Stationary Markovian equilibria in Altruistic Stochastic OLG models with Limited Commitment*

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Abstract

We introduce a new class of infinite horizon altruistic stochastic OLG model with capital and labor, but without commitment between overlapping generations. Under mild regularity conditions, for economies with both bounded and unbounded state spaces, continuous monotone Markov perfect Nash equilibria (MPNE) are shown to exist, and form an antichain. For each such MPNE, we also construct corresponding stationary Markovian equilibrium invariant distributions. We then show that for many parameterizations of our economies used in applied work in macroeconomics, unique MPNE exist relative to the space of bounded measurable function. We can then directly related this result to those obtain by promised utility/continuation methods based upon the work of Abreu, Pearce, and Stacchetti. As our results are constructive, we can provide characterizations of numerical methods for approximating MPNE, and we construct error bounds. Finally, a series of examples show potential applications and limitations of our results.

1 Introduction and related literature

Over the last two decades, there has been a great deal of interest in studying dynamic equilibrium models without commitment. Examples of this work include models of sustainable plans, altruistic growth, Ricardian equivalence, endogenous borrowing constraints, sovereign debt, monetary policy games,

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savings with hyperbolic discounting, and Ramsey taxation. A central issue that has emerged in this work has been the question on developing tools appropriate for characterizing the structure of the set of subgame perfect equilibrium that arise in such economies. Many methodological proposals have been made. For example, beginning with the work of Kydland and Prescott (1977, 1980) on time consistent optimal Ramsey taxation, and continuing in many recent papers (e.g. Atkeson (1991), Chari and Kehoe (1993), Sleet (1998), and Phelan and Stacchetti (2001) among others), the "promised utility" approach to constructing subgame perfect equilibrium has been proposed. In this method, one constructs a set of sustainable values for each player in the game in a subgame perfect equilibrium by applying strategic dynamic programming arguments.[1] In related work, other have appealed to the use of traditional dynamic programming frameworks where the (dynamic) incentive constraints introduce a recursive structure of the constraint side (in addition to the objective). In some cases, a direct approach to constructing Markov perfect Nash equilibrium has been implemented (e.g, see Amir (1989, 1996b, 2002, 2005), Curtat (1996), and Nowak (2006)).[2] In other cases, where worst equilibrium are easily defined, extended set of recursive methods have been proposed (e.g. Marcet and Marimon (2009) and Rustichini (1998)). Finally, appealing to the theory of generalized functions and/or implicit function theorem, Harris and Laibson (2001) and Klein, Krusell, and Ríos-Rull (2008) have proposed a new first order theory for MPNE via an "generalized" Euler equation approach.

Although these approaches are promising, each suffer also from well-known technical limitations. For example, when using promised utility methods, it is difficult to rigorously characterize the set of equilibrium pure strategies that sustain the set of sustainable equilibrium values.[3] When applying traditional dynamic programming methods, aside from the problem of finding suitable function spaces to solve the resulting fixed point problems, issues relating the structure and/or uniqueness of stationary Markovian equilibrium arise. Further, when using the dynamic programming methods of Marcet and Marimon (2009) and Rustichini (1998), punishment schemes that sustain subgame perfect equilibrium must be imposed in an *ad hoc* manner. More troubling, for Marcet and Marimon recursive saddlepoint methods, important counterexamples exist. Finally, when applying GEE methods, the question of relating the first order theory (and the assumed

[1]Its bears mentioning that in Kydland and Prescott's original work, Markov perfect Nash equilibrium were the focus. In this later work in promised utility methods, Markov perfection was not necessarily the focus. For a interesting survey of strategic dynamic programming methods, see the work of Pearce and Stacchetti (1997) and Sleet and Yeltekin (2003).

[2]The recursive saddlepoint methods of Marcet and Marimon (2009), as well as the generalized dynamic programming methods of Rustichini (1998), can also be viewed abstractly as within this tradition.

[3]Promised utility methods generally also need discounting.

smoothness of equilibrium solutions) to the equilibrium value function that solve agents dynamic programs in the game have yet to be rigorously developed.[4] So the usefulness or applicability of these collections of methods is still yet to be firmly established.

In this paper, we propose an extension of the approach first developed in Amir (1996b) and Nowak (2006) to models of stochastic growth without commitment. The class of stochastic intergererational altruism models in their deterministic incarnation date back to the work of Phelps and Pollak (1968), and have been studied extensively in the literature (see Amir (1996b) and Nowak (2006) or references within). The model consists of a sequence of identical generations, each living one period and deriving utility from its own consumption and leisure, as well as the consumption and leisure of its successor generation. Given the lack of commitment assumed between generations, the agents in this environment face a time-consistency problem as each current generation has an incentive to deviate from a given sequence of bequests, consume a disportionate amount of current bequests, leaving little (or nothing) for subsequent generations. A key novelty of our collection of stochastic game is that unlike all this existing work on stochastic approaches to strategic altruism, we introduce paternalistic altruism over two objects (namely, the next generations consumption-leisure pair, as opposed to just consumption). This extension is important as a great deal of work in macroeconomics and dynamic public finance, for example, allow for agents preferences to be defined over consumption and leisure, and allow for elastic labor supply. Although the introduction of this feature does complicate matters a great deal relative to previous work on stochastic game approaches to economies without commitment, we are still able to obtain existence results under very general conditions, and also extend the uniqueness results known for the this class of models without elastic labor supply. Therefore, we show that by introducing a stochastic transition structure into the game's state variables, and allowing for more general forms of intergenerational altruism than previously studied, we can still obtain a very tractible model of strategic interaction among a countable collection of generations from the viewpoint of existence and computation of MPNE.

We prove many interesting results about this class of models in the paper. We first address the question of existence of MPNE. Although existence of MPNE has been studied in versions of the model with inelastic labor supply in both the deterministic (e.g., Leininger (1986), Kohlberg (1976)) and stochastic setting (Amir (1996b), Nowak (2006), Balbus, Reffett, and Woźny (2009)), these proofs do not apply the case of elastic labor supply and more

[4]For example, Klein, Krusell, and Ríos-Rull (2008) apply the implicit function theorem at the steady-state on the agents Euler equation to construct a local GEE. Unfortunately, it is not proven that on the open set near this steady state, the Euler equation is sufficient (as the equilibrium value function need not be concave). A similar issues arises in the first order theory of Harris and Laibson (2001).

general paternalistic altruism. Next, given the importance of numerical characterizations of MPNE in the existing macroeconomics literature, we next address the question of sufficient conditions under which there exist globally stable constructive iterative procedures for constructing unique MPNE. We establish existence of MPNE in the space of continuous MPNE, while we should give sufficient conditions guaranteeing uniqueness of MPNE relative to the space of a Borel measurable strategies. Therefore, under our conditions for uniqueness, our global stability result for iterative methods applies relative to any initial Borel measurable function that is pointwise feasible in the game. Additionally, we show conditions for existence of a unique corresponding nontrivial invariant distribution. These results hold for both the bounded or unbounded state space case. Finally, as our methods are constructive, we are able to provide two approximation results that allow us to construct error bounds of MPNE set, and provide a rigorous numerical approach to approximating the unique MPNE. The paper concludes with a number of examples which show both the applications and limitations of our results.

Our results are important for a numerous additional reasons. First, they provide a rigorous set of tools for quantitative study of stochastic OLG economies with limited commitment, as we can easily tie our results to the rigorous numerical characterization (including obtaining uniform error bounds) of MPNE in this class of economies. We actually provide two such procedures. Second, the methods developed in this paper are general, and, hence, can shed some new light on other dynamic economies with time consistency issues (Atkeson 1991, Phelan and Stacchetti 2001). For example, the methods can be extended to models with hyperbolic discounting (Krusell and Smith 2003, Peleg and Yaari 1973)), or more general stochastic discounted supermodular games Curtat (1996) (e.g., games of multigenerational altruism with capital accumulation, (Amir 2002) fishwars and other dynamic resource extraction problems (Levhari and Mirman 1980)), Ramsey taxation problems, among others.

From a technical perspective, we build on the approach first proposed in Balbus, Reffett, and Woźny (2009) to a similar partial commitment economies with inelastic labor supply. That is, we again build a theory of existence and computation based upon *decreasing* operators. Relative to the existence question, we extend the tools first discussed in this earlier work to a more general paternalistic altruism model. The existence result, hence, can be seen as a contribution to the literature of monotone operators similar to the one proposed (in case of elastic labor supply) by Coleman (1997) and Datta, Mirman, and Reffett (2002), but only using decreasing operators (for which the existence questions per fixed points are more complicated), instead of increasing operators (where existence can be established via various versions of Tarski's theorem. Relative to uniqueness conditions, our methods in this paper are also related to those in Balbus, Reffett, and

Woźny (2009), where the fixed point theorems used here to show uniqueness are base on geometrical properties of monotone mappings defined in abstract cones found in the work of Guo and Lakshmikantham (1988) or Guo, Cho, and Zhu (2004). But as is clear from the paper, the existence of 2-dimensional altruism greatly complicates the characterization of sufficient conditions for global stability for the case of altruism over consumption and leisure (relative to simply consumption as in this previous work).

Finally, given the two dimensional nature of the altruism in our model, it bears mentioning the discrepancies between methods developed here or in Balbus, Reffett, and Woźny (2009), as opposed to the ones applied by Amir (1996b) or Nowak (2006). Specifically the operator used in our proof of uniqueness theorem is defined on the set of bounded functions on the states spaces and assigns for any expected utility of the next generation its best response expected utility of a current generation. This operator is hence a value function operator defined on the space of functions, whose construction is motivated by the Abreu, Pearce, and Stacchetti (1990) (APS henceforth) operator that could be defined (in this example) on the set of subsets of the value functions. This is a striking difference to the "direct methods" applied by Amir (1996b) or Nowak (2006). As a result, on the one hand, methods we develop in this paper can be seen as a generalization to a two dimensional setting of an "inverse procedure" linking choice variables with their values proposed by Coleman (2000) (see also Nowak (2007)). But on the other, and more important, hand our contribution can be seen as a consistent way of sharpening the equilibrium characterization results obtained by Kydland and Prescott (1980) or other correspondence-based approaches to the study of time consistency problems. Specifically when applying APS techniques, although existence arguments can be addressed under very general conditions, the rigorous characterization of the set of dynamic equilibrium policies (either theoretically or numerically) is typically weak. Further, it has not yet been shown how to apply APS to obtain any characterization of the long-run stochastic properties of stochastic games (i.e., equilibrium invariant distributions and/or ergodic distributions). Once again value function methods proposed in this paper should be seen as a way of circumventing the mentioned APS predicaments.

The rest of the paper is organized as follows: in section 2 we present the formal model and state our assumptions, in section 3 we state our main results, and some examples of applications of our results. That last section 4 gives the proofs of all the theorems in the paper.

2 Economy and its model

We consider an infinite horizon stochastic game with a countable number of players. The model is a dynastic production economy with capital and labor, each generation living one period, caring about the consumption and leisure plans of the successor generation, but without commitment between the generations. Time is discrete and indexed by $t = 0, 1, 2, \ldots$ For simplicity, the size of each generation is assumed to be equal and normalized to unity, and there is no population growth. Apart from elastically supplying labor services $1 - l$ (where l denotes leisure), any given generation divides also its (inherited) output s between current consumption c and investment $s - c$ for the next generation. The current generation receives utility from both consumption and leisure $U(c, l)$, as well as utility from its immediate successor consumption and leisure $v(c', l')$ (where (c', l') denotes the next period consumption and leisure). There is a stochastic production technology summarized by stochastic transition Q that maps current savings, labor supply, and output $(s - c, 1 - l, s)$ into next period output s'.

Let K be a set of capital stock and $L := [0, 1]$ be a set of possible levels of labor. We will consider two cases for the capital stock, namely the unbounded case ($K := \mathbb{R}_+$) or the bounded case ($K := [0, S]$, where $S \in \mathbb{R}_{++}$). For a Markov stationary, measurable policy of the next generation for consumption and leisure, $h := (h_1, h_2)$ where $h : K \to K \times L$, the objective function for the current generation is:

$$U(c, l; h, s) := u(c, l) + \int_K v(h_1(y), h_2(y)) Q(dy | s - c, 1 - l, s).$$

Before stating our assumptions let us introduce some notation: for $s \in K$ let $A(s) := \{(c, l) \in K \times L, c \leq s\}$ and by $\mathrm{Int}(A(s))$ denote interior of that set. Let δ_0 be a probability measure on K concentrated at point zero. We now state a series of assumptions on preferences and transition.

Assumption 1 (Preferences 1) *We assume that:*

- $u : K \times L \to \mathbb{R}_+$ *is twice continuously differentiable, increasing in both arguments, supermodular and concave function,*
- u *is strictly concave and strictly increasing on* $Int(A(s))$ *for* $s > 0$,
- $v : K \times L \to \mathbb{R}_+$ *is increasing, measurable and* $\int_K v(s', 1) \lambda_k(ds'|s) < \infty$ *for all* $k = 1, \ldots, m$, *where each* λ_k *is a measure from assumptions 3,4,5.*

A special case of these preferences, we shall often consider the case of additive separability between current consumption and leisure:

Assumption 2 (Preferences 2) *given by:*

- $u(c, l) = u_1(c) + u_2(l)$, *where* u_1 *and* u_2 *are twice continuously differentiable functions, strictly increasing, strictly concave and*

- $u_1(0) = u_2(0) = 0$,
- *for* $i = 1, 2$ *the function* u_i *satisfies* $u_i'(0^+) = \infty$,
- $v : K \times L \to \mathbb{R}_+$ *is increasing, measurable and* $\int\limits_K v(s', 1)\lambda_k(ds'|s) < \infty$ *for all* $k = 1, \ldots, m$.

The model of stochastic production we adopt is a special case of that studied in a series of papers by Amir (1996a, 1996b, 1997), Nowak (2006), and Magill and Quinzii (2009). For our purposes, it proves to be convenient to adopt a simple mixing formulation for the stochastic transition structure that is similar to that studied in Nowak (2006). We simply extend that specification to the case of elastic labor supply, and the case of simple mixing between two distributions:

Assumption 3 (Transition: general case) *In the model without absorbing state, let transition*

- Q *be given by:*

  $$Q(\cdot|s - c, 1 - l, s) := g(s - c, 1 - l)\lambda_1(\cdot|s) + (1 - g(s - c, 1 - l))\lambda_2(\cdot|s),$$

  where the function $g : K \times L \to [0, 1]$ *is twice continuously differentiable, increasing in both arguments, supermodular and concave,*
- *we have* $(\forall c, l \in A(s))\, g(c, 0) = g(0, l) = 0$,
- g *is strictly concave and strictly increasing on* $Int(A(s))$ *for* $s > 0$,
- *there exists a measure* μ *such that each measure* $\lambda_k(\cdot|s)$ *has a density* $\rho_k(\cdot, s)$ *with respect to a common measure* μ, *i.e. can be described as* $\lambda_k(A|s) = \int\limits_A \rho_k(s', s)\mu(ds')$.

For some of the results in the paper, it proves useful to consider slight modifications to the assumption above on the stochastic transition Q for production. In particular, we have two modifications that we shall often use; one for the case that allows an absorbing state, the other with additive separability. These two different case are given the following two assumptions.

Assumption 4 (Transition: absorbing state) *In the model with absorbing state, let transition*

- Q *be given by:*

  $$Q(\cdot|s - c, 1 - l, s) := \sum_{i=1}^{m} g_i(s - c, 1 - l)\lambda_i(\cdot|s) + g_0(s - c, 1 - l)\delta_0(\cdot),$$

where for $i = 1, \ldots, m$ functions $g_i : K \times L \to [0,1]$ are twice continuously differentiable, increasing in both arguments, supermodular, concave and $\sum_{i=0}^{m} g_k(\cdot) = 1$.

- *for $i = 1, \ldots, m$ we have $(\forall c, l \in A(s))\, g_i(c, 0) = g_i(0, l) = 0$,*
- *for $i = 1, \ldots, m$ function g_i is strictly concave and strictly increasing on $Int(A(s))$ for $s > 0$,*
- *there exists a measure μ such that each measure $\lambda_k(\cdot|s)$ has a density $\rho_k(\cdot, s)$ with respect to a common measure μ, i.e can be described as $\lambda(A|s) = \int_A \rho_k(s', s)\mu(ds')$.*

Assumption 5 (Transition: separated variables) *In the model with absorbing state, let transition*

- *Q be given by:*

$$Q(\cdot|s-c, 1-l, s) := g(s-c, 1-l)\lambda(\cdot|s) + (1 - g(s-c, 1-l))\delta_0(\cdot),$$

  where the function $g : K \times L \to [0,1]$ is of the form $g(a,b) = g_1(a) + g_2(b)$ and each g_i is twice continuously differentiable, strictly concave and strictly increasing on K,
- *there exists a measure μ such that $\lambda(\cdot|s)$ has a density $\rho(\cdot, s)$ with respect to measure μ, i.e. can be described as $\lambda(A|s) = \int_A \rho(s', s)\mu(ds')$,*
- *moreover, the collection of the measures $\lambda(\cdot|s)$ is stochastically decreasing with s on K.*

Our assumption on preferences are standard, while assumptions on transition structure and state space require some comment. In related work by Nowak (2006) or Balbus, Reffett, and Woźny (2009), the authors assume that K is a compact interval in $\mathbb{R}_+$, while in Amir (1996b), the author takes the state space $K = \mathbb{R}_+$. In particular, to show the existence of MPNE using Amir (1996b) methods the assumption of unbounded state space is very important (while this is not the case Nowak (2006)). In this paper, we can work with either bounded and unbounded state spaces. One reason in our work we allow for both bounded and unbounded state space cases comes from an observation first made Balbus, Reffett, and Woźny (2009) concerning the nondegeneracy of stationary Markov perfect equilibrium. In particular, for the assumption of a bounded state space, existence of an absorbing state 0, strict monotonicity of g, and interiority of a MPNE implies one ends up with a positive probability of reaching an absorbing state 0 each period; hence any Stationary Markov Equilibrium admits a trivial invariant distribution. By allowing for unbounded state spaces, we can work out transitions with and without absorbing states, and obtain conditions where we can avoid this outcome. We should also remark, in the existing literature, very little attention has been focused on the structure of stationary Markov equilibrium in the class of games we study (the one exception, being Balbus, Reffett, and Woźny (2009) for the case of inelastic labor, and a more restrictive form of intergenerational altruism.

Additionally, following Nowak (2006), our transition Q is a convex combination of a finite number of measures λ_i (and in assumption 4 and 5 also δ_0) depending jointly on the state s, as well as the decision variables $s-c, l$. This type of transition structure for production has not been studied in the existing literature. In particular, the functions g_i are viewed as the "weights" placed on probability measures that govern the stochastic structure of production. In what follows, we shall analyze cases with, and cases without an absorbing state. The former case is obtained by taking one of the measures (namely δ_0)) to be a delta Dirac measure concentrated at point zero. The examples of transitions satisfying these assumptions (but without elastic labor supply) can be found a.o. in Nowak (2006). Also, it bears mentioning that our supermodularity assumptions on the primitives of preferences $u(c,l)$ and production g are critical for showing monotonicity of a best response operator in a model with an absorbing state.

To understand our assumptions in the context of the existing literature, in related work on stochastic bequest economies (with inelastic labor supply), Amir (1996b) uses a different approach to characterizing the stochastic transition Q. Apart on the assumptions of the state space K that we have already discussed, the main differences between our case (following Nowak) and Amir assumptions are the following: (i) Amir assumes that transition Q, parameterized by current decisions, is (weakly) continuous, stochastically increasing and stochastically concave, while (ii) Nowak takes Q to depend on both current decisions and current state, and lets Q be given by a convex combination of a finite number of measures, where weights are given by the production process g_i.

Therefore, on the one hand, Nowak does not require stochastic monotonicity and stochastic concavity of Q, while on the other hand, Amir do not require the particular (convex combination) structure of Q. The critical results obtained in this paper follow from the monotonicity of a best response operator, where sufficient conditions can be given for Nowak structure of Q. It is not clear how such results can be generalized to the case of Amir's stochastic transition structure on Q. To see this, think of an transition given by assumption 3 when a measures λ_1 is stochastically dominating λ_2. This can generate Amir's transitions (see his example 2 and following comments); but, unfortunately as we argue later, this set of assumptions is not sufficient to show monotonicity of the best response operator we study (and, hence, not sufficient for uniqueness using methods developed in this paper).

By D, we denote a set of all measurable pure strategies

$$D := \{h : K \to K \times L : (\forall s \in S)\, h(s) \in A(s), h \text{ is measurable}\}$$

endowed with standard pointwise (product) order $\leq$ by:

$$(\forall \xi, \eta \in D) \quad \xi \leq \eta \quad \text{iff } (\forall s \in K)\, \xi_1(s) \leq \eta_1(s) \quad \text{and} \quad \xi_2(s) \leq \eta_2(s).$$

For $h \in D$, given continuity of the primitive data of the model (i.e., utility and stochastic production), we can define the best response map $BR(h)(s)$ as follows:

$$BR(h)(s) := \arg \max_{(c,l) \in A(s)} U(c, l; h, s).$$

Then, a *Markov perfect Nash equilibrium* (MPNE) in D is any function $h^* \in D$ such that $h^* \in BR(h^*)$.

3 Existence and approximation of MPNE

We begin our analysis of MPNE in this model by considering the case of stochastic production with an absorbing state. In this setting, we prove the existence of MPNE in the set of bounded, measurable strategies D. In particular, under assumptions 1 and 4, we can write the objective for a typical generation as:

$$U(c, l; h, s) := u(c, l) + \sum_{k=1}^{m} \int_K v(h_1(y), h_2(y)) \lambda_k(dy|s) g_k(s - c, 1 - l).$$

In Theorem 3.1, we are ready to state our first major existence result under the assumption of an absorbing state for stochastic production.

Theorem 3.1 (Existence of MPNE) *Under assumptions 1 and 4 there exists a stationary perfect equilibrium. Moreover the set of MPNE in D is an anti-chain (i.e. has no ordered elements).*

The existence result in the paper like those in theorem 3.1 (and later in 3.4) are obtained under very general conditions on technology and preferences. They extend the existence results previously obtained in Amir (1996b) and Nowak (2006) to the case of altruistic stochastic growth with more general paternalistic altruism (namely, altruism defined over multi-dimension strategies by the successor generation (in our case, consumption and leisure), as well as elastic labor supply.

As for corollaries of theorem 3.1, we can provide a further characterization of the continuity and monotonicity properties of any MPNE policy in D. We first begin in corollary 1 with an additional result on the continuity properties of MPNE in the set of measurable strategies D.

Corollary 1 (Continuous MPNE) *Let assumptions 1 and 4 be satisfied. Assume additionally that*

i) there exists a μ- measurable and integrable function $\bar{\rho}$ such that $\rho_j(s', s) \leq \bar{\rho}(s')$ for each $s', s \in K$ and $j = 1, ..., m$,

ii) for each function $f : K \to K$ such that $\int_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ the integral $\int_K f(s')\rho(s', s)\mu(ds')$ is continuous as a function of s,

iii) $\int_K v(s', 1)\bar{\rho}(s')\mu(ds') < \infty$.

Then there exists a MPNE (c^*, l^*) where $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions.

Notice, unlike the results in the case of inelastic labor supply and 1 period altruism (e.g., Amir (1996b)), the result is continuous MPNE. So with elastic labor and more general intergenerational altruism, we are unable to obtain MPNE that are Lipschitzian (see discussion below). In particular, investment decisions are only continuous, but not increasing everywhere. This is also in contrast to the case of stochastic growth models with elastic labor supply and perfect commitment (where investment can be show to be increasing in the current period capital stock/output).

Finally, we can consider the monotonicity properties of MPNE (aside from simply the issues of investment noted above). In corollary 2, under an additional separability condition for stochastic production, we give monotonicity properties of MPNE in D

Corollary 2 (Monotone MPNE) *Let assumptions 1 and 5 be satisfied. Then, there exists a MPNE (c^*, l^*) with both $c^*(\cdot)$ and $l^*(\cdot)$ increasing functions.*

Combining results of theorem 3.1, corollaries 1 and 2, we obtain conditions for existence of a continuous and monotone MPNE under very general (complementarity) conditions. Observe, however, that our result provide *weaker* characterization of MPNE than obtained in Amir (1996b) and Nowak (2006) for models with inelastic labor supply. In particular, our characterization of a MPNE policies do not guarantee existence of an Lipschitz continuous MPNE. So by introducing two choice variables (consumption and leisure) into the game, we cannot guarantee this important additional characterization of MPNE. In Example 3.3 below, we shall also provide an example where MPNE exist, but are neither monotone nor Lipschitz continuous in our game. In this sense, this example shall show both that (i) Lipschitzian MPNE cannot be generally expected in this class of stochastic games, and further (ii) assumption 5 cannot be dropped in corollary 2.

To complete our characterization of MPNE in our baseline model, we now address the question of approximating MPNE in this game. In theorem 3.2, in particular, we prove an important theorem concerning an approximation of a MPNE. We do this by a simple truncation/iteration argument, and studying the structure of pointwise limits. In particular, for $n \geq 1$, and given $s \in K$, we can recursively define the sequences: $\phi_{2n}(s) = BR(\phi_{2n-1})(s)$, $\phi_{2n+1}(s) = BR(\phi_{2n})(s)$ with $(\forall s \in K)\, \phi_1(s) = (0, 0)$. Similarly we let $\psi_{2n}(s) = BR(\psi_{2n-1})(s)$, $\psi_{2n+1}(s) = BR(\psi_{2n})(s)$ with $(\forall s \in K)\, \psi_1(s) = (s, 1)$. Observe, this can be done as under our assumptions, BR is a function (see lemma 4.3).

With this notation, we present first existence of fixed edges (ϕ^d, ϕ^u) and (ψ^d, ψ^u) with a (pointwise) approximation result for a set of MPNE.

Theorem 3.2 (Approximation of MPNE set) *Under assumptions 1 and 4 the following holds:*

i) there exist limits

$$(\forall s \in K) \quad \phi^d(s) = \lim_{n\to\infty} \phi_{2n-1}(s) \text{ and } \phi^u(s) = \lim_{n\to\infty} \phi_{2n}(s), \tag{1}$$

ii) as well as

$$(\forall s \in K) \quad \psi^u(s) = \lim_{n\to\infty} \psi_{2n-1}(s) \text{ and } \psi^d(s) = \lim_{n\to\infty} \psi_{2n}(s), \tag{2}$$

iii) $\phi^u = BR\left(\phi^d\right)$, $\phi^d = BR\left(\phi^u\right)$, *and* $\psi^u = BR\left(\psi^d\right)$, $\psi^d = BR\left(\psi^u\right)$,

iv) if h^* *is a MPNE then* $(\forall s \in K) \quad \phi^d(s) \leq h^*(s) \leq \psi^u(s)$,

v) if $\phi^d(s) = \psi^u(s)$ *for all* $s \in K$ *then there is a unique MPNE* h^**. Moreover* $h^*(s) = \phi^d(s) = \psi^u(s) = \phi^u(s) = \psi^d(s)$.

An important remark on this theorem. The existence of fixed edges for iterations, as well as the limiting results in the theorem follow directly from the monotonicity of a BR operator in the model with an absorbing state (see 4.3). It then turns out one way of showing these results is to adapt the methods in (Guo, Cho, and Zhu (2004), chapter 3.2) to our problem with decreasing operators. Therefore, Theorem 3.2.(iv) states our first approximation result, i.e. pointwise bounds for a set of MPNE. Theorem 3.2.(v) then provides a type of numerical stability result for iterative methods.

To obtain a further characterization of the set of MPNE, we need more assumptions on preferences (namely separability of utility with respect to consumption and leisure), as well as an Inada type assumptions on function g. We should mention, these assumptions are often used in the applied macroeconomics literature:

Theorem 3.3 (Uniqueness of a MPNE) *Let assumption 2 and 4 with $m = 1$ be satisfied i.e.*

$$Q(\cdot|s-c, 1-l, s) := g(s-c, 1-l)\lambda(\cdot|s) + (1 - g(s-c, 1-l))\delta_0(\cdot), \tag{3}$$

and for each $(c, l) \in Int(A(s))$. Assume additionally that g is on the form $g(c, l) = g_1(c) + g_2(l)$, where both g_i are increasing, concave and twice continuously differentiable. Moreover, $g_1'(0) = g_2'(0) = \infty$. Finally assume that there exists a number $\tau \in (0, 1)$ such that $\forall (c, l) \in Int(A(s))$ (with $s > 0$) we have:

$$-\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(l)}{u_2'(l)} + \frac{g_2''(1-l)}{g_2'(1-l)}} \leq \tau. \tag{4}$$

Then there exists a unique MPNE h^* in D. Moreover let $h_0 \in D$ be an arbitrary starting point in the sequence of iterations $\varphi_{n+1} = BR(\varphi_n)$ with $\varphi_1 = h_0$ then

$$\lim_{n\to\infty} ||\varphi_n - h^*|| = 0,$$

and

$$||\varphi_n - h^*|| \leq M(1 - \tau^{r^n}),$$

where M, τ are constants dependent on a choice of h_0.

A number of important things about Theorem 3.3. First, the theorem give conditions under which there exist a unique MPNE in D. These conditions, although restrictive, are actually met often in applications. Second, combining this with corollaries 1 and 2, we obtain existence of a unique, continuous and monotone MPNE. Our method of proving the uniqueness result is bases on the uniqueness of a fixed point of a particular operator defined an a (normal and solid) cone. The result is obtained by showing that under condition (4) particular operator, corresponding to best response map, is decreasing and e-convex (in the terminology of Guo and Lakshmikantham (1988)). Hence, by applying theorem 3.2.5 in Guo, Cho, and Zhu (2004), one has a unique fixed point and convergence and approximation results in theorem 3.2.(v) follow.

Third, also observe that although in theorem 3.2, we obtain approximation results for pointwise limits, in our context, we now get uniform convergence results. The reason for this follows again from condition (4), which guarantees that our cone in not only normal but also regular (see Guo, Cho, and Zhu (2004) for discussion). Fourth, although it is not obvious to verify whether the operator used in the proof of theorem in a contraction, by a converse to the contraction mapping theorem (see e.g. Leader (1982)), one obtains this link indirectly. This argument can be made explicit using exactly the same argument in Balbus, Reffett, and Woźny (2009) adapted to our setting, and it can give additional computation procedures and uniform error bounds for a (step function) approximation relative to unique MPNE.

Finally, let us mention that the operator used in the proof of this theorem is defined on the set of bounded measurable functions on K, and assigns for any expected utility of the next generation its best response expected utility of a current generation (see proof of theorem 3.3 for the details). Hence, interesting, this operator is a operator defined on the space of *value functions*, and whose construction can be equivalent motivated by the correspondence-based strategic dynamic programming methods of Kydland and Prescott (1980) and Abreu, Pearce, and Stacchetti (1990), but only adapted to stochastic OLG models with discounting. That is, it could be defined (in this example) as a selection in strategic dynamic programming approach defined in spaces of (measurable) correspondences of continuation value functions. In this sense, we have proven that if we restrict our attention to strategic dynamic programming methods that select measurable continuation structures (ala Sleet (1998)) for our environment, this mapping would produce iterations (eventually) that are described in theorem 3.2. That is, in some cases, strategic dynamic programming methods (at least local to a greatest fixed point) can possess geometric structure. Additionally, the way we calculate strategies associated with a particular value function is based on the (generalized) inverse procedure proposed by[5] Coleman (2000). This indicates how all these methods can be unified in the context of our stochastic OLG model without commitment under additional conditions.

Remark 1 *Note that the proof and hence result of 3.3 would still hold if we replace condition of nonsingularity of second order derivatives by condition*

$$\forall (c,l) \in Int(A(s)) \quad u_1''(c) < 0 \quad \textit{and} \quad u_2''(l) < 0.$$

Finally observe that under conditions of theorem 3.3 we immediately obtain the following corollary.

Corollary 3 (Finite horizon approximation) *Consider a finite horizon version of our economy, i.e. for a finite T let $(\forall t < T)$ preferences be given by $u_1(c_t) + u_2(l_t) + (g_1(s_t - c_t) + g_2(1 - l_t)) \int_K v(c_{t+1}(y), l_{t+1}(y))\lambda(dy|s_t)$ and for the last generation by $u_1(c_T) + u_2(l_T)$. Let assumptions of theorem 3.3 be satisfied and denote by h_T^* the unique perfect equilibrium strategy of the first generation in T horizon game, then*

$$\lim_{T\to\infty} ||h_T^* - h^*|| = 0,$$

where h^* is the unique MPNE from theorem 3.3.

[5] See also Nowak (2007) section 5 for a similar argument.

We should mention additive separability in consumption and leisure is a typical assumption in applied lifecycle models. Further, in Balbus, Reffett, and Woźny (2009), authors show uniqueness condition (with inelastic labor supply), similar to our condition (4). One can note per our present uniqueness results, by multiplying the numerator and the denominator in inequality (4) by c, we obtain the corresponding "elasticities" interpretation for our uniqueness theorem. Hence, coming up with examples of where our theorems apply are quite simple. For example, consider an example which explains our arguments.

Example 3.1 *Let $u_1(c) = c^{\alpha_1}$, $u_2(l) = l^{\alpha_2}$ and $v(c,l) = c^{\beta_1} l^{\beta_2}$. We find parameters α_i,$\beta_i \in (0,1)$ such that this model satisfies condition 4:*

$$
\begin{aligned}
& -\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(c)}{u_2'(c)} + \frac{g_2''(1-l)}{g_2'(1-l)}} = \\
&= \frac{\beta_1}{1-\alpha_1 - \frac{g_1''(s-c)}{g_1'(s-c)}} + \frac{\beta_2}{1-\alpha_2 - \frac{g_2''(1-l)}{g_2'(1-l)}} \leq \\
&\leq \frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2}.
\end{aligned}
$$

Hence condition of theorem 3.3 is satisfied if $\frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2} < 1$ and g_1 and g_2 are arbitraries functions satisfying assumption 4 and conditions of theorem 3.3.

Assumption on existence of an absorbing state may be restrictive especially in when the state space K is bounded (see discussion in Balbus, Reffett, and Woźny (2009)) for this reason we state a MPNE existence result for a model without absorbing point. Under assumptions 1 and 3 for a given strategy $h \in D$ the objective becomes now:

$$
U(c,l;h,s) := u(c,l) + \beta(h,s) g(s-c, 1-l) + \gamma(h,s),
$$

with

$$
\begin{aligned}
\gamma(h,s) &:= \int_K v(h_1(y), h_2(y)) \lambda_2(dy|s) \quad \text{and} \\
\beta(h,s) &:= \int_K v(h_1(y), h_2(y)) \lambda_1(dy|s) - \int_K v(h_1(y), h_2(y)) \lambda_2(dy|s).
\end{aligned}
$$

We state the following theorem.

Theorem 3.4 (Existence of a continuous MPNE) *Under assumptions 1 and 3 there exists a MPNE. If in addition:*

i) *there exists a* μ- *measurable function* $\bar{\rho}$ *such that* $\rho_j(s', s) \leq \bar{\rho}(s')$ *for each* $s', s \in K$ *and* $j = 1, ..., m$*,*

ii) *for each function* $f : K \to K$ *such that* $\int_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ *the integral* $\int_K f(s')\rho(s', s)\mu(ds')$ *is continuous as a function*[6] *of* s*,*

iii) $\int_K v(s', 1)\bar{\rho}(s')\mu(ds') < \infty$.

Then $MPNE = (c^*, l^*)$*, where* $c^*(\cdot)$ *and* $l^*(\cdot)$ *are continuous functions.*

We continue this section by presenting two additional examples showing application for results 3.1 and 3.2. We assume however that $K = [0, S]$, where $S \in \mathbb{R}_{++}$.

Example 3.2 *In this example, assumptions 1 and 4 are satisfied. Let* $u(x, l) = \sqrt{s}\sqrt[4]{l}$ *and* $v(c, l) = \sqrt{cl}$*, and*

$$Q(\cdot|s - c, 1 - l, s) = \sqrt{s - c}\sqrt[4]{1 - l}\lambda(\cdot) + \left(1 - \sqrt{s - c}\sqrt[4]{1 - l}\right)\delta_0(\cdot),$$

where, λ *is a uniform distribution on* $[0, 1]$*. Then, we have*

$$U(c, l; h, s) = \sqrt{c}\sqrt[4]{l} + \xi(h)\sqrt{s - c}\sqrt[4]{1 - l},$$

with $\xi(h) := \int_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$*. The best response map* BR *is a well defined function, and can be described:*

$$BR(h)(s) := \left(\frac{s}{1 + \xi^4(h)}, \frac{1}{1 + \xi^4(h)}\right)$$

By theorem 3.2, we conclude that each perfect equilibrium is the limit ϕ^d *and* ψ^u*, where*

$$\phi^d(s) := \lim_{n\to\infty} \phi_{2n-1}$$

and

$$\psi^u(s) := \lim_{n\to\infty} \psi_{2n-1},$$

where, the above sequences are in the form $\phi_1(s) = (0, 0)$*, and, for* $n > 1$

$$\phi_{n+1}(s) = \left(\frac{s}{1 + \xi^4(\phi_n)}, \frac{1}{1 + \xi^4(\phi_n)}\right).$$

At the same time $\psi_1(s) = (s, 1)$ *and for* $n > 1$

$$\psi_{n+1}(s) = \left(\frac{s}{1 + \xi^4(\psi_n)}, \frac{1}{1 + \xi^4(\psi_n)}\right).$$

[6] Obviously this condition is satisfied if for all $s' \in K$ the function $\rho(s', \cdot)$ is continuous.

We can compute $\xi(\phi_n)$ by the following recursive formula: $\xi(\phi_1) = 0$; for $n > 1$, we have

$$\xi\left(\phi_{n+1}\right) = \frac{\int\limits_K \sqrt{s}ds}{1+\xi^4\left(\phi_n\right)} = \frac{\frac{2}{3}}{1+\xi^4\left(\phi_n\right)}.$$

The same recursive formula is satisfied for $\xi\left(\psi_n\right)$. Only initial value is different, i.e, $\xi\left(\psi_1\right) = \frac{2}{3}$. Note, the function $f(x) = \frac{\frac{2}{3}}{1+x^4}$ is decreasing, hence $\xi\left(\phi_n\right)$ and $\xi\left(\psi_n\right)$ have at most two cumulation points, with both cumulation points the cumulation point of

$$f(f(x)) = \frac{\frac{2}{3}}{1+\left(\frac{\frac{2}{3}}{1+x^4}\right)^4}.$$

Since $f(f(x))$ has exactly one fixed point $\xi^* \approx 0,5932$, this is also a unique fixed point of f. Hence, $\xi\left(\phi_n\right) \to \xi^*$ and $\xi\left(\psi_n\right) \to \xi^*$. Further, the functions ϕ^d and ψ^u from theorem 3.2 are equal and

$$\phi^d(s) = \psi^u(s) = \left(\frac{s}{1+(\xi^*)^4}, \frac{1}{1+(\xi^*)^4}\right) \approx \left(\frac{s}{1.12}, \frac{1}{1.12}\right) \approx (0.89\, s, 0.89).$$

Then by theorem 3.2, this strategy above is an unique MPNE. Observe, however, that the conditions of theorem 3.3 are not satisfied by this example. On the other hand, we show an application of approximation from theorem 3.2.

The next example shows that strict concavity assumptions on u and g (on the interior of their domain) are necessary for BR to be a function.

Example 3.3 *Let $u(c,l) = \sqrt{cl}$, $v(c,l) = 3\sqrt{cl}$. Transition probability is of the form*

$$Q(\cdot|s-c, 1-l, s) = \sqrt{(s-c)(1-l)}\lambda(\cdot) + \left(1 - \sqrt{(s-c)(1-l)}\right)\delta_0(\cdot),$$

where λ is a uniform distribution on $[0,1]$. Note, neither u neither g is strictly concave in the interior of any $A(s)$, since both functions are linear on the diagonals $d(s) := \{(c,l) : c = ls\}$. We have

$$U(c,l;h,s) = \sqrt{cl} + \xi(h)\sqrt{(s-c)(1-l)},$$

with $\xi(h) := 3\int\limits_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$. Then, the best response map $BR : D \to 2^D$ is a multifunction describe as:

$$BR(h)(s) = \begin{cases} \{(1,s)\} & \textit{if} \quad \xi(h) < 1, \\ \{(sl,l) : l \in [0,1]\} & \textit{if} \quad \xi(h) = 1, \\ \{(0,0)\} & \textit{if} \quad \xi(h) > 1. \end{cases} \tag{5}$$

Hence, the maximal best response (pointwise order) is

$$\overline{BR}(h)(s) = \begin{cases} (1, s) & \text{if} \quad \xi(h) \le 1, \\ (0, 0) & \text{if} \quad \xi(h) > 1. \end{cases}$$

while, the minimal best response is

$$\underline{BR}(h)(s) = \begin{cases} (1, s) & \text{if} \quad \xi(h) < 1, \\ (0, 0) & \text{if} \quad \xi(h) \ge 1. \end{cases}$$

By (5), each strategy of the form $h^*(s) = (s\, l^*(s), l^*(s))$ such that $\xi(h^*) = 1$ is a MPNE. This mean $l^* : K \to [0, 1]$ is arbitrary Borel-measurable function satisfying:

$$\xi(h) = 3 \int_S \sqrt{s'} l^*(s') ds' = 1,$$

or, equivalently,

$$\int_S \sqrt{s'} l^*(s') ds' = \frac{1}{3}.$$

Hence, there is many examples of perfect equilibria, for example: $h^1(s) = \left(\frac{s}{2}, \frac{1}{2}\right)$, $h^2(s) = \left(\frac{5}{6}s^2, \frac{5}{6}s\right)$ and $h^3(s) = \left(\frac{5}{6}s\sqrt{s(1-s)}, \frac{5}{6}\sqrt{s(1-s)}\right)$. Note that h^3 is neither increasing with respect to s, neither Lipschitz continuous. Finally, note $\phi^d(s) = (0, 0)$ and $\psi^u(s) = (s, 1)$, hence, in this case, our approximation from theorem 3.2 becomes trivial.

Theorems 3.1, 3.3 and 3.4 guarantee existence of MPNE in the models with and without absorbing state. We end up this section by analyzing a proces generated by these MPNE and find conditions for existence of a associated invariant distribution and hence Stationary Markov Equilibria.

Theorem 3.5 (Existence of a SME) *Assume 1 and 3. If additionally $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions, λ_2 does not depend on s (i.e. $\lambda_2(\cdot|s) \equiv \lambda_2(\cdot)$) and $\sup_{s \in K} g(s, 1) < 1$, then there exists a unique invariant distribution.*

To see the role of assumptions in theorem 3.5 follow the example:

Example 3.4 *Let $\lambda_1(\cdot|s) = \lambda_2(\cdot|s)$ and it is a Dirac delta in $s + 1$. Then $Q(\{s + 1\}|s) = 1$. Note that the assumption of theorem 3.5, i.e. $\lambda_2(\cdot|s)$ does not depend on s is not satisfied. It is easy to notice that the invariant distribution does not exist.*

4 Proofs

4.1 Proofs in the model with absorbing state

We start by extending to the set of strategies D to a set of randomized policies: $\mathcal{D}$, i.e. if $\bar{h} \in \mathcal{D}$ then $\bar{h}$ is a transition probability from K to $K \times L$ such that $\bar{h}\left(A(s)|s\right) = 1$. Under assumptions 1 and 4 we define $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$ in the following way

$$\mathcal{BR}\left(\bar{h}\right)(s) := \arg \max_{(c,l) \in A(s)} U(c, l; \bar{h}, s),$$

where

$$U(c, l; \bar{h}, s) := u(c, l) + \sum_{k=1}^{m} \int_K \int_{K \times L} v(c', l') \bar{h}(dc', dl'|s) \lambda_k(dy|s) g_k(s - c, 1 - l).$$

Following Balbus and Nowak (2008) we endow $\mathcal{D}$ with the weak-star topology. By a Caratheodory function $w : C \to R$ on $C := K \times A$ with $A := K \times L$ we mean a function such that $w(s, \cdot)$ is continuous on $A(s)$ for each $s \in K$, $w(\cdot, a)$ is Borel measurable for each $a \in A(s)$ and $s \to \max_{a \in A(s)} |w(s, a)|$ is μ-integrable over K. Since all the sets $A(s)$ are compact, $\mathcal{D}$ is compact and metrizable when endowed with the weak-star topology. For the details we refer the reader to Balder (1980) or Chapter IV in Warga (1972). Here, we only mention that a sequence $\bar{h}_n$ converges to $\bar{h}$ if and only if for every Caratheodory function

$$\int_K \int_{A(s)} w(s, a) \bar{h}_n(da|s) \mu(ds) \to \int_K \int_{A(s)} w(s, a) \bar{h}(da|s) \mu(ds).$$

Observe that $\mathcal{D}$ could be treated as the set of equivalence class of μ correlated strategies equal μ a.e. Note that each function $h \in D$ can be treated as a member of $\mathcal{D}$ (say $\bar{h}$) which has property $\bar{h}(s) \in \{h(s)\}$ μ a.e.

Now define an auxiliary function

$$F(c, l, \xi, s) := u(c, l) + \sum_{k=1}^{m} \xi_k g_k(s - c, 1 - l),$$

with $\xi := (\xi_1, ..., \xi_m)$. We formulate a lemma.

Lemma 4.1 *Under assumptions 1 and 4, the functions*

$$C_0(l, \xi, s) := \arg \max_{c \in [0,s]} F(c, l, \xi, s),$$

and

$$L_0(c, \xi, s) := \arg \max_{l \in [0,1]} F(c, l, \xi, s),$$

are well defined. Moreover, $c \to L_0(c, \xi, s)$ is increasing and continuous, $l \to C_0(l, \xi, s)$ is increasing and continuous.

Proof of lemma 4.1 Step 1: *Fix* $l \in L$ *and* ξ *and* $s \in K$. *Note that if* $s = 0$ *then* $C_0(l, \xi, s) = 0$ *and* $L_0(c, \xi, s) = 0$. *Assume that* $s > 0$.
Let $C_0(l) := C_0(l, \xi, s)$ *We show that* C_0 *is well defined. By assumption 4 we obtain that the function above is strictly concave on* $c \in (0, s)$ *and hence on* $c \in [0, s]$ *if* $l \in (0, 1)$. *Hence* $C_0(l)$ *is well defined on* $l \in (0, 1)$. *Suppose that* $C_0(0)$ *is not well defined. Then by concavity the function* $c \to F(c, 0; \xi, s)$ *would be constant on some interval say* $c \in (c_1, c_2)$, *where both* $c_i \in C_0(0)$. *But it is impossible since each* $g_i(s - c, 1)$ *is strictly concave in* c. *Hence* $C_0(0)$ *posses a single element. If* $l = 1$ *then*

$$F(c, 1; \xi, s) := u(c, 1).$$

Hence $C_0(1) = s$ *and* $C_0(l)$ *is well defined.*

We show that $C_0(\cdot)$ *is decreasing function. Note that for all* $s \in K$ *the set* $I(s) \times L$ *is complete lattice. Define* $\eta(c, l) = U(c, l; h, s)$.

$$\eta^{(1,2)}(c, l) = u^{(1,2)}(c, l) + \sum_{k=1}^{m} \xi_k g^{(1,2)}(s - c, 1 - l) > 0.$$

Hence η *is supermodular and by Topkis (1978) and Berge (1997) maximum theorem* C_0 *is increasing and continuous in* l.

Step 2: *Fix* $s \in K$ $c \in I(s)$. *Let*

$$L_0(c) = \arg\max_{l \in L} U(c, l, h, s).$$

We show that L_0 *is well defined function Let* $c = s$. *Then by assumptions 1 and 4 we have*

$$F(s, l, \xi, s) := u(s, l).$$

Hence $L_0(s) = 1$. *Let* $c = 0$. *We obtain that*

$$F(0, l, \xi, s) = u(0, l) + \sum_{k=1}^{m} \xi_k g^{(1,2)}(s, 1 - l) > 0.$$

Since the function above is strictly concave, hence $L_0(0)$ *is well defined as well. If* $c \in (0, s)$, *by assumption 1 and 4 we know that* $\zeta(c, l) := U(c, l; h, s)$ *is strictly concave. Hence* $L_0(c)$ *is also well defined in this case. Moreover by assumptions 1 and 4 we have*

$$\zeta^{(1,2)}(c) := u^{(1,2)}(c, l) + \sum_{k=1}^{m} \xi_k g^{(1,2)}(s - c, 1 - l) \geq 0.$$

Hence ζ *is supermodular on the complete lattice* $I(s) \times L$. *Hence by Topkis (1978) and Berge (1997) maximum theorem* $L_0(\cdot)$ *is increasing and continuous in* c.

Lemma 4.2 *The functions $\xi \to C_0(l, \xi, s)$ and $\xi \to L_0(c, \xi, s)$ are decreasing in product order sense.*

Proof of lemma 4.2 *Note that if at least one $\xi_k > 0$, then by assumption 1 and 4 we know that both functions $c \to F(c, l, \xi, s)$ and $l \to F(c, l, \xi, s)$ are strictly concave. Let $\xi^1 \leq \xi^2$ in product order sense. Then by assumption 4 we have*

$$
\begin{aligned}
F^{(1)}\left(c, l, \xi^1, s\right) &= u^{(1)}(c, l) - \sum_{k=1}^{m} \xi_k^1 g_k^{(1)}(s - c, 1 - l) \\
&\geq u^{(1)}(c, l) - \sum_{k=1}^{m} \xi_k^2 g_k^{(1)}(s - c, 1 - l) \\
&= F^{(1)}\left(c, l, \xi^2, s\right).
\end{aligned}
$$

Hence we easily conclude that $C_0(l, \xi^1, s) \geq C_0(l, \xi^2, s)$. Similarly we prove that $\xi \to L_0(c, \xi, s)$ is decreasing.

Lemma 4.3 *For each $h \in D$ function $BR(h)$ is well defined. Moreover BR is decreasing.*

Proof of lemma 4.3 Step 1: *Without loss of generality assume $s > 0$. Note that by assumption 1 function $(c, l) \to U(c, l; h, s)$ is strictly concave. Hence $BR(h)$ is well defined function $BR : D \to D$.*

Step 2: *We show monotonicity of BR. Note that $BR(h)(s) = (s, 1)$ if*

$$
U^{(1)}(c, l; h, s) \geq 0 \quad \text{and} \quad U^{(2)}(c, l; h, s) \geq 0,
$$

with strict inequality in at least one position of this system above. Next $BR(h)(s) = (0, 0)$ if

$$
U^{(1)}(c, l; h, s) \leq 0 \quad \text{and} \quad U^{(2)}(c, l; h, s) \leq 0,
$$

with strict inequality in at least one position of this system above. If the system has a solution, then $BR(h)(s)$ solves it. Note that if $BR(h)(s) = (c^h, l^h)$ then

$$
c^h = C_0(l^h, \xi(h, s), s) \quad \text{and} \quad l^h = L_0(c^h, \xi(h, s), s),
$$

where $\xi_k(h, s) := \int_S v(h_1(s'), h_2(s'))\lambda_k(ds'|s)$ and $\xi = (\xi_1, ..., \xi_m)$. Now we show that (c^h, l^h) is decreasing in h. By lemma 4.2 both functions $\xi \to C_0(l, \xi, s)$ and $\xi \to L_0(c, \xi, s)$ are decreasing. Define $\kappa_1(l, h, s) := C_0(l, \xi(h, s), s)$ and $\kappa_2(c, h, s) := L_0(c, \xi(h, s), s)$. Clearly for $i = 1, 2$ the function $h \to \kappa_i(c, h, s)$ is decreasing by definition of κ_i and $\xi(h, s)$ and *lemma 4.2. Let p and $r \in D$ and $p \leq r$. Then $\xi(p,s) \leq \xi(r,s)$. Note that if $(c^r, l^r) = (0,0)$ then clearly $(0,0) = (c^r, l^r) \leq (c^p, l^p)$. Let $(c^r, l^r) = (s,1)$. Then for $i = 1,2$ and each (c,l) we have*

$$0 \leq u^{(i)}(c,l) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(i)}(s-c, 1-l) \leq u^{(i)}(c,l) - \sum_{k=1}^{m} \xi_k(p,s) g_k^{(i)}(s-c, 1-l).$$

with at least strict inequality above. Hence we also have $(c^p, l^p) = (s,1)$. Hence desired inequality holds also in this case. Suppose for $i = 1,2$ we have

$$u^{(i)}(c^r, l^r) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(i)}(s - c^r, 1 - l^r) = 0$$

and

$$u^{(i)}(c^p, l^p) - \sum_{k=1}^{m} \xi_k(p,s) g_k^{(i)}(s - c^p, 1 - l^p) = 0.$$

We obtain

$$\begin{aligned} 0 &= u^{(1)}(c^p, l^p) - \sum_{k=1}^{m} \xi_k(p,s) g_k^{(1)}(s - c^p, 1 - l^p) && (6) \\ &\geq u^{(1)}(c^p, \kappa_2(c^p, p, s)) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(1)}(s - c^p, 1 - \kappa_2(c^p, p, s)) \\ &\geq u^{(1)}(c^p, \kappa_2(c^p, r, s)) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(1)}(s - c^p, 1 - \kappa_2(c^p, r, s)). && (7) \end{aligned}$$

The last inequality follows from properties of κ_2, supermodularity of u and each g_k on all sets $A(s)$, and lemma 4.2. Note that by definition of κ_2 c^r is a zero element of the function

$$f_0(c) := u^{(1)}(c, \kappa_2(c,r,s)) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(1)}(s - c, 1 - \kappa_2(c,r,s)).$$

We claim that c^r is unique solution of equation above. On the contrary suppose $f_0(c') = 0$ and $c' \neq c^r$. Then $(c', \kappa_2(c',r,s))$ holds

$$u^{(1)}(c', \kappa_2(c',r,s)) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(1)}(s - c', 1 - \kappa_2(c',r,s)) = 0.$$

By definition of $\kappa_2(c',r,s)$ at the same time we have

$$u^{(2)}(c', \kappa_2(c',r,s)) - \sum_{k=1}^{m} \xi_k(r,s) g_k^{(2)}(s - c', 1 - \kappa_2(c',r,s)) = 0,$$

and hence by assumption 1 we obtain that $(c', \kappa_2(c', r, s))$ *is argument maximizing* $U(c, l; r, s)$ *over* $(c, l) \in A(s)$. *But* (c^r, l^r) *also maximize* $U(c, l; r, s)$ *over* $(c, l) \in A(s)$. *It is contradiction because* $(c, l) \to U(c, l; r, s)$ *is strictly concave in the interior of* $A(s)$. *Hence* $f_0(c) = 0$ *has unique solution* c^r. *Note that by 7* $f_0(c^p) < 0$. *Moreover, by strict concavity of* $(c, l) \to F(c, l; \xi, s)$ *for each* $(c, l) \in A(s)$ *we have*

$$\begin{aligned} 0 &\leq F(c^r, l^r; \xi(r, s), s) - F(c, l; \xi(r, s), s) \\ &< F^{(1)}(c, l; \xi(r, s), s)(c^r - c) + F^{(2)}(c, l; \xi(r, s), s)(l^r - l). \end{aligned} \tag{8}$$

If we put to 8 $c := c^p$ *and* $l := \kappa_2(c^p, r, s)$ *we obtain*

$$F^{(2)}(c^p, \kappa_2(c^p, r, s); \xi(r, s), s) = 0 \text{ and } F^{(1)}(c^r, \kappa_2(c^p, r, s); \xi(r, s), s) = f_0(c^p). \tag{9}$$

Therefore $0 \leq f_0(c^p)(c^r - c^p)$. *Since* $f_0(c^p) < 0$, *hence* $c^p > c^r$. *Hence, by definition of* κ_2 *and lemmas 4.1 and 4.2 we conclude* $l^p = \kappa_2(c^p, p, s) \geq \kappa_2(c_r, r, s) = l^r$, *which means that in product order sens* $BR(p)(s) = (c^p, l^p) \geq (c^r, l^r) = BR(r)(s)$.

Lemma 4.4 *BR is a continuous function i.e. if* $h_n \to h$ *pointwise, then* $BR(h_n) \to BR(h)$ *pointwise as well.*

Proof of lemma 4.4 *Since* $(c, l) \to U(c, l; h, s)$ *is concave it is sufficient to show that* $h \to U(c, l; h, s)$ *is continuous. Note that*

$$U(c, l; h, s) = F(c, l, \xi(h, s), s).$$

Clearly F *is continuous. It is sufficient to show that* $\xi(\cdot, s)$ *is continuous in the pointwise topology. If* $h_n \to h$ *pointwise we obtain* $\xi(h_n, s) \to \xi(h, s)$ *by assumption 1 and Lebesgue Dominance theorem. Hence* $U(c, l; h, s)$ *is continuous in* h *as superposition of continuous functions. By strict concavity of* $U(\cdot, \cdot; h, s)$ *the proof is complete.*

Proof of theorem 3.1: First we show that $\mathcal{BR}$ is well defined function mapping $\mathcal{D}$ to $\mathcal{D}$. Note that

$$U(c, l; \bar{h}, s) = F(c, l; \xi(\bar{h}, s), s),$$

where

$$\begin{aligned} \xi_k(\bar{h}, s) &:= \int_K \int_{A(s)} v(c', l')\bar{h}(dc', dl'|s')\lambda_k(ds'|s) \\ &= \int_K \int_{A(s)} v(c', l')\rho_k(s', s)\bar{h}(dc', dl'|s')\mu(ds'). \end{aligned}$$

Note that by definition of U, assumption 1 we immediately obtain that U is strictly concave in $(c, l) \in A(s)$. Hence there is unique optimal solution of maximization problem of $U(c, l; \bar{h}, s)$. Hence we have shown that $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$. Moreover, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D}) \subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n \to \bar{h}$ in the weak star topology. Note that if $a = (c, l)$ then for each $s \in K$ the function

$$w_k(s', a) := v(a)\rho_k(s', s)$$

is a Caratheodory function. Hence

$$\xi_k(\bar{h}_n, s) \to \xi_k(\bar{h}, s) \quad \text{as } n \to \infty,$$

and $U(c, l; \bar{h}_n, s) \to U(c, l; \bar{h}, s)$. Since for each $\bar{h}$ and s the function $(c, l) \to U(c, l; \bar{h}, s)$ is strictly concave in the interior of its domain, hence optimal solution of $U(c, l; \bar{h}_n, s)$ must converge to the optimal solution of $U(c, l; \bar{h}, s)$. Hence there $\mathcal{BR}(\bar{h}_n) \to \mathcal{BR}(\bar{h})$ pointwise and $\mathcal{BR}$ is continuous. Hence by Schauder-Tikhonov theorem we conclude that there exists fixed point $h^* = \mathcal{BR}(h^*)$ μ a.e. Let $h^o := \mathcal{BR}(h^*)$. Since $\mathcal{BR} : \mathcal{D} \to D$, hence h^o must be a stationary strategy. Since $h^o = h^*$ μ a.e. by definition of the function $\xi(h, s)$ we conclude that $\xi(h^*, s) = \xi(h^o, s)$ for each $s \in K$ and hence for each (c, l) we have $U(c, l; h^o, s) = U(c, l; h^*, l)$. Hence $h^o = h^*$ for each $s \in K$ and $h^*(s) = \mathcal{BR}(h^*)(s)$ for each $s \in K$. ∎

Proof of corollary 1: We show that a stationary MPNE $(c^*(s), l^*(s))$ is a continuous function. It is sufficient to show that BR maps D into the set of bounded, continuous functions on K. Fix $h \in D$. Let $s_n \to s_0$ as n tends to ∞. By condition (iii) of this corrolary we have

$$\int_K v(h_1(s'), h_2(s'))\rho_j(s', s_n)\mu(ds') \quad \leq \quad \int_K v(s', 1)\bar{\rho}_j(s')\mu(ds'). \qquad (10)$$

By condition (iii) we obtain $\int_K v(s', 1)\bar{\rho}_j(s')\mu(ds') < \infty$. Hence, by (ii) and Lebesgue Dominance theorem we immediately obtain

$$\int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_n) \to \int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_0). \qquad (11)$$

Let $(c_n, l_n) := BR(h)(s_n)$ and (c_0, l_0) be an arbitrary cummulation point of (c_n, l_n). We have

$$U(c_n, l_n; h, s) \geq U(c, l; h, s)$$

for all $(c,l) \in A(s)$ and $n \in N$. Hence and by (11) we immediately obtain

$$U(c_0, l_0; h, s) \geq U(c, l; h, s),$$

for all $(c,l) \in A(s)$. Hence $(c_0, l_0) := BR(h)(s_0)$. Hence BR maps D into set of continuous functions. Therefore $(c^*(\cdot), l^*(\cdot))$ must be a continuous function. ■

Proof of corollary 2: Note that the utility has now a form

$$U(c,l;h,s) = u(c,l) + g_1(s-c) \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)$$

$$+g_2(1-l) \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s).$$

Fix arbitrary $s > 0$, and $h \in D$. If $\int_K v(h_1(s'), h_2(s'))\lambda(ds'|s) > 0$, then by assumption 1 and 5 this function above is strictly concave on $A(s)$, and hence maximization problem of $(c,l) \to U(c,l;h,s)$ has unique solution. If $\int_K v(h_1(s'), h_2(s'))\lambda(ds'|s) = 0$, then only $(s,1)$ is optimal solution. Hence best response map BR is well defined function. Further if we follow the reasoning from theorem 3.4 we obtain that BR has a fixed point.

Finally observe that by assumptions for each $h \in D$ where h is increasing, function $U(\cdot,\cdot,;h,\cdot)$ is supermodular in (c,l) and has increasing differences in (c,s) and (l,s). Hence by Topkis (1978) theorem for each increasing $h \in D$ function $BR(h)(\cdot)$ in increasing on K. Hence BR maps increasing (bounded, measurable) functions into increasing (bounded, measurable) functions and a fixed point of BR, i.e. a MPNE is increasing on K. ■

Proof of theorem 3.2: *Step 1.* We prove (i). We show that ϕ_{2n-1} is increasing and ϕ_{2n} is decreasing. Clearly $\phi_1 \leq \phi_3$ and $\phi_1 \leq \phi_2$. By lemma 4.3 and definition of sequence ϕ_n we obtain

$$\phi_2 = BR(\phi_1) \geq BR(\phi_3) = \phi_4.$$

Suppose that for some n hold $\phi_{2n} \geq \phi_{2(n+1)}$ and $\phi_{2n-1} \leq \phi_{2n+1}$. By lemma 4.3 and definition of sequence ϕ_n we obtain

$$\phi_{2n+1} = BR(\phi_{2n}) \leq BR(\phi_{2n+2}) = \phi_{2n+3}.$$

Therefore

$$\phi_{2(n+2)} = BR(\phi_{2n+3}) \leq BR(\phi_{2n+1}) = \phi_{2(n+1)}.$$

Finally we obtain that both sequences ϕ_{2n} and ϕ_{2n-1} are monotone and bounded. Hence there exist limits in (1).

Step 2. We prove (ii). We show that ψ_{2n-1} is decreasing and ψ_{2n} is increasing. Clearly $\psi_1 \geq \psi_3$ and $\psi_1 \leq \psi_2$. By lemma 4.3 and definition of sequence ψ_n we obtain

$$\psi_2 = BR(\psi_1) \leq BR(\psi_3) = \psi_4.$$

Suppose that for some n hold $\psi_{2n} \leq \psi_{2(n+1)}$ and $\psi_{2n-1} \geq \psi_{2n+1}$. By lemma 4.3 and definition of sequence ψ_n we obtain

$$\psi_{2n+1} = BR\left(\psi_{2n}\right) \geq BR\left(\psi_{2n+2}\right) = \psi_{2n+3}.$$

Therefore

$$\psi_{2(n+2)} = BR\left(\psi_{2n+3}\right) \geq BR\left(\psi_{2n+1}\right) = \psi_{2(n+1)}.$$

Finally we obtain that both sequences ψ_{2n} and ψ_{2n-1} are monotone and bounded. Hence their limits exists.

Step 3. We prove (iii). Note that $\phi_{2n+1} = BR(\phi_{2n})$. By Step 1 and by lemma 4.4 we obtain $\phi^u = BR\left(\phi^d\right)$. By analogous reasoning we obtain the rest of results.

Step 4. We prove (iv). By definition of h^* we know that $h^* = BR\left(h^*\right)$. By definition of BR and ϕ_n and ψ_n we immediately obtain

$$\phi_1 \leq BR(h^*) = h^* \leq \psi_1. \tag{12}$$

Assume that for some n holds

$$\phi_{2n-1} \leq h^* \leq \psi_{2n-1}. \tag{13}$$

Note that

$$\phi_{2n+1} = BR\left(BR(\phi_{2n-1})\right) \quad \text{and} \quad \psi_{2n+1} = BR\left(BR(\psi_{2n-1})\right). \tag{14}$$

Moreover, by (12) and lemma (4.3) we obtain

$$BR\left(BR(h^*)\right) = h^* \tag{15}$$

Observe that by lemma 4.3 function $BR \circ BR(\cdot)$ is increasing. Hence, combining (12), (13), (14) we obtain

$$\begin{aligned}
\psi_{2n+1} &= BR\left(BR(\psi_{2n-1})\right) \geq BR\left(BR(h^*)\right), \\
&= h^*, \\
&= BR\left(BR(h^*)\right) \geq BR\left(BR(\phi_{2n-1})\right), \\
&= \phi_{2n+3}.
\end{aligned} \tag{16}$$

To finish the proof we just take a limit in (16).

Step 5. Proof of (v) is immediate from theorem 3.1 and from (iv). ■

4.2 Proofs in the model with separated variables utility and absorbing state

By assumptions 2 and assumption of theorem 3.3 the objective becomes:

$$U(c,l;h,s) := u_1(c) + u_2(l) + \xi(h,s)(g_1(s-c) + g_2(1-l)),$$

with $\xi(h,s) := \int_K v(h_1(y), h_2(y))\lambda(dy|s)$.

Proof of theorem 3.3: Let $\mathcal{P}$ be a set of bounded, Borel measurable functions $p : K \to R_+$with a pointwise partial order and the sup norm. Clearly $\mathcal{P}$ is a normal solid cone. Define an operator $T : \mathcal{P} \to \mathcal{P}$:

$$T(p)(s) = \int_K v(c_p(s'), l_p(s'))\lambda(ds'|s),$$

where $(c_p(s), l_p(s))$ is a measurable solution (refer to Brown and Purves (1973) theorem 2) of optimization problem of the function

$$H(c,l;p,s) := u_1(c) + u_2(l) + p(s)(g_1(s-c) + g_2(1-l)).$$

Clearly H has decreasing differences in (c,l) and p, hence by Topkis (1978) theorem (c^p, l^p) is decreasing. By assumption 1 $T(\cdot)$ is decreasing.

Now we show that the function $J(t) := t^\tau T(tp)$ $t \in (0,1)$ is increasing for each $p \in Int\,(\mathcal{P})$ and τ from 4. Showing continuity of J at t=1 we obtain that $T(tp) \le t^{-\tau}T(p)$, i.e. the e-convexity condition for T in theorem 3.2.5 of Guo, Cho, and Zhu (2004). Fix p from interior of $\mathcal{P}$ and $s \in K \setminus \{0\}$. Define $c(t) := c_{tp}$ and $l(t) := l_{tp}$.

Step 1. First note that by $u_i'(0^+) = g_i'(0^+) = \infty$ $(i = 1,2)$ for $t \in (0,1)$ we have $(c(t), l(t)) \in Int(A(s))$.

Hence the equalities are satisfied

$$H^{(1)}(c(t), l(t); tp(s), s) = u_1'(c(t)) - tp(s)g_1'(s - c(t)) = 0$$

and

$$H^{(2)}(c(t), l(t); tp(s), s) = u_2'(l(t)) - tp(s)g_2'(1 - l(t)) = 0.$$

By implicit function theorem both $c(t)$ and $l(t)$ are differentiable and:

$$-c'(t) = \frac{p(s)g_1'(s - c(t)}{-\left(u_1''(c(t)) + tp(s)g_1''(s - c(t))\right)},$$

and

$$-l'(t) = \frac{p(s)g_2'(1 - l(t))}{-\left(u_2''(l) + tp(s)g_2''(1 - l(t))\right)}.$$

Then we have

$$
\begin{aligned}
-\frac{d}{dt}v(c(t),l(t)) &= -v^{(1)}(c(t),l(t))c'(t)-v^{(2)}(c(t),l(t))l'(t), \\
&= v^{(1)}(c(t),l(t))\frac{p(s)g_1'(s-c(t))}{-\left(u_1''(c(t))+tp(s)g_1''(s-c(t))\right)}, \\
&+ v^{(2)}(c(t),l(t))\frac{p(s)g_2'(1-l(t))}{-\left(u_2''(l(t))+g_2''(1-l(t))\right)}, \\
&= \frac{1}{t}\frac{\frac{v^{(1)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_1''(c(t))}{u_1'(c(t))}+\frac{g_1''(s-c(t))}{g_1'(s-c(t))}}v(c(t),l(t)), \\
&+ \frac{1}{t}\frac{\frac{v^{(2)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_2''(l(t))}{u_2'(l(t))}+\frac{g_2''(1-l(t))}{g_2'(1-l(t))}}v(c(t),l(t)), \\
&\leq \frac{\tau}{t}v(c(t),l(t)) \leq \frac{\tau}{t}v(s,1). \qquad (17)
\end{aligned}
$$

The last inequality follows directly from 4. Hence if $c(t,s):=c(t)$ then the derivative of $v(c(t,s),l(t,s))$ is integrable with respect to probabilistic measure $\lambda(\cdot|s)$ since $v(s,1)$ is integrable by assumption 2. Hence by (17) and (4) we obtain:

$$
\begin{aligned}
t\frac{\partial}{\partial t}\left(\int_K v(c(t,s'),l(t,s'))\lambda(ds'|s)\right) &= t\int_K \frac{\partial}{\partial t}v(c(t,s'),l(t,s'))\lambda(ds'|s) \\
&\geq -\tau\int_K v(c(t,s),l(t,s))\lambda(ds'|s) \\
&= -\tau T(tp).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J'(t) &= t^{\tau-1}\left(\tau T(tp)+t\left(\int_K \frac{\partial}{\partial t}v(c(t,s'),l(t,s'))\lambda(ds'|s)\right)\right) \\
&\geq t^{\tau-1}\left(\tau T(tp)-\tau T(tp)\right) \\
&= 0.
\end{aligned}
$$

Hence $J(t)$ is increasing on $(0,1)$. Since J is continuous, J is decreasing on all $[0,1]$. Hence by Guo, Cho, and Zhu (2004) we obtain that T posses unique fixed point say p^*. Moreover, each sequence of iterations $p_{n+1}=T(p_n)$ (with p_0 arbitrary starting point) converges to p^*. Hence there exists unique $h^*:=(c^*,l^*)$ such that $p^*(s)=\int_K v(c^*(s'^*(s'))\lambda(ds'|s)$, where the pair $(c^*(s), l^*(s))$ solves optimization problem of the function $(c, l) \to H(c, l; p^*, s)$. Moreover, (c^*, l^*) is a unique perfect equilibrium. Obviously $\varphi_n \to h^*$ (refer to Kall (1986) theorem 2). ■

4.3 Proofs in the model without absorbing state

We now turn to a transition without an absorbing state (see assumption 3). By assumptions 1 and 3 the objective becomes:

$$U(c, l; h, s) := u(c, l) + \beta(h, s)g(s - c, 1 - l) + \gamma(h, s),$$

with $\beta(h, s) := \int_K v(h_1(y), h_2(y))\lambda_1(dy|s) - \int_K v(h_1(y), h_2(y))\lambda_2(dy|s)$, and $\gamma(h, s) := \int_K v(h_1(y), h_2(y))\lambda_2(dy|s)$.

Define $G(c, l; \beta, \gamma, s) := u(c, l) + \beta g(s - c, 1 - l) + \gamma$ with $\beta \in R$ and $\gamma \in R_+$. We start with some preliminary lemmas.

Lemma 4.5 *For each $\beta \in R$, $\gamma \in R_+$ and $s \in K$ the function $(c, l) \to G(c, l; \beta, \gamma, s)$ has a unique maximum.*

Proof of lemma 4.5 *Since $G(\cdot, \cdot; \beta, \gamma, s)$ is continuous on $A(s)$, hence the set of maximization problem must be nonempty. We show that optimal solution is unique. If $\beta \geq 0$ then we obtain uniqueness of optimal solution, since in this case $G(\cdot, \cdot; \beta, \gamma, s)$ is strictly concave. If $\beta < 0$ by assumption 3 we obtain unique solution as well. Moreover it is $(s, 1)$.*

Lemma 4.6 *Let $\beta_n \to \beta$ and $\gamma_n \to \gamma$. Let $h^n = \arg\max_{(c,l)\in A(s)} G(c, l; \beta_n, \gamma_n, s)$ and $h = \arg\max_{(c,l)\in A(s)} G(c, l; \beta, \gamma, s)$. Then $h^n \to h$.*

Proof of lemma 4.6 *Since $A(s)$ is compact from each sequence contained in $A(s)$ we can choose convergent subsequence. Hence without loss of generality we can assume that $h^n \to h^o$ where h^o is some point in $A(s)$. Then for arbitrary $(c, l) \in A(s)$ we have*

$$G(h_1^n, h_2^n; \beta_n, \gamma_n, s) \geq G(c, l; \beta_n, \gamma_n, s).$$

Taking a limit as $n \to \infty$ we obtain

$$G(h_1^o, h_2^o; \beta, \gamma, s) \geq G(c, l; \beta, \gamma, s).$$

Hence $h^o = \arg\max_{(c,l)\in A(s)} G(c, l; \beta, \gamma, s)$. By definition of h we obtain that $h = h^o$.

Proof of theorem 3.4: By lemma 4.5 and assumption 1 we immediately obtain that there is unique optimal solution of maximization problem of

$U(c,l;\bar{h},s)$. Hence we have shown that $\mathcal{BR}:\mathcal{D}\to\mathcal{D}$. Moreover, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D})\subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n\to\bar{h}$ in the weak star topology. Note that if $a=(c,l)$ then for each $s\in K$ the function

$$w_k(s',a):=v(a)\rho_k(s',s),$$

is a Caratheodory function. Hence

$$\beta(\bar{h}_n,s)\to\beta(\bar{h},s)\quad \text{as } n\to\infty,$$

and

$$\gamma(\bar{h}_n,s)\to\gamma(\bar{h},s)\quad \text{as } n\to\infty,$$

and hence $U(c,l;\bar{h}_n,s)\to U(c,l;\bar{h},s)$. By lemma 4.6 optimal solution of $U(c,l;\bar{h}_n,s)$ must converge to the optimal solution of $U(c,l;\bar{h},s)$. Hence there $\mathcal{BR}(\bar{h}_n)\to\mathcal{BR}(\bar{h})$ pointwise and hence in the weak star topology. Hence $\mathcal{BR}$ is continuous. Therefore by Schauder-Tikhonov theorem we conclude that there exists a fixed point $h^*=\mathcal{BR}(h^*)$ μ a.e. Let $h^o:=\mathcal{BR}(h^*)$ pointwise. Since $\mathcal{BR}:\mathcal{D}\to D$, hence h^o must be a stationary strategy. Since $h^o=h^*$ μ a.e. by definition of the functions $\beta(h,s)$ and $\gamma(h,s)$ we conclude that $\beta(h^*,s)=\beta(h^o,s)$ and $\gamma(h^*,s)=\gamma(h^o,s)$ for each $s\in K$ and hence for each (c,l) we have $U(c,l;h^o,s)=U(c,l;h^*,l)$. Hence $h^o=h^*$ for each $s\in K$ and $h^*(s)=\mathcal{BR}(h^*)(s)$ for each $s\in K$. Finally to show continuity of a MPNE follow the reasoning in the proof of corollary 1. ■

Proof of theorem 3.5: Let (c^*,l^*) be given MPE. For a transition probability $Q(\cdot|s-c^*(s),1-l^*(s),s)$ let us define a corresponding Markov operator $H:\mathcal{C}(K)\to\mathcal{C}(K)$as

$$H(f)(s):=g(s-c^*(s),1-l^*(s))\int\limits_K f(s')\lambda_1(ds'|s)$$

$$+(1-g(s-c^*(s),1-l^*(s)))\int\limits_K f(s')\lambda_2(ds').$$

Observe that operator H is stable hence $Q(\cdot|s-c^*(s),1-l^*(s),s)$ has a Feller property. We now show that H is also quasi-compact[7]. To see that let us also define an operator L:

$$L(f)(s):=(1-g(s-c^*(s),1-l^*(s)))\int\limits_K f(s')\lambda_2(ds').$$

[7]Endow $\mathcal{C}(K)$ with the sup norm. An operator $H:\mathcal{C}(K)\to\mathcal{C}(K)$ is said to be quasi-compact if there exists a natural number n and a compact operator L such that $||H^n-L||<1$

in $\mathcal{C}(K)$. Endow $\mathcal{C}(K)$ with the sup norm and denote a unit ball in $\mathcal{C}(K)$ by $\mathcal{B}$. Note that:

$$L(\mathcal{B}) = \left\{ (1 - g(s - c^*(s), 1 - l^*(s))) \int_K f(s')\lambda_2(ds') : f \in \mathcal{B} \right\}.$$

Note that

$$L(\mathcal{B}) = \{(1 - g(s - c^*(s), 1 - l^*(s)))\, \alpha :\in [0,1]\}.$$

is the compact set. Hence L is a compact operator. Then:

$$\begin{aligned} |H(f)(s) - L(f)(s)| &= \left| g(s - c^*(s), 1 - l^*(s)) \int_K f(s')\lambda_1(ds'|s) \right|, \\ &\leq g(s - c^*(s), 1 - l^*(s)) \int_K \left| f(s') \right| \lambda_1(ds'|s), \\ &\leq \sup_{s \in K} g(s, 1) < 1. \end{aligned}$$

This completes that H is quasi-compact. Finally applying theorem 3.3 from Futia (1982) we get that H is equicontinuous. We take arbitrary element $s_0 \in \mathrm{supp}(\lambda_2)$. Then $U_\varepsilon := (s_0 - \varepsilon, s_0 + \varepsilon) \cap Int(\mathrm{supp}(\lambda_2)) \neq \emptyset$ for all ε. Hence $Q(U_\varepsilon | s - c^*(s), 1 - l^*(s)) > 0$. Hence Q satisfies uniqueness criterion 2.11 in Futia (1982) . Therefore thesis of this theorem follows directly from his theorem 2.12. ■

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