\frac{p_2^{k_2}}{p_1^{k_2}}} \geq 0,\ R_3 \geq \Delta_{31} + R_1\frac{p_3^{k_1}}{p_1^{k_1}},$$

$$\alpha(\Delta_{13} - R_1) + (1-\alpha)(\Delta_{23} - R_2) + R_3 \min_k \left[\alpha\frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0,\ \forall\alpha \in [0,1].$$

Case 10: $\theta_i^3 \longleftarrow \theta_i^2 \rightleftarrows \theta_i^1$ (non-generic case)

From $R_2 = \Delta_{21} + R_1 \min_k \frac{p_2^k}{p_1^k} = \Delta_{23}$, $R_1 = \Delta_{12} + R_2 \min_k \frac{p_1^k}{p_2^k}$, $R_3 = 0$, information rents are

$$R_1 = \frac{\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{21}}{1 - \min_k \frac{p_2^k}{p_1^k}\min_k \frac{p_1^k}{p_2^k}},\ R_2 = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \min_k \frac{p_2^k}{p_1^k}\min_k \frac{p_1^k}{p_2^k}} = \Delta_{23},\ R_3 = 0$$

This is a non-generic case because of the second equality.

Type θ_i^1 receives positive rent at state $k_1 = \operatorname{argmin} \frac{p_1^k}{p_2^k}$, and θ_i^2 at state $k_2 = \operatorname{argmin} \frac{p_2^k}{p_1^k}$. $k_1 \neq k_2$ is required for finite information rent. This case is characterized by the following.

$$R_1 \geq 0,\ R_2 \geq 0,\ R_1 \geq \Delta_{13},\ R_3 = 0 \geq \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}}R_1,\ R_3 = 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}R_2.$$

Case 11: $\theta_i^3 \rightleftarrows \theta_i^2 \rightleftarrows \theta_i^1$

Information rents are decided by:

$$\min_{R_1, R_2(k), R_3} p_1R_1 + p_2\sum_k R_2(k)p_2^k + p_3R_3 \text{ s.t. } R_1 = \Delta_{12} + \sum_k R_2(k)p_1^k,\ R_3 = \Delta_{32} + \sum_k R_2(k)p_3^k$$

$$\sum_k R_2(k)p_2^k = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k}R_1 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k}R_3.$$

By eliminating R_1 and R_3, we get

$$\min_{R_2(k)} p_1\Delta_{12} + p_3\Delta_{32} + \sum_k R_2(k)(p^{1k} + p^{2k} + p^{3k})$$

$$\text{s.t. } \sum_k R_2(k)\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k\Big) = \Delta_{21},\ \sum_k R_2(k)\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_3^{\tilde{k}}}p_3^k\Big) = \Delta_{23}$$

The solution exists unless the following two vectors are parallel, but not identical.

$$\frac{1}{\Delta_{21}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k\Big)_{1 \leq k \leq M} \quad \text{and} \quad \frac{1}{\Delta_{23}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_3^{\tilde{k}}}p_3^k\Big)_{1 \leq k \leq M}.$$

Clearly we can find a solution $\mathcal{R}_2$ such that there are at most two states k where $R_2(k) > 0$. Denote such k to be θ_{-i}^1 and θ_{-i}^3. Define

$$\underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_1^k} = 1,\ \underset{k}{\operatorname{argmin}} \frac{p_2^k}{p_3^k} = 3,\ C(2,3) = \frac{\Delta_{23} + \frac{p_2^3}{p_3^3}\Delta_{32}}{1 - \frac{p_2^3}{p_3^3}\frac{p_3^1}{p_2^1}},\ C(2,1) = \frac{\Delta_{21} + \frac{p_2^1}{p_1^1}\Delta_{12}}{1 - \frac{p_2^1}{p_1^1}\frac{p_1^3}{p_2^3}}.$$

Then information rents are:

$$\begin{aligned}
&R_2(1) = \frac{1}{p_2^1}C(2,3),\ R_2(2) = \frac{1}{p_2^3}C(2,1),\ R_2 = C(2,3) + C(2,1),\\
&R_1 = \Delta_{12} + \frac{p_1^1}{p_2^1}C(2,3) + \frac{p_1^3}{p_2^3}C(2,1),\ R_3 = \Delta_{32} + \frac{p_3^1}{p_2^1}C(2,3) + \frac{p_3^3}{p_2^3}C(2,1).
\end{aligned}$$

For the computed R_1, R_2 and R_3, the conditions characterizing this case are:

$$R_1 \geq 0,\ R_2 \geq 0,\ R_3 \geq 0,\ R_3 \geq \Delta_{31} + \frac{p_3^1}{p_1^1}R_1,\ R_1 \geq \Delta_{13} + \frac{p_1^3}{p_3^3}R_3.$$

Case 12: $\begin{tikzcd} \theta_i^3 & \theta_i^2 \arrow[l] \arrow[r, shift left] & \theta_i^1 \arrow[l, shift left] \arrow[ll, bend left] \end{tikzcd}$ (non-generic case)

From $R_1 = \Delta_{13} = \Delta_{12} + \min_k \frac{p_1^k}{p_2^k}R_2$, $R_2 = \Delta_{23} = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k}R_1$, $R_3 = 0$, we get:

$$R_1 = \Delta_{13} = \frac{\Delta_{12} + \min_k \frac{p_1^k}{p_2^k}\Delta_{21}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}},\ R_2 = \Delta_{23} = \frac{\Delta_{21} + \min_k \frac{p_2^k}{p_1^k}\Delta_{12}}{1 - \min_k \frac{p_2^k}{p_1^k} \min_k \frac{p_1^k}{p_2^k}},\ R_3 = 0.$$

Note that this is a measure zero case. Also, for the information rent to be finite, $k_2 = \operatorname{argmin} \frac{p_2^k}{p_1^k} \neq k_1 = \operatorname{argmin} \frac{p_1^k}{p_2^k}$ is required. This case is characterized by the following.

$$\Delta_{13} = \frac{\Delta_{12} + \frac{p_1^{k_1}}{p_2^{k_1}}\Delta_{21}}{1 - \frac{p_2^{k_2}}{p_1^{k_2}}\frac{p_1^{k_1}}{p_2^{k_1}}},\ \Delta_{23} = \frac{\Delta_{21} + \frac{p_2^{k_2}}{p_1^{k_2}}\Delta_{12}}{1 - \frac{p_2^{k_2}}{p_1^{k_2}}\frac{p_1^{k_1}}{p_2^{k_1}}},\ 0 \geq \Delta_{32} + \frac{p_3^{k_2}}{p_2^{k_2}}R_2,\ 0 \geq \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}}R_1,\ k_1 \neq k_2$$

Case 13: $\begin{tikzcd} \theta_i^3 \arrow[r] & \theta_i^2 \arrow[r, shift left] & \theta_i^1 \arrow[l, shift left] \arrow[ll, bend left] \end{tikzcd}$

Similarly to case 11, information rents are calculated by:

$$\begin{aligned}
\min_{R_1, R_2(k), R_3}\ p_1R_1 + p_2\sum_k R_2(k)p_2^k + p_3R_3 \ \text{ s.t. }\ R_1 = \Delta_{12} + \sum_k R_2(k)p_1^k = \Delta_{13} + \min_k \frac{p_1^k}{p_3^k}R_3,\\
\sum_k R_2(k)p_2^k = \Delta_{21} + \min_k \frac{p_2^k}{p_1^k}R_1,\ R_3 = \Delta_{32} + \sum_k R_2(k)p_3^k.
\end{aligned}$$

As in case 11, the solution exists unless the following two vectors are parallel, but not identical.

$$\frac{1}{\Delta_{21}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}} p_1^k\Big)_{1\le k\le M} \quad \text{and} \quad \frac{1}{\Delta_{23}}\Big(p_2^k - \min_{\tilde{k}} \frac{p_1^{\tilde{k}}}{p_3^{\tilde{k}}} p_3^k\Big)_{1\le k\le M}.$$

For $k_1 = \operatorname{argmin} \frac{p_2^k}{p_1^k}$ and $k_3 = \operatorname{argmin} \frac{p_3^k}{p_1^k}$, this case is characterized by:

$$R_1 \ge 0,\ R_2 \ge 0,\ R_3 \ge 0,\ R_3 \ge \Delta_{31} + \frac{p_3^{k_1}}{p_1^{k_1}} R_1,\ R_2 \ge \Delta_{23} + \frac{p_2^{k_3}}{p_3^{k_3}} R_3.$$

Case 14: $\begin{tikzcd} \theta_i^3 \arrow[r] \arrow[rr, bend right] & \theta_i^2 \arrow[r, shift left] & \theta_i^1 \arrow[l, shift left] \end{tikzcd}$

Information rents are characterized by:

$$\min_{R_1(k),R_2(k),R_3} p_1\mathcal{R}_1 \cdot \mathcal{P}_1 + p_2\mathcal{R}_2 \cdot \mathcal{P}_2 + p_3 R_3 \ s.t.\ \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1,\ \mathcal{R}_2 \cdot \mathcal{P}_2 = \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2,$$

$$R_3 = \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3 = \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.$$

Suppose $M = 2$, i.e., $\Theta_{-i} = \{\theta_{-i}^1, \theta_{-i}^2\}$. The constraints are simplified into

$$[R_1(1) - R_2(1)]p_1^1 + [R_1(2) - R_2(2)]p_1^2 = \Delta_{12},\ [R_1(1) - R_2(1)]p_2^1 + [R_1(2) - R_2(2)]p_2^2 = -\Delta_{21},$$
$$[R_1(1) - R_2(1)]p_3^1 + [R_1(2) - R_2(2)]p_3^2 = -\Delta_{31}.$$

Generically, a solution $(R_1(1) - R_2(1), R_1(2) - R_2(2))$ does not exist. However, if there are more than two states, a solution exists generically. For example, if there are three states, $\Theta_{-i} = \{\theta_{-i}^1, \theta_{-i}^2, \theta_{-i}^3\}$, the constraints are

$$A \begin{pmatrix} R_1(1) - R_2(1) \\ R_1(2) - R_2(2) \\ R_1(3) - R_2(3) \end{pmatrix} = \begin{pmatrix} \Delta_{12} \\ -\Delta_{21} \\ -\Delta_{31} \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} p_1^1 & p_1^2 & p_1^3 \\ p_2^1 & p_2^2 & p_2^3 \\ p_3^1 & p_3^2 & p_3^3 \end{pmatrix}.$$

The unique solution $(R_1(1) - R_2(1), R_1(2) - R_2(2), R_1(3) - R_2(3))$ exists generically since matrix A is invertible generically. Plugging these into the objective function, $p_1(R_1(1)p_1^1 + R_1(2)p_1^2 + R_1(3)p_1^3) + p_2(R_2(1)p_2^1 + R_2(2)p_2^2 + R_2(3)p_2^3) + p_3R_3$, we can compute information rent. (The extension to the case of $M > 3$ is straightforward, so we omit it.) Once we calculate R_3, $\mathcal{R}_1$, and $\mathcal{R}_2$, we can derive the incentive compatibility constraints characterizing this case:

$$\mathcal{R}_1 \cdot \mathcal{P}_1 \ge 0,\ \mathcal{R}_2 \cdot \mathcal{P}_2 \ge 0,\ R_3 \ge 0,\ \mathcal{R}_1 \cdot \mathcal{P}_1 \ge \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1,\ \mathcal{R}_2 \cdot \mathcal{P}_2 \ge \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2.$$

Similarly to case 2, the following is a necessary and sufficient condition for case 14.

$$
\begin{aligned}
&\mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0,\ \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0, \\
&\alpha(\Delta_{13} - R_1) + (1-\alpha)(\Delta_{23} - R_2) + R_3 \min_k \left[\alpha \frac{p_1^k}{p_3^k} + (1-\alpha)\frac{p_2^k}{p_3^k}\right] \leq 0,\ \forall \alpha \in [0,1].
\end{aligned}
$$

Case 15: $\begin{tikzcd} \theta_i^3 \arrow[r, shift left] \arrow[rr, bend right] & \theta_i^2 \arrow[l, shift left] \arrow[r, shift left] & \theta_i^1 \arrow[l, shift left] \end{tikzcd}$

Information rent is calculated by:

$$
\begin{aligned}
\min_{\mathcal{R}_1, \mathcal{R}_2, R_3} p_1 \mathcal{R}_1 \cdot \mathcal{P}_1 + p_2 \mathcal{R}_2 \cdot \mathcal{P}_2 + p_3 R_3 \ s.t.\ &\mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1, \\
&R_3 = \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3 = \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3. \\
&\mathcal{R}_2 \cdot \mathcal{P}_2 = \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2 = \Delta_{23} + \min_k \frac{p_2^k}{p_3^k} R_3.
\end{aligned}
$$

For the calculated information rent, the following characterizes case 15:

$$
\mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0,\ \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0,\ R_3 \geq 0,\ \mathcal{R}_1 \cdot \mathcal{P}_1 \geq \Delta_{13} + \frac{p_1^{\bar{k}}}{p_3^{\bar{k}}} R_3 \ \text{ where } \ \bar{k} = \operatorname{argmin} \frac{p_2^k}{p_3^k}.
$$

Note that if $M = 2$, the solution of the above program does not exist in general.[4]

Case 16: $\begin{tikzcd} \theta_i^3 \arrow[r, shift left] \arrow[rr, bend right] & \theta_i^2 \arrow[l, shift left] \arrow[r, shift left] & \theta_i^1 \arrow[l, shift left] \arrow[ll, bend right] \end{tikzcd}$

Information rents are derived by:

$$
\begin{aligned}
&\min_{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3} \ \mathcal{R}_1 \cdot \mathcal{P}_1 + \mathcal{R}_2 \cdot \mathcal{P}_2 + \mathcal{R}_3 \cdot \mathcal{P}_3 \\
&\quad s.t.\ \mathcal{R}_1 \cdot \mathcal{P}_1 = \Delta_{12} + \mathcal{R}_2 \cdot \mathcal{P}_1 = \Delta_{13} + \mathcal{R}_3 \cdot \mathcal{P}_1,\ \mathcal{R}_2 \cdot \mathcal{P}_2 = \Delta_{21} + \mathcal{R}_1 \cdot \mathcal{P}_2 = \Delta_{23} + \mathcal{R}_3 \cdot \mathcal{P}_2, \\
&\qquad \mathcal{R}_3 \cdot \mathcal{P}_3 = \Delta_{31} + \mathcal{R}_1 \cdot \mathcal{P}_3 = \Delta_{32} + \mathcal{R}_2 \cdot \mathcal{P}_3.
\end{aligned}
$$

The conditions characterizing case 16 are:

$$
\mathcal{R}_1 \cdot \mathcal{P}_1 \geq 0,\ \mathcal{R}_2 \cdot \mathcal{P}_2 \geq 0,\ \mathcal{R}_3 \cdot \mathcal{P}_3 \geq 0
$$

[4] After eliminating R_3 by replacement, and assuming $\frac{p_2^1}{p_3^1} < \frac{p_2^2}{p_3^3}$ without loss of generality, the constraints are reduced to the following if $M = 2$.

$$
p_1^1[R_1(1) - R_2(1)] + \frac{p_1^2}{p_2^2} \frac{\frac{p_1^1}{p_3^1}(\Delta_{31} - \Delta_{32}) - \Delta_{21}}{1 - \frac{p_2^1}{p_3^1}\frac{p_3^2}{p_2^2}} = \Delta_{12},\ p_2^1[R_1(1) - R_2(1)] + \frac{\frac{p_1^1}{p_3^1}(\Delta_{31} - \Delta_{32}) - \Delta_{21}}{1 - \frac{p_2^1}{p_3^1}\frac{p_3^2}{p_2^2}} = -\Delta_{21}
$$

Generically, $[R_1(1) - R_2(1)]$ cannot satisfy both of the equations.

Again, if $M = 2$, the domain of the linear program is generically empty.[5]

8 Conclusions

[To be added]

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[5] If $M = 2$, the constraints are reduced to

$$[R_1(1) - R_2(1)]p_1^1 + [R_1(2) - R_2(2)]p_1^2 = \Delta_{12},\ [R_1(1) - R_3(1)]p_1^1 + [R_1(2) - R_3(2)]p_1^2 = \Delta_{13},$$

$$[R_2(1) - R_1(1)]p_2^1 + [R_2(2) - R_1(2)]p_2^2 = \Delta_{21},\ [R_2(1) - R_3(1)]p_2^1 + [R_2(2) - R_3(2)]p_2^2 = \Delta_{23},$$

$$[R_3(1) - R_1(1)]p_3^1 + [R_3(2) - R_1(2)]p_3^2 = \Delta_{31},\ [R_3(1) - R_2(1)]p_3^1 + [R_3(2) - R_2(2)]p_3^2 = \Delta_{32}.$$

Generically, there will be no solution.

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A Appendix

A.1 Proof of Proposition 1

"If" part: For each i, let $U_i^*(\theta)$ be a solution to (9) and suppose that (11) hold. Now consider a profile of players expected surpluses, $(\hat{U}_1(\theta), U_2^*(\cdot), ..., U_n^*(\cdot))$, where $\hat{U}_1(\theta) = U_1^*(\theta) + S - \sum_{i=1}^n V_i$ for all $\theta \in \Theta$. Obviously, this profile of expected surpluses satisfies all constraints in (6)-(8).
"Only If" part: The proof is obvious, and is therefore omitted.

A.2 Proof of Proposition 2

Consider the Program $[LP_D(s_i)]$. Since its objective of is increasing in $\alpha(s_i)$, it is optimal to set $\alpha(s_i)$ at the maximum i.e., $\alpha(s_i) = \min_\theta \frac{p(\theta)}{p(\theta)-q(\theta|s_i,s_{-i}^*)}$, which gives us expression (14).

A.3 Proof of Proposition 3

At first, we establish the following Lemma:

Lemma 2 *$\gamma_i(\theta_i, \theta_i') = s_i(\theta_i, \theta_i')\alpha_i$ for an $\alpha_i \geq 0$ and $s_i(\theta_i, \theta_i') \geq 0$ such that $\sum_{\theta_i'} s_i(\theta_i, \theta_i') = 1$.*

Proof. Suppose not. Then, $\sum_{\theta_i'} \gamma_i(\theta_i, \theta_i') = f(\theta_i)$ where $f(\cdot)$ is not a constant function. Let $\alpha_i = \max_{\theta_i} f(\theta_i)$. Note that $\gamma_i(\theta_i, \theta_i)$ could be an arbitrary non-negative number since the incentive compatibility constraint associated with $\gamma_i(\theta_i, \theta_i)$ is a trivial constraint. Redefine $\gamma_i(\theta_i, \theta_i)$ as $\tilde{\gamma}_i(\theta_i, \theta_i) = \gamma_i(\theta_i, \theta_i) + (\alpha_i - f(\theta_i))$. Then we derive the condition $\sum_{\theta_i'} \gamma_i(\theta_i, \theta_i') = \alpha_i$ for all θ_i. Simply by defining $s_i(\theta_i, \theta_i') := \gamma_i(\theta_i, \theta_i')/\alpha_i$, we prove the Lemma. ∎

To prove the Proposition, first, notice that $V_i \geq V_i(s_i)$ for any $s_i \in S_i$. So it is sufficient to establish that $V_i \leq V_i(s_i)$ for some $s_i \in S_i$.

By Lemma 2, we can take $s_i(\theta_i, \theta_i') := \gamma_i(\theta_i, \theta_i')/\alpha_i$ and $\sum_{\theta_i'} s_i(\theta_i, \theta_i') = 1$. Since the domain for the above maximization problem is compact, there is $s_i \in S_i$ that maximizes the above program. For such s_i, we get

$$V_i \leq g_i(s_i)\frac{p(\theta)}{p(\theta) - q(\theta|s_i)} \quad \text{for all } \theta \text{ such that } q(\theta|s_i) < p(\theta).$$

If there is no such θ, $V_i = 0$ trivially. Take $\bar{\theta}$ such that $\bar{\theta} = \underset{\theta}{\operatorname{argmin}}\{\frac{q(\theta|s_i)}{p(\theta)}\}$. Then $V_i \leq g_i(s_i)\frac{p(\bar{\theta})}{p(\bar{\theta}) - q(\bar{\theta}|s_i)} = V_i(s_i)$.

A.4 Comparison with min cost flow problem

Consider the following operation research question. There are $|\Theta_i|$ cities. There is marginal benefit of transferring resource from city θ_i to city θ_i', denoted by $\Delta u_i(\theta_i, \theta_i')p_i(\theta_i)$. If its sign is negative, it is marginal cost. Let the flow from city θ_i to θ_i' to be $\gamma(\theta_i, \theta_i')$. Then the objective function is $\max_{\gamma(\cdot,\cdot)} \sum_{\theta_i, \theta_i'} [\Delta u_i(\theta_i, \theta_i')p_i(\theta_i)]\gamma(\theta_i, \theta_i')$. Let θ_{-i} be the variable denoting the weather distribution over the cities. Each city produces unit amount of resource. The transfer technology is not perfect: e.g., there could be leakage in transferring water, or there could be "leakage" from the underground water into the pipe. Let us say the leakage is at the rate of $1 - \frac{p(\theta_i', \theta_{-i})}{p(\theta_i, \theta_{-i})}$ when water is transferred from city θ_i' to city θ_i in weather θ_{-i}. If $1 - \frac{p(\theta_i', \theta_{-i})}{p(\theta_i, \theta_{-i})}$ is negative, it means that the water is "leaked" from the underground to the pipe during the transfer. Thus the inflow to city θ_i is $\sum_{\theta_i'} \gamma(\theta_i', \theta_i)\frac{p(\theta_i', \theta_{-i})}{p(\theta_i, \theta_{-i})}$, and the outflow is $\sum_{\theta_i'} \gamma(\theta_i, \theta_i')$. Thus the constraint city θ_i faces is

$$\underbrace{\sum \gamma(\theta_i', \theta_i)\frac{p(\theta_i', \theta_{-i})}{p(\theta_i, \theta_{-i})}}_{inflow} - \underbrace{\sum \gamma(\theta_i, \theta_i')}_{outflow} + \underbrace{1}_{production} \geq 0,$$

which is equivalent to the constraint of $[LP_D]$. Note that the planning of the transfer (the decision of $\gamma(\cdot,\cdot)$) should be arranged before knowing the weather.

Our problem is more general than min cost flow problem in the following senses: the coefficients of the objective function, $\Delta u_i(\theta_i,\theta_i')p_i(\theta_i)$ could be positive or negative depending on (θ_i,θ_i'), and there are more constraints because of "weather" θ_{-i}. Our problem is more special in the following senses: there are no capacity constraints on arcs, and the *gain/loss function* $g(\theta_i,\theta_i') := \frac{p(\theta_i',\theta_{-i})}{p(\theta_i,\theta_{-i})}$ is antisymmetric for given θ_{-i}, i.e., $g(\theta_i,\theta_i') = 1/g(\theta_i',\theta_i)$.

A.5 Proof for case 2

For the existence of $\mathcal{R}_2 \geq 0$ such that $\mathcal{P}_2 \cdot \mathcal{R}_2 = \Delta_{23},\ \ \mathcal{P}_1 \cdot \mathcal{R}_2 \leq -\Delta_{12},\ \mathcal{P}_3 \cdot \mathcal{R}_2 \leq -\Delta_{32}$, let us consider the following linear program:

$$\max_{\mathcal{R}_2 \geq 0} \mathcal{R}_2 \cdot \mathbf{0} \ \text{ s.t. } \ \mathcal{P}_1 \cdot \mathcal{R}_2 \leq -\Delta_{12},\ \mathcal{P}_3 \cdot \mathcal{R}_2 \leq -\Delta_{32},\ \ \mathcal{P}_2 \cdot \mathcal{R}_2 = \Delta_{23}. \tag{19}$$

Let ρ_1, ρ_3, and γ be the dual variables for constraints of (19). Dual linear program is:[6]

$$\min_{\rho_1 \geq 0, \rho_2 \geq 0, \gamma} (-\Delta_{12}, -\Delta_{32}, \Delta_{23}) \cdot (\rho_1, \rho_3, \gamma) \ \text{ s.t. } \ p_1^k \rho_1 + p_3^k \rho_3 + p_2^k \gamma \geq 0 \ \text{ for all } k. \tag{20}$$

Clearly, $\rho_1 = \rho_3 = \gamma = 0$ is a feasible solution, and the value of the dual linear program is zero at it. The optimal value of the primal linear program is zero, as long as the domain of the primal linear program is non-empty. Thus, $\mathcal{R}_2$ exists if and only if the optimal value of the dual linear program is zero, i.e., if and only if $\rho_1 = \rho_3 = \gamma = 0$ is an optimal solution.

Each constraint $p_1^k \rho_1 + p_3^k \rho_3 + p_2^k \gamma \geq 0$ (indexed by k) represents a half-space in three-dimensional Euclidean space $\mathbb{R}^3$ passing the origin. Let us denote each by H_k. Also $\rho_1, \rho_3 \in \mathbb{R}_+$ can be represented by half planes, $1\cdot\rho_1 + 0\cdot\rho_3 + 0\cdot\gamma \geq 0$ and $0\cdot\rho_1 + 1\cdot\rho_3 + 0\cdot\gamma \geq 0$; let the first half-space be Q_1, and the second half-space be Q_3. Intersection of half-spaces passing the origin is a convex polyhedral cone. Thus the feasibility of the dual linear program is summarized by convex polyhedral cone $\left(\bigcap_{k \in \Theta_{-i}} H_k\right) \cap Q_1 \cap Q_2$.

If the minimum is achieved at $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$, the value of the objective function (weakly) decreases by moving from $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$ to some other point in the convex cone. For $\alpha \in [0,1]$, consider the following point:

$$\left(\rho_1 = \epsilon\alpha, \rho_3 = \epsilon(1-\alpha), \gamma = -\epsilon \min_k \left[\alpha \frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right]\right).$$

[6] The first two constraints for the primal LP are inequality constraints; thus, the dual variables for the constraints will be non-negative, i.e., $\rho_1 \geq 0$ and $\rho_3 \geq 0$. The last constraint for the primal LP is equality constraint; thus, the dual variable can be negative or positive, i.e., $\gamma \in \mathbb{R}$. Also the primal LP restricts that $R_2(k)$ is non-negative; thus the each dual constraint corresponding to each $R_2(k)$ is an inequality constraint.

Point $(\rho_1 = \epsilon\alpha, \rho_3 = \epsilon(1-\alpha))$ is away from $(0,0)$ in the direction of $(\alpha, 1-\alpha)$, and γ was minimally changed so that all the constraints in (20) are still satisfied, and at least one constraint is binding.[7] For $k \in \text{argmin}\left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right]$, constraint k is still binding after this change.

From this change, the value of the objective function (see (20)) increases by

$$\epsilon\left[-\alpha\Delta_{12} - (1-\alpha)\Delta_{32} - \Delta_{23}\min_k\left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right]\right].$$

This change of the value is non-negative if and only if $(\rho_1 = 0, \rho_3 = 0, \gamma = 0)$ is the minimum of the dual linear program. Thus, the condition for the existence of $\mathcal{R}_2 \geq 0$ is:

$$\alpha\Delta_{12} + (1-\alpha)\Delta_{32} + \Delta_{23}\min_k\left[\alpha\frac{p_1^k}{p_2^k} + (1-\alpha)\frac{p_3^k}{p_2^k}\right] \leq 0, \ \forall\alpha \in [0,1].$$

(Notice that a local minimum of the dual linear program is the global minimum.)

A.6 Derivation of information rent for case 9

Binding incentive compatibility constraints were enough to characterize information rent so far. However, concern on minimization is required in this case. Also, unlike Example 1, there could be more than one state where type θ_i^2 receives positive rent as there are two types trying to imitate type θ_i^2. The following minimization problem characterizes information rent.

$$\min_{R_1, R_2(k), R_3} \ p_1R_1 + p_2\sum_k R_2(k)p_2^k + p_3R_3$$

$$\text{s.t.} \ \sum_k R_2(k)p_2^k = \Delta_{21} + R_1\min_k\frac{p_2^k}{p_1^k}, \ R_1 = \Delta_{12} + \sum_k R_2(k)p_1^k, \ R_3 = \Delta_{32} + \sum_k R_2(k)p_3^k.$$

Plugging the second and the third constraints into the first and the objective function, we get:

$$\min_{R_2(k)} p_1\Delta_{12} + p_3\Delta_{32} + \sum_k R_2(k)(p^{1k} + p^{2k} + p^{3k}) \text{ s.t. } \sum_k R_2(k)\left[p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k\right] = \Delta_{21} + \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}\Delta_{12}.$$

(Note $p^{1k} := p_1 \times p_1^k$.) The minimum is achieved when

$$R_2(k) = \left(\Delta_{21} + \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}\Delta_{12}\right)\Big/\left(p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k\right) \ \text{ if } \ k = \text{argmin}\,\frac{p^{1k} + p^{2k} + p^{3k}}{p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k}, \ 0 \ \text{ otherwise.}$$

Thus, for $k = \text{argmin}(p^{1k} + p^{2k} + p^{3k})/(p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k)$,

$$R_1 = \Delta_{12} + p_1^k\frac{\Delta_{21} + \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}\Delta_{12}}{p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k}, \ R_2 = \frac{\Delta_{21} + \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}\Delta_{12}}{1 - \frac{p_1^k}{p_2^k}\min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}}, \ R_3 = \Delta_{32} + p_3^k\frac{\Delta_{21} + \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}\Delta_{12}}{p_2^k - \min_{\tilde{k}}\frac{p_2^{\tilde{k}}}{p_1^{\tilde{k}}}p_1^k}.$$

[7] Note $\rho_1p_1^k + \rho_3p_3^k + \gamma p_2^k \geq 0 \Leftrightarrow \gamma \geq -\left[\rho_1p_1^k/p_2^k + \rho_3p_3^k/p_2^k\right]$.

A.7 Proof that B is invertible

Let $A^{(\theta_i,\theta_i')}$ be the (θ_i,θ_i')-th column of matrix A, B^θ be the θ-th column of matrix B, and $x_{B\theta}$ be the θ-th row of x_B. Since $x_N = \mathbf{0}$, (15) becomes

$$\sum_{\tilde\theta} B^{\tilde\theta} x_{B\tilde\theta} = b.$$

After increasing $\gamma_i(\theta_i^j,\theta_i^k)$ up to the maximum, we have

$$A^{(\theta_i^j,\theta_i^k)}\gamma_i(\theta_i^j,\theta_i^k) + \sum_{\tilde\theta\neq\theta} B^{\tilde\theta}\xi_{B\tilde\theta} = b$$

where ξ_B is the value changed from x_B by the change in $\gamma_i(\theta_i^j,\theta_i^k)$, and θ is such that the θ-th row of x_B hits zero the earliest.

From the two equations, we derive

$$A^{(\theta_i^j,\theta_i^k)} x_{N(\theta_i^j,\theta_i^k)} + \sum_{\tilde\theta\neq\theta} B^{\tilde\theta}(\xi_{B\tilde\theta} - x_{B\tilde\theta}) - B^\theta x_{B\theta} = 0.$$

If $\gamma_i(\theta_i^j,\theta_i^k) > 0$, $x_{B\theta} > 0$ (otherwise, $x_{B\theta}$ cannot be the row of x_B that becomes zero). Thus, $A^{(\theta_i^j,\theta_i^k)}$ is spanned by $\{B^{\tilde\theta} : \tilde\theta \in \Theta\}$, and the coefficient for B^θ is non-zero. As long as $\{B^{\tilde\theta} : \tilde\theta \in \Theta\}$ is a basis, $\{B^{\tilde\theta} : \tilde\theta \neq \theta\} \cup \{A^{(\theta_i^j,\theta_i^k)}\}$ is a basis too. Therefore, the replacement of B^θ with $A^{(\theta_i^j,\theta_i^k)}$ makes matrix B invertible as long as B was invertible before the replacement. On the other hand, if $x_{N(\theta_i^j,\theta_i^k)} = 0$, consider the situation of making $x_{N(\theta_i^j,\theta_i^k)} = \epsilon > 0$. Then $\xi_{B\theta}$ will become a strictly negative number. Thus we can show in the same way that $A^{(\theta_i^j,\theta_i^k)}$ is spanned by $(B^{\tilde\theta})_{\tilde\theta\in\Theta}$, and the coefficient for B^θ is non-zero. Thus B is again invertible after the replacement.

Since the algorithm starts with $B = I$, B remains invertible along the ongoing steps.